Lessons from massive S-matrix

Yu-tin Huang (National Taiwan University)

with Nima Arkani-Hamed, Tzu-Chen huang

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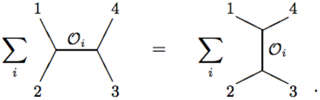
Often by studying consistency conditions directly on physical observables yield model independent constraint:

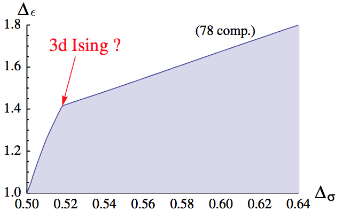
- Tree-level unitarity constraint for $W_L W_L \to W_L W_L$
- Positivity for higher dimensional operators in EFT: A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi

$$A(s,t)|_{t\to 0} = \sum_i c_i s^{2i} \qquad c_i > 0$$

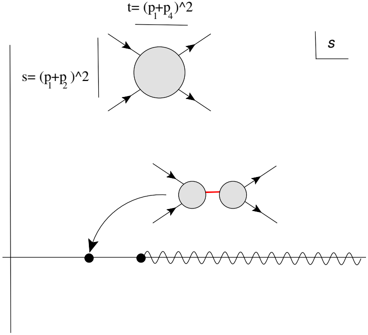
- Bounds on gap for operator dimensions in CFT Sheer El-Showk, Miguel F. Paulos, David Poland, Slava Rychkov, David Simmons-Duffin, Alessandro Vichi



$\rightarrow$



- Consistent massive amplitudes in the IR
- Towards the spectrum for large N QCD
- Constraints for higher dimension operators



Consistent QFTs from unitarity and locality of S-matrix

Kinematics:

- Massless:

$$p^2 = 0 \to det[p^{\alpha\dot{\alpha}}] = 0, \quad p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

  Particle species are labelled by their representation under the U(1) little group

$$\lambda^\alpha \to e^{i\frac{\theta}{2}} \lambda^\alpha, \quad \tilde{\lambda}^{\dot{\alpha}} \to e^{-i\frac{\theta}{2}} \tilde{\lambda}^{\dot{\alpha}}$$

- Massive:

$$p^2 = m^2 \to det[p^{\alpha\dot{\alpha}}] = m^2, \quad p^{\alpha\dot{\alpha}} = \lambda^{\alpha I} \tilde{\lambda}_I^{\dot{\alpha}}$$

  Particle species are labelled by their representation under the SU(2) little group

$$\lambda^{\alpha I} \to g^I{}_J \lambda^{\alpha J}, \quad \tilde{\lambda}^{\dot{\alpha} I} \to g^I{}_J \tilde{\lambda}^{\dot{\alpha} J}$$

Consistent QFTs from unitarity and locality of S-matrix

The S-matrix is a Lorentz invariant function that transforms covariantly under the little group

- Massless:

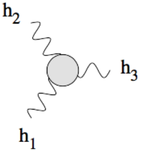
$$M_n^{h_1 h_2 \cdots h_n}(e^{i\frac{\theta}{2}}\lambda_i, e^{-i\frac{\theta}{2}}\tilde{\lambda}_i) = \left(\prod_i e^{i\frac{h_i\theta_i}{2}}\right) M_n^{h_1 h_2 \cdots h_n}(\lambda_i, \tilde{\lambda}_i)$$

- Massive: if the i-th leg is spin-L

$$M_n^{h_1 \cdots \{I_1 I_2, \cdots I_{2L}\} \cdots h_n}(\lambda_i^I, \tilde{\lambda}_i^I, \lambda_i, \tilde{\lambda}_j)$$

Massless warm up:

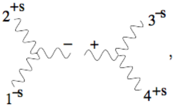
- Kinematics: the three point amplitude for arbitrary massless particles is fixed by its helicity (h_1, h_2, h_3)



$$M^{h_1h_2h_3} = \begin{array}{ll} \tilde{g}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} & \text{when } h_1+h_2+h_3>0 \\ g\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} & \text{when } h_1+h_2+h_3<0 \end{array}$$

where $\langle 12\rangle = \lambda_1^{\alpha}\lambda_{2\alpha}, \quad p_i = \lambda_i\tilde{\lambda}_i.$

- Dynamics: the residue of the four point function is fixed

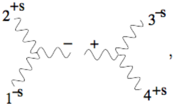


$$, \quad R_s = \left(\frac{\langle 1I\rangle^3}{\langle 12\rangle\langle I2\rangle}\right)^s\left(\frac{[I4]^3}{[I3][43]}\right)^s = \left(\frac{\langle 13\rangle^2[24]^2}{u}\right)^s$$

$$R_s = \left(\frac{\langle 13\rangle^2[24]^2}{u}\right)^s, R_t = \left(\frac{\langle 13\rangle^2[24]^2}{s}\right)^s, R_u = \left(\frac{\langle 13\rangle^2[24]^2}{t}\right)^s$$

Massless warm up:

- Dynamics: the residue of the four point function is fixed



$$R_s = \left(\frac{\langle 1I\rangle^3}{\langle 12\rangle\langle I2\rangle}\right)^{s}\left(\frac{[I4]^3}{[I3][43]}\right)^{s} = \left(\frac{\langle 13\rangle^2[24]^2}{u}\right)^{s}$$

$$R_s = \left(\frac{\langle 13\rangle^2[24]^2}{u}\right)^{s}, R_t = \left(\frac{\langle 13\rangle^2[24]^2}{s}\right)^{s}, R_u = \left(\frac{\langle 13\rangle^2[24]^2}{t}\right)^{s}$$

Consistency of the four-point function in all channels impose theory constraints:

$$\langle 13\rangle^2[24]^2\left(\frac{A^{a_1a_2a_3a_4}}{st} + \frac{B^{a_1a_2a_3a_4}}{tu} + \frac{C^{a_1a_2a_3a_4}}{us}\right)$$

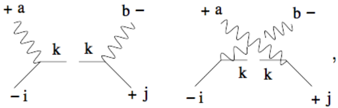
$$\begin{aligned} C^{a_1a_2a_3a_4} - A^{a_1a_2a_3a_4} &= f^{a_1a_2e}f^{ea_3a_4} \\ A^{a_1a_2a_3a_4} - B^{a_1a_2a_3a_4} &= f^{a_2a_3e}f^{ea_4a_1} \\ B^{a_1a_2a_3a_4} - C^{a_1a_2a_3a_4} &= f^{a_3a_1e}f^{ea_2a_4} \end{aligned}$$

The solution:

$$f^{a_1a_2e}f^{ea_3a_4} + f^{a_2a_3e}f^{ea_4a_1} + f^{a_3a_1e}f^{ea_2a_4} = 0$$

An interaction vector theory has a global Lie-2 algebra structure

Further constraints from coupling mixed spins:



$$\left\{ \begin{array}{l} R_s = \frac{(\langle 13\rangle[24])^{2S}\langle 3|p_1 - p_4|2]^{4-2S}}{u^2} \\ R_u = \frac{(\langle 13\rangle[24])^{2S}\langle 3|p_1 - p_4|2]^{4-2S}}{s^2} \end{array} \right.$$

which leads to the following universal form of coupling to gravity

$$\kappa^2 \frac{(\langle 13\rangle[24])^{2s}[2|(p_1 - p_4)|3\rangle^{4-2s}}{stu} .$$

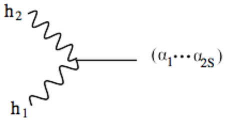
Note that for $S > 2$ there is unphysical poles $\rightarrow$ there can be no fundamental massless particle with $S > 2$ in flat space

Now for massive:

$$M_3^{\{I_1\cdots I_{2S}\},h_1,h_2} = \lambda_{3,\alpha_1}^{I_1}\cdots\lambda_{3,\alpha_{2S}}^{I_{2S}} M_3^{\{\alpha_1\cdots\alpha_{2S}\},h_1,h_2}$$

Find two vectors $(v_\alpha u_\beta)$ to span the SL(2,C) space. All possible massive amplitudes are just all possible polynomial function of $(v_\alpha u_\beta)$

- 2 massless 1 massive



$$M^{h_1h_2}{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}}$$

Since both legs $1, 2$ are massless, their spinors can serve as a natural basis:

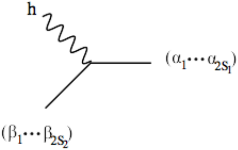
$$(v_\alpha, u_\alpha) = (\lambda_{1\alpha}, \lambda_{2\alpha}) \tag{4.2}$$

The helicity weight (h_1, h_2) then completely fixes the degree-$2S$ polynomial in λ_1, λ_2 up to an overall coupling constant:

$$M^{h_1h_2}{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}} = \frac{g}{m^{2S+h_1+h_2-1}}\left(\lambda_1^{S+h_2-h_1}\lambda_2^{S+h_1-h_2}\right)_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}}[12]^{S+h_1+h_2}\,, \tag{4.3}$$

Unique!

- 1 massless 2 massive



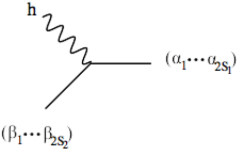
$$M^h{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}}$$

For unequal mass:

$$(v_\alpha, u_\alpha) = \left(\lambda_{3\alpha},\ p_{2\alpha\dot{\beta}}\tilde{\lambda}_3^{\dot{\beta}}\right)$$

$$M^h{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}} = \sum_{i=1}^{C} g_i(u^{S_1+S_2+h}v^{S_1+S_2-h})_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}}$$

- 1 massless 2 massive



$$M^h{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}}$$

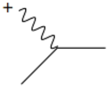
For equal mass: v, u becomes colinear

$$v^\alpha u_\alpha = 0 \rightarrow \quad x\lambda_{3\alpha} = \frac{p_{2\alpha\dot{\alpha}}}{m}\tilde{\lambda}_3^{\dot{\alpha}},$$

$$\begin{aligned} M^h{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}} &= \sum_{i=|S_1-S_2|}^{(S_1+S_2)} g_i x^{h+i} (\lambda_3^{2i} \varepsilon^{S_1+S_2-i})_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}} \\ &= \sum_{i=|S_1-S_2|}^{(S_1+S_2)} g_i\, x^h \left[\lambda_3^i \left(\frac{p_2\tilde{\lambda}_3}{m}\right)^i \varepsilon^{S_1+S_2-i}\right]_{\{\alpha_1\alpha_2\cdots\alpha_{2S_1}\},\{\beta_1\beta_2\cdots\beta_{2S_2}\}} \end{aligned}$$

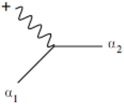
Exposes the simplicity of the interactions:

Minimal coupling is just x!

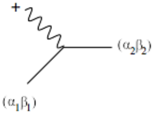


$$\text{Scalars}: \quad \epsilon_3 \cdot p_1 = \frac{\langle \xi | p_1 | 3]}{\langle 3\xi \rangle} = -mx\,, \tag{4.14}$$

where we've used the identity $xm\lambda_3 = p_1|3]$. Similarly for spin-$\frac{1}{2}$ and 1, we have:



$$\text{Fermons}: \quad \mathbf{u}_1 \not{\epsilon}_3 v_2 = \left(\frac{p_2^{\dot\gamma\alpha_2}}{m}, \delta_\gamma^{\alpha_2}\right)\begin{pmatrix} 0 & \frac{\tilde\lambda_{3\dot\gamma}\xi_\beta}{\langle 3\xi\rangle} \\ -\frac{\tilde\lambda_3^{\dot\beta}\xi^\gamma}{\langle 3\xi\rangle} & 0 \end{pmatrix}\begin{pmatrix} \frac{p_{2\dot\beta}{}^{\alpha_1}}{m} \\ \varepsilon^{\alpha_1\beta} \end{pmatrix}$$
$$= x\varepsilon^{\alpha_1\alpha_2} \tag{4.15}$$



$$\text{Vectors}: \quad \frac{p_1^{\beta_1\dot\alpha_1}}{m}\left[\epsilon_3 \cdot p_1 \varepsilon_{\alpha_1\alpha_2}\varepsilon_{\dot\alpha_1\dot\alpha_2} + p_{2\alpha_1\dot\alpha_1}\epsilon_{3\alpha_2\dot\alpha_2} - p_{1\alpha_2\dot\alpha_2}\epsilon_{3\alpha_1\dot\alpha_1}\right]\frac{p_2^{\beta_2\dot\alpha_2}}{m}$$
$$= -mx\left(\varepsilon^{\alpha_1\alpha_2}\varepsilon^{\beta_1\beta_2} + sym(\alpha \leftrightarrow \beta)\right). \tag{4.16}$$

Consistent factorisation at four-point leads to similar theory constraints:

Compton scattering for spin-3/2

$$-\frac{\langle 3|p_1|2]^2}{t}\left(\frac{\langle 43\rangle[12]+\langle 13\rangle[42]}{\langle 3|p_1|2]}\right)^3 .$$

Leads to

$$M(1^{\frac{3}{2}},\gamma_2^{+1},\gamma_3^{-1},4^{\frac{3}{2}})=\frac{\langle 3|p_1|2]^2}{(u-m^2)(s-m^2)}A^3-\left\{\frac{\langle 3|p_1|2][21]\langle 34\rangle}{2m^2(s-m^2)}\times\right.$$
$$\left.\left(3A^2-3A\frac{\langle 43\rangle[32]\langle 21\rangle}{2m^3}+\frac{\langle 43\rangle^2[32]^2\langle 21\rangle^2}{4m^6}\right)+(1\leftrightarrow 4)\right\}$$

At high energies:

$$\frac{\langle 3|p_1|2]^2}{(u-m^2)(s-m^2)}\frac{[14]^3}{m^3}\to\frac{\langle 31\rangle^2[12]^2}{us}\frac{[14]^3}{m^3}.$$

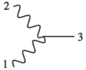
There are no isolated fundamental charged particles beyond spin-1! No fundamental particles beyond spin-2

Higgs mechanism as an IR unification

If particles are fundamental, the IR massive amplitude must be consistent with massless UV ones $\rightarrow$ recovers all features of the Higgs mechanism

Shows unambiguously an obstruction for spin-2

$$= \frac{1}{M_{pl}}\langle \mathbf{1}|p_2 - p_3|\mathbf{1}]^2$$



$$= \frac{1}{M_{pl}m^2}\langle \mathbf{12}\rangle^2[\mathbf{12}]^2$$

$$= \frac{1}{M_{pl}m^6}\left[\langle \mathbf{12}\rangle[\mathbf{12}]\langle \mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cyc.}\right]^2 ,$$

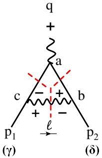
At high energies

$$\frac{1}{M_{pl}m^6}\left[\langle \mathbf{12}\rangle[\mathbf{12}]\langle \mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cyc.}\right]^2 \xrightarrow{HE} \begin{cases} (-2,-2,+2): & \frac{1}{M_{pl}}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2} \\ (0,-2,0): & \frac{3}{M_{pl}}\frac{\langle 12\rangle^2\langle 23\rangle^2}{\langle 13\rangle^2} \end{cases}$$

Applications

Streamline loop calculations:

Lets consider the $e^+, e^- \to \gamma$ at one loop. The diagram we want to build is:



$$\sim e^3 m^3 x_a \varepsilon_{\alpha\beta} \left[\varepsilon^{\beta\gamma} \frac{x_b}{x_c} \left(\varepsilon + x_c \frac{\lambda_\ell \lambda_\ell}{m} \right)^{\alpha\delta} + \varepsilon^{\alpha\delta} \frac{x_c}{x_b} \left(\varepsilon - x_b \frac{\lambda_\ell \lambda_\ell}{m} \right)^{\beta\gamma} \right]$$

The pure ε piece is identical to a charged scalar. The magnetic moment is then

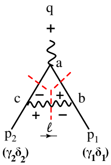
$$e^2 m^2 x_a (x_b - x_c) \lambda_\ell^\delta \lambda_\ell^\gamma = -m x_a q^\delta{}_{\dot\alpha} \ell^{\dot\alpha\beta}$$

This is a simple linear integrand:

$$-m x_a \int \frac{d^4\ell}{(2\pi)^4} \frac{q^\delta{}_{\dot\alpha} \ell^{\dot\alpha\beta}}{\ell^2((\ell - p_2)^2 - m^2)((\ell + p_1)^2 - m^2)} = \frac{e^2}{(4\pi)^2} 2 x_a \frac{q^\delta{}_{\dot\alpha} p_1^{\dot\alpha\beta}}{m} = \frac{\alpha}{2\pi} x^a \lambda_q^\gamma \lambda_q^\delta$$

Applications

$W^+, W^- \to \gamma$ fits on a page:



$$\sim e^3 m^3 x_a \varepsilon_{(\alpha_1\alpha_2}\varepsilon_{\beta_1)\beta_2}\Bigg[\varepsilon^{\beta_1(\gamma_1}\varepsilon^{\alpha_1\delta_1)}\frac{x_b}{x_c}\left(\varepsilon + x_c\frac{\lambda_\ell\lambda_\ell}{m}\right)^{\alpha_2(\delta_2}\left(\varepsilon + x_c\frac{\lambda_\ell\lambda_\ell}{m}\right)^{\beta_2\gamma_2)}$$

$$+\varepsilon^{\beta_2(\gamma_2}\varepsilon^{\alpha_2\delta_2)}\frac{x_c}{x_b}\left(\varepsilon - x_b\frac{\lambda_\ell\lambda_\ell}{m}\right)^{\alpha_1(\delta_1}\left(\varepsilon - x_b\frac{\lambda_\ell\lambda_\ell}{m}\right)^{\beta_1\gamma_1)}\Bigg].$$

Leaving behind the electric coupling:

$$\underbrace{-4e^2 x_a m\left[\varepsilon^{\delta_1(\delta_2}q^{\gamma_1}{}_{\dot\alpha}\ell^{\dot\alpha\gamma_2)} + \varepsilon^{\gamma_1(\delta_2}q^{\delta_1}{}_{\dot\alpha}\ell^{\dot\alpha\gamma_2)}\right]}_{f_1(q)} + \underbrace{\frac{2e^2x_a}{3m}\left[(p_{1\dot\alpha}{}^{\delta_1}\ell^{\dot\alpha\gamma_1})(p_{2\dot\alpha}{}^{\delta_2}\ell^{\dot\alpha\gamma_2}) + perm\right]}_{f_2(q)}$$

Which leads to:

$$F_1(q) = I_3[f_1(q)] = 4\frac{\alpha}{2\pi}x^a\left(\varepsilon^{\delta_1(\delta_2}\lambda_q^{\gamma_1}\lambda_q^{\gamma_2)} + \varepsilon^{\gamma_1(\delta_2}\lambda_q^{\delta_1}\lambda_q^{\gamma_2)}\right).$$

$$F_2(q) = I_3[f_2(q)] = \frac{\alpha}{(4\pi)9m^3}\left(\mathcal{O}_{1,2}^{\delta_1\gamma_1}\mathcal{O}_{1,2}^{\delta_2\gamma_2} + perm: \delta_1,\gamma_1,\delta_2,\gamma_2\right),$$

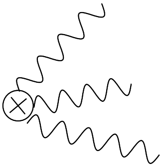
where we've defined $\mathcal{O}_{i,j}^{\alpha\beta} \equiv p_{i\dot\alpha}{}^{\alpha}p_j^{\dot\alpha\beta}$.

Applications

Correlation functions:

$$\langle \phi(x_1) \cdots\cdots \phi(x_n) \rangle$$

In real measurements we are really considering



$$\langle \tilde{\phi}(k_1) \cdots\cdots \tilde{\phi}(k_n) \rangle \equiv \textit{Fourier}\,[\langle \phi(x_1) \cdots\cdots \phi(x_n) \rangle]$$

For arbitrary k! (This is why local operators cannot exist in gravity)

Instead of Fourier transform position space calculations, we can compute massive S-matrix with $\tilde{\phi}(k_1)$ as a massive higher spin particle!

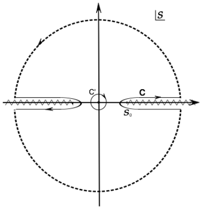
Can one learn anything from S-matrix constraint of consistent UV completions?

- UV completion of gravity
- Spectrum of bound states in SU(N) YM

For SU(N) QCD the spectrum contains stable glueballs and mesons

- From analyticity of the S-matrix if $\frac{1}{s^L} A(s,t)|_{s \to \infty} = 0$ for some t and L, then after subtraction A'

$$A'(s,t) = \oint_C \frac{dv}{v-s} A'(v,t) =$$



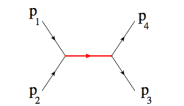
$$-\oint_{C'} \frac{dv}{v-s} A'(v,t)$$

This implies an alternative representation:

$$A'(s,t) = \sum_{i=1} \frac{n(m_i,t)}{s-m_i^2}$$

To reproduce the poles in t we must let the spectrum tend to infinity!

Unitarity requires that



$$\rightarrow A_3(\phi_1, \phi_2, h^\ell) \sim i c_\ell (p_1 - p_2)^{\mu_1} (p_1 - p_2)^{\mu_2} \cdots (p_1 - p_2)^{\mu_\ell} \epsilon_{\mu_1 \mu_2 \cdots \mu_\ell}$$

The residue must take the simple form:

$$[(p_1 - p_2) \cdot (p_3 - p_4)]^{2n} = (t - u)^{2n} \rightarrow n(m_i, t) = P_\ell(1 + 2t/m_i^2)$$

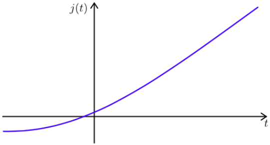
In other words

$$A'(s,t) = \sum_{i=1} \frac{c_i^2 P_{\ell_i}(1 + 2t/m_i^2)}{s - m_i^2}$$

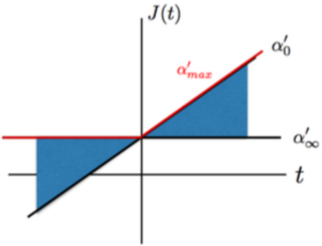
Using this representation, at $1 << s < t$

$$A(s,t)|_{s \to \infty} s^{2+J(t)}$$

the Regge trajectory must be linear! Simon Caron-Huot, Zohar Komargodski, Amit Sever, and Alexander Zhiboedov



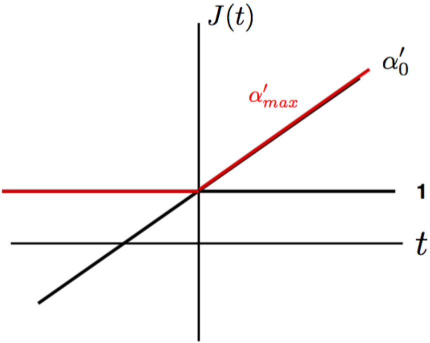
Can we find a function that behaves string like at $t > 0$ but polynomial at $t < 0$?
String in AdS! Polchinski and Strassler (Siegel and Andreev):



We can do even better!

$$A^{\pi}(s,t) = (s+t)\frac{\Gamma[-s]\Gamma[-t]}{\Gamma[1-s-t]} + \frac{1}{s} + \frac{1}{t}$$

We would obtain



Reproduces linear at $t > 0$, and hard scattering at $t < 0$!

Can we gather information with respect to the spectrum from the IR ?

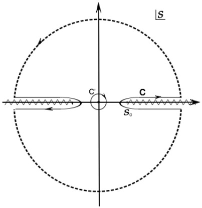
In the IR again these UV degrees of freedom are encoded in the higher dimensional operators:

$$A(s,t) = \sum_{i,j} g_{i,j} s^i t^j$$

There are already known constraints from unitarity A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi

For $t = 0$, we can write

$$g_{i,0} = \oint \frac{ds}{s^{i+1}} A(s,0) =$$

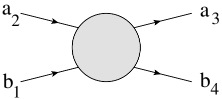


$$= \int_{s_0}^{\infty} Im[A(s,0)] \quad i \in even$$

But from the optical theorem we know that $Im[A(s,0)] = s\sigma > 0 \rightarrow g_{i,0} > 0$

There are more!

For fixed t we have



$$A(s,t) = -\left(\frac{1}{s-m_i^2} + \frac{1}{u-m_i^2}\right) c_i^2 G_{\ell_i}^D \left(1 + \frac{2t}{m_i^2}\right)$$

$G_{\ell_i}^D$ are Gegenbauer polynomials with $G_{\ell_i}^D(1) > 0$.

The low energy expression can be expanded in a way that reflects the $s \leftrightarrow u$ symmetry

$$s = -\frac{t}{2} + z, \quad u = -\frac{t}{2} - z$$

Then

$$A(z,t) = \sum_{i,j} \tilde{g}_{i,j} z^{2i} t^j$$

$$A(z,t) = \sum_{i,j} \tilde{g}_{i,j} z^{2i} t^{j}$$

For a fixed mass dimension L, the space of possible higher dimensional operators has dimension that correspond to the number of (i,j)s that satisfies $2i + j = L$. Any particular theory lives on a particular point in this subspace:

Exp: for $L = 4$ we have

$$\tilde{g}_{0,4} t^4 + \tilde{g}_{1,2} z^2 t^4 + \tilde{g}_{2,0} z^4$$

A given theory is represented as a specific point $(\tilde{g}_{0,4}, \tilde{g}_{1,2}, \tilde{g}_{2,0})$ in this three-dimensional space

Within this space, what is the subspace that has a UV completion of the form?

$$A(z,t) = -\sum_i \left(\frac{1}{-\frac{t}{2} + z - m_i^2} + \frac{1}{-\frac{t}{2} - z - m_i^2} \right) c_{i,\ell_i}^2 G_{\ell_i}^D \left(1 + \frac{2t}{m_i^2}\right)$$

We can also expand in low energy. For fixed L the coefficient for $z^k t^{L-k}$

$$z^k t^{L-k}: \quad \sum_i c_{i,\ell_i}^2 \left\{ \sum_{q=0}^{L-k} \begin{pmatrix} L-k \\ k \end{pmatrix} 2^{2q-L+k} (-)^{L-q} G_{\ell,q}^D \right\} = \sum_i c_{i,\ell_i}^2 \hat{G}_{\ell,k}^D$$

where $G_{\ell,q}^D = \frac{\partial^q}{\partial x^q} G_\ell^D(1+x)\Big|_{x=0}$. For $L=4$ we have infinite set of three-dimensional vectors

$$\mathcal{V}_\ell = \begin{pmatrix} \hat{G}_{\ell,4}^D \\ \hat{G}_{\ell,2}^D \\ \hat{G}_{\ell,0}^D \end{pmatrix}$$

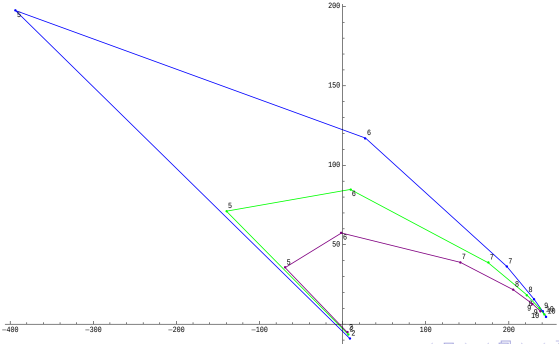
The allowed space is spanned by the convex hull of $\mathcal{V}_\ell$s.

$$\mathcal{V}_\ell = \begin{pmatrix} \hat{G}^D_{\ell,4} \\ \hat{G}^D_{\ell,2} \\ \hat{G}^D_{\ell,0} \end{pmatrix}$$

The allowed space is spanned by the convex hull of $\mathcal{V}_\ell$s.

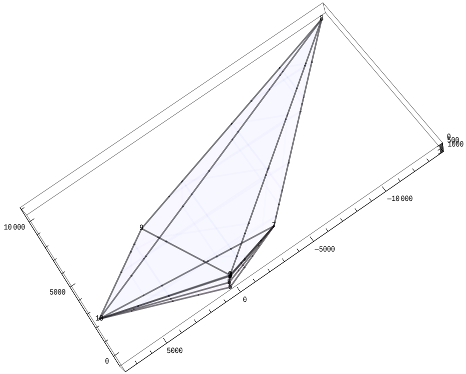
Remarkably, it does not span the full three-dimensional space ! Let's consider the space projectively. Taking

$$v_\ell = \left(\frac{\mathcal{V}^1_\ell - \mathcal{V}^2_\ell}{\mathcal{V}^3_\ell}, \frac{\mathcal{V}^2_\ell - \mathcal{V}^3_\ell}{\mathcal{V}^3_\ell} \right)$$

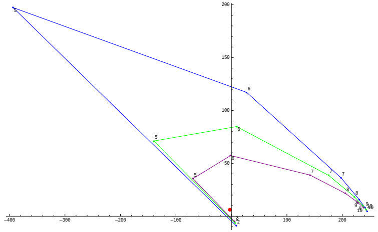


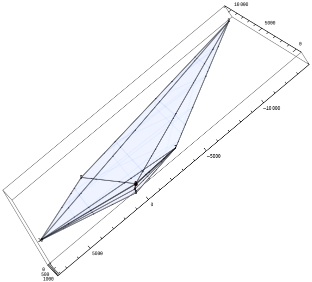
$L = 6$ is spanned by a four-dimensional space: the projective three-dimensional polytope:

$$v_\ell = \left(\frac{\mathcal{V}_\ell^1 - \mathcal{V}_\ell^2}{\mathcal{V}_\ell^4}, \frac{\mathcal{V}_\ell^2 - \mathcal{V}_\ell^3}{\mathcal{V}_\ell^4}, \frac{\mathcal{V}_\ell^3 - \mathcal{V}_\ell^4}{\mathcal{V}_\ell^4} \right)$$



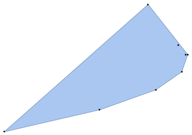
We can ask where does super string theory lies:

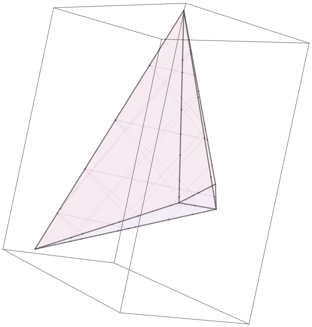




What about spinning external state?

In four-dimensions we also have a universal polynomial, $G_\ell^D \to \mathcal{J}(\ell + 4h, 0, -4h, \cos\theta)$





Conclusion

Consistency conditions (Lorentz Invariance and Unitarity) on the S-matrix imposes stringent constraint

- The general structure of renormalizable interactions (no need for UV completion) are fully determined
- Higher dimensional operators are constrained to live with in a small region of available coupling.