Accessing nucleon structure from Euclidean spacetime

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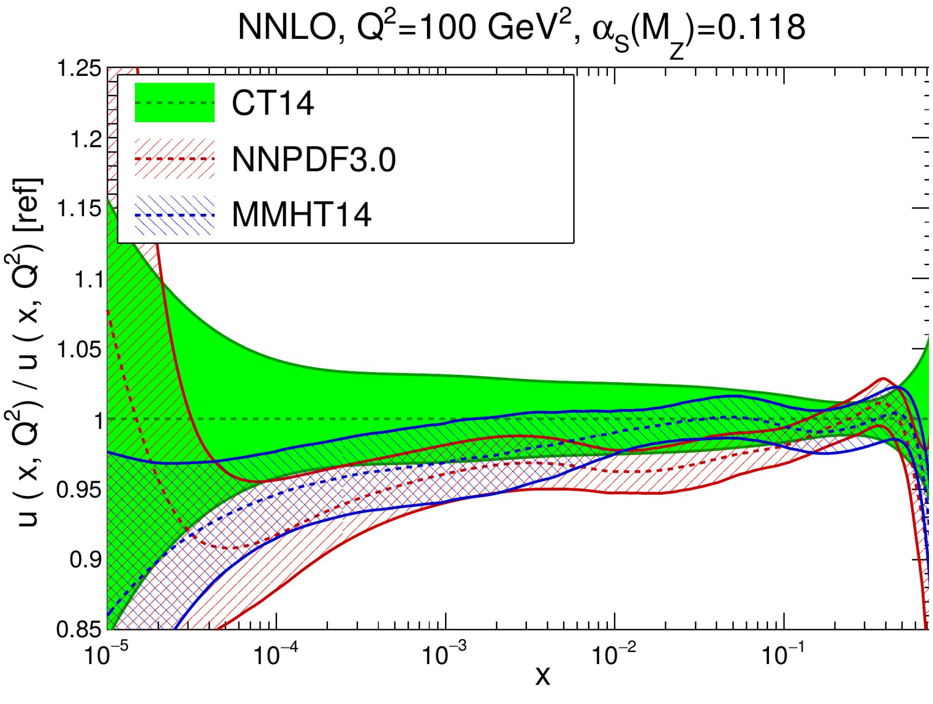
HOW FAST DO PARTONS TRAVEL?

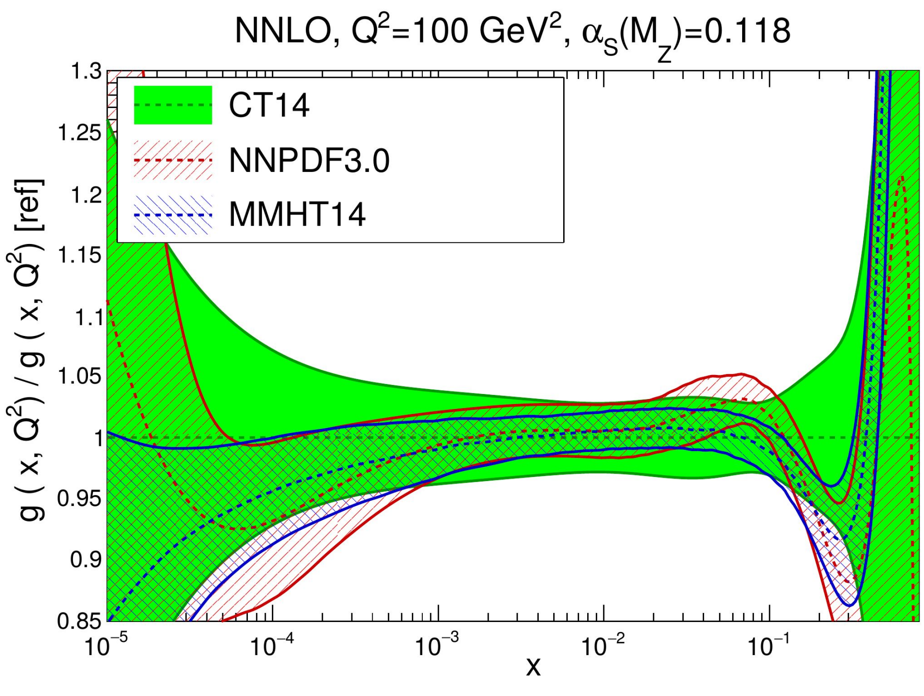
How is the momentum of a fast-moving nucleon distributed amongst its constituents?

WHERE DOES THE SPIN OF A PROTON COME FROM?

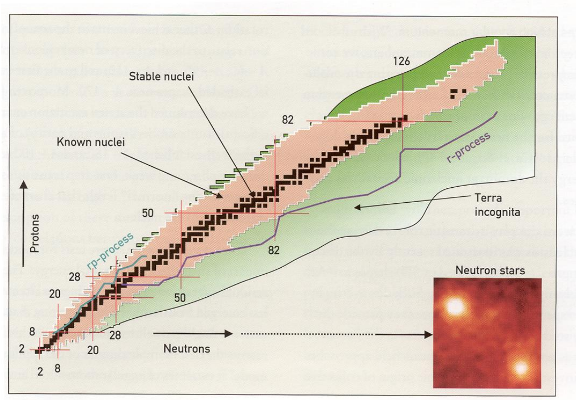
How do position and longitudinal momentum of a parton correlate in a fast-moving nucleon?

PDF UNCERTAINTIES

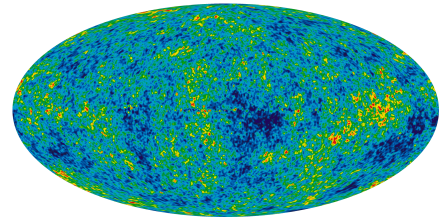


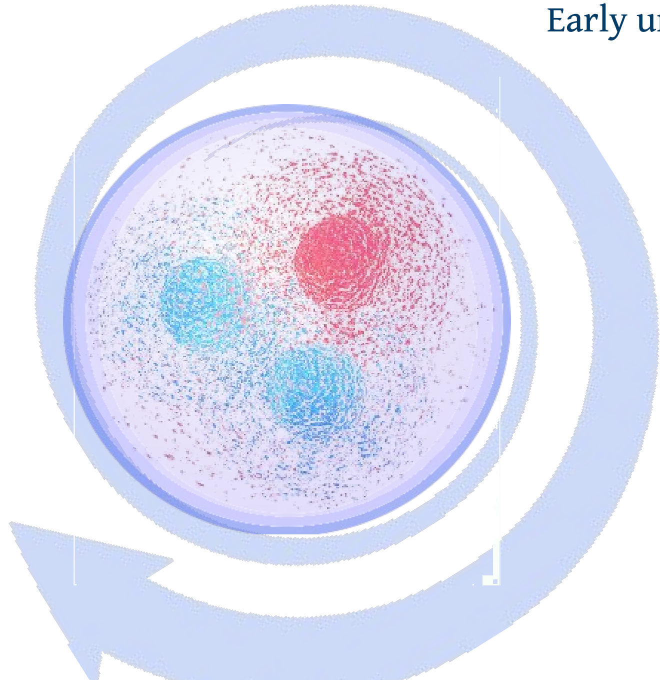


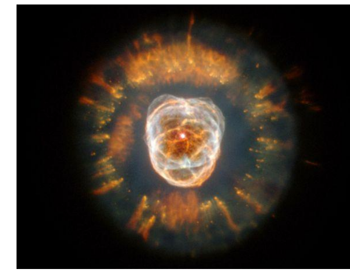
From J. Butterworth *et al.*, J.P.G 43 (2016) 023001

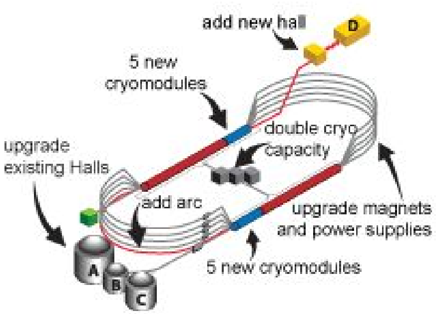


Nuclear landscape

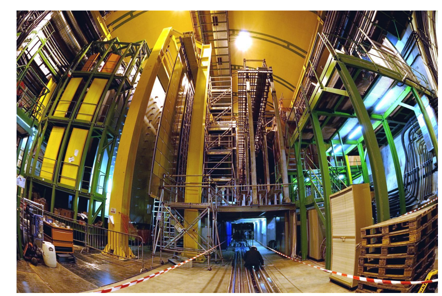

Early universe

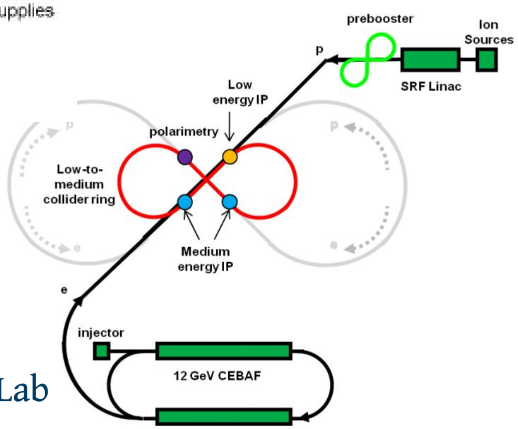



Neutron stars



12 GeV

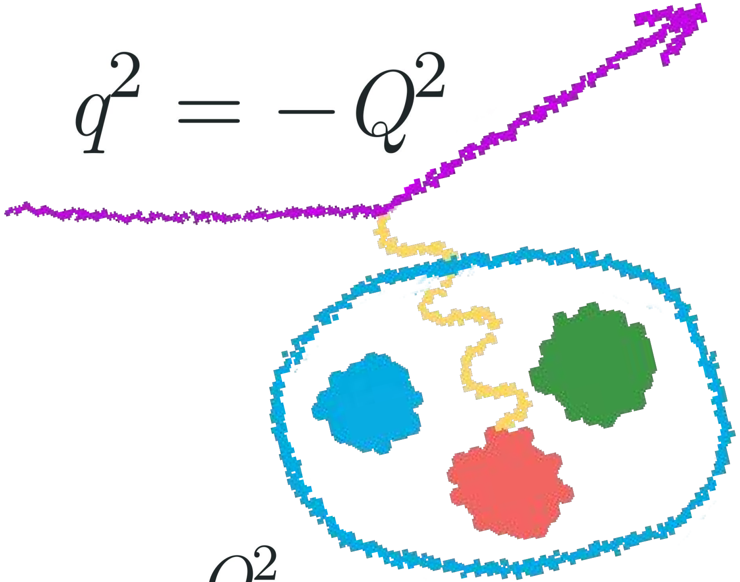

LHCb



EIC @ JLab

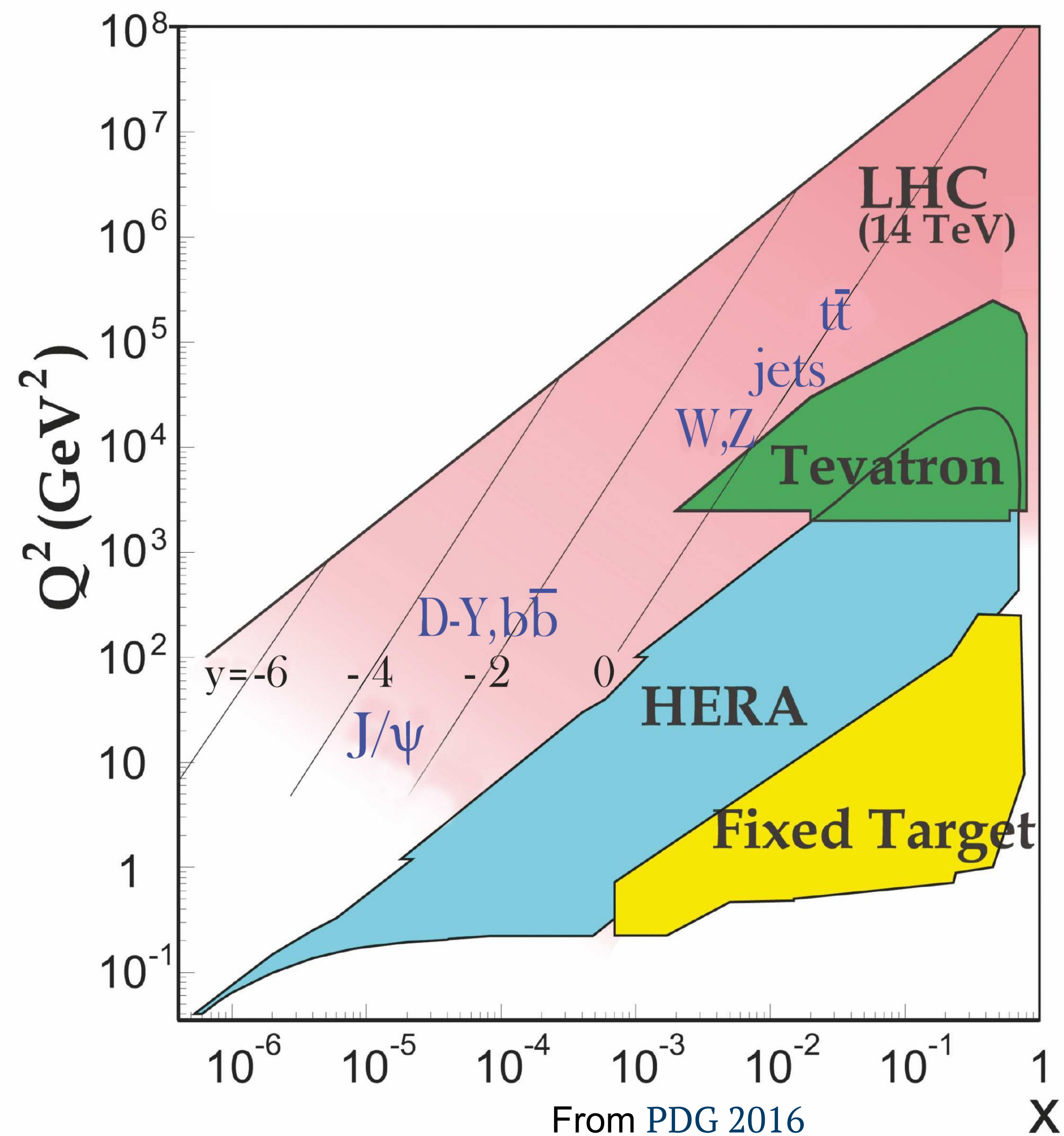

LUX

EXPERIMENTAL EXTRACTION



$$q^2 = -Q^2$$

$$x = \frac{Q^2}{2P \cdot q}$$



From PDG 2016

EXPERIMENTAL EXTRACTION

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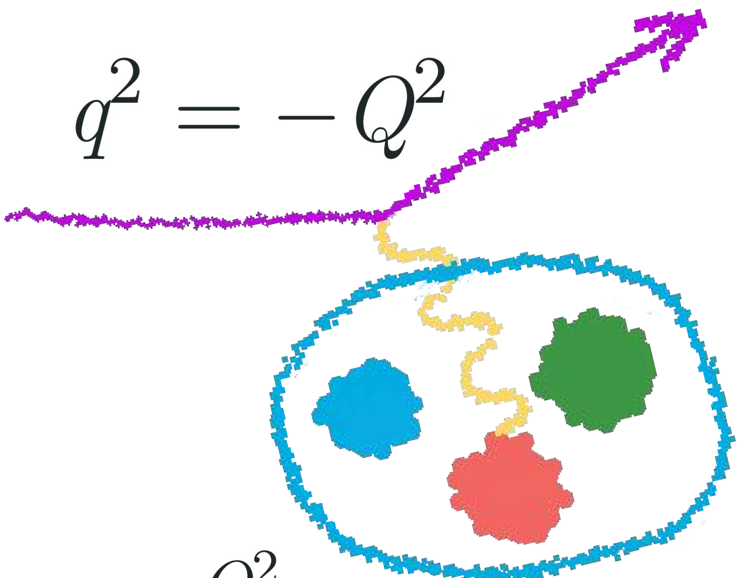

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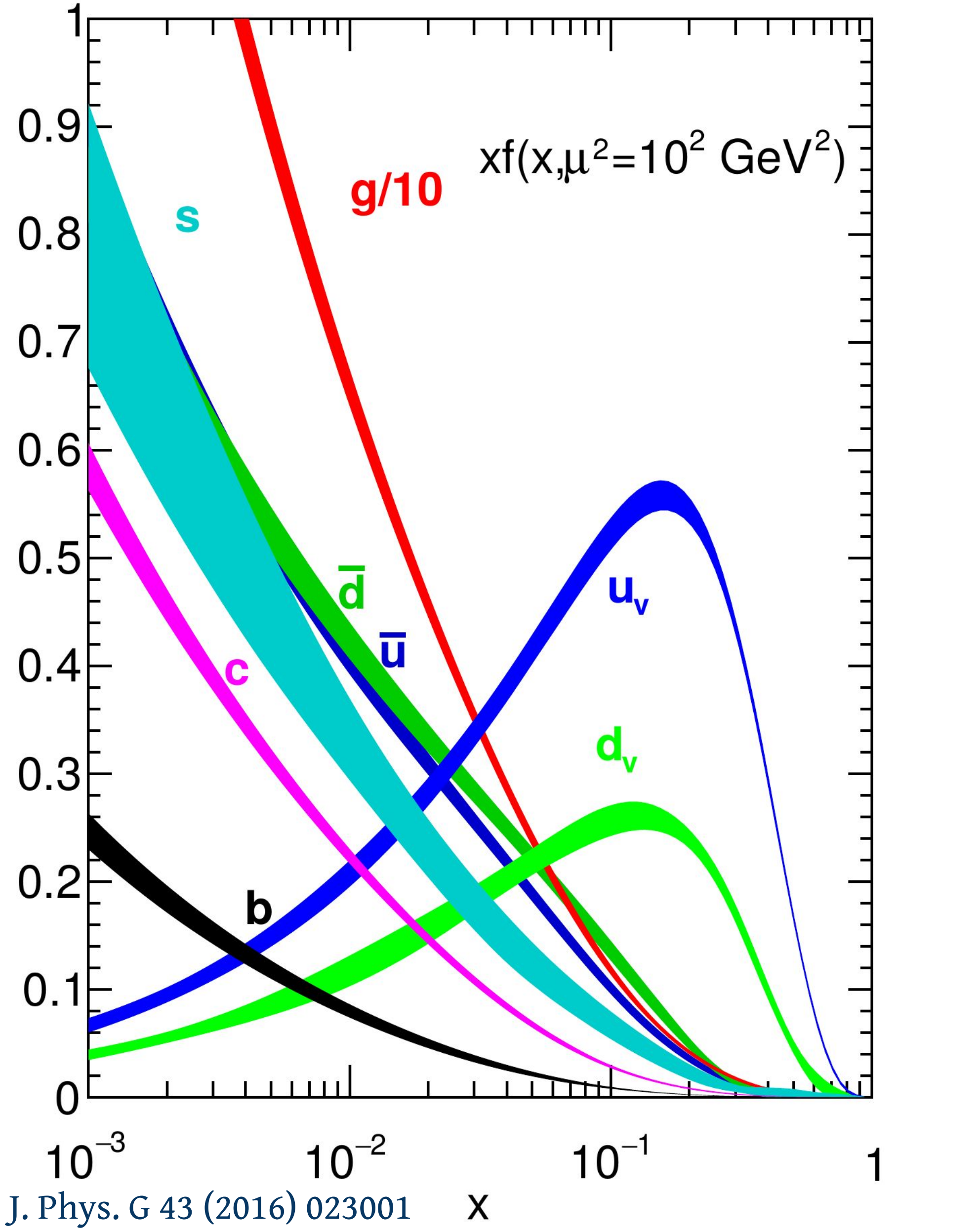


PDF4LHC15 (NNLO), J. Phys. G 43 (2016) 023001

PDFs FROM EUCLIDEAN SPACETIME

An ~~unsolved~~ almost-solved challenge

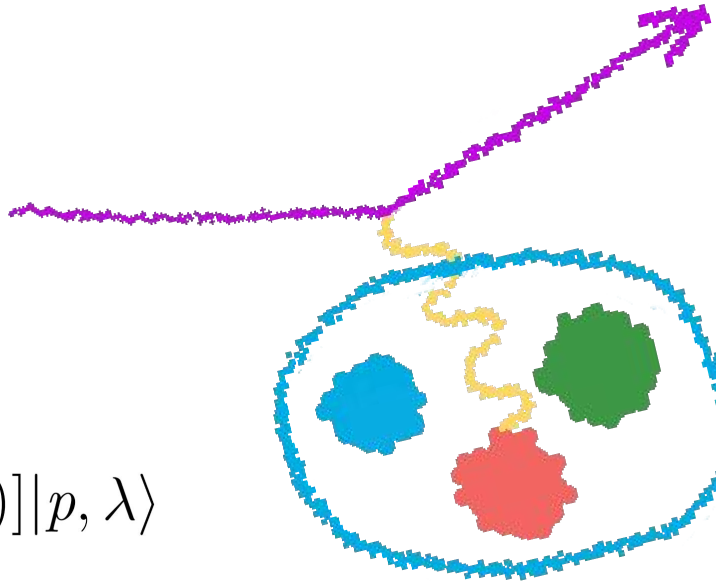
DIS

Decompose cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega\mathrm{d}E} = \frac{e^4}{16\pi^2 Q^4}\ell^{\mu\nu} W_{\mu\nu}$$

Hadronic contribution

$$W_{\mu\nu}(p,q) = \frac{1}{4\pi}\int \mathrm{d}^x e^{iq\cdot x}\,\langle p,\lambda'|[j_\mu(x),j_\nu(x)]|p,\lambda\rangle$$



in turn, expressed in terms of structure functions

$$F(x,Q^2) = \int \mathrm{d}y\, C\left(\frac{x}{y},\frac{Q^2}{\mu^2}\right) f(x,\mu^2)$$

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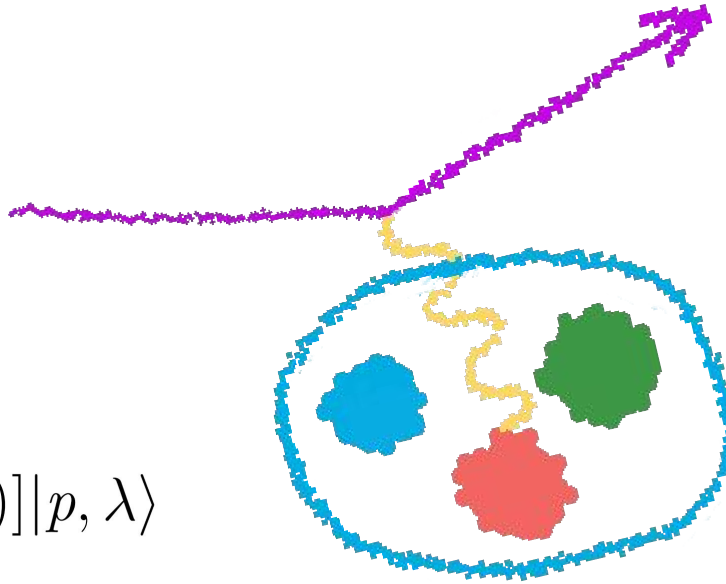

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in turn, expressed in terms of structure functions

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parton distribution functions (PDFs)

PDFs (GPDs)

Defined as

$$f^{(0)}(\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega^-}{4\pi} e^{-i\xi P^+ \omega^-} \left\langle P \left| T\, \overline{\psi}(0, \omega^-, \mathbf{0}_\mathrm{T}) W(\omega^-, 0) \gamma^+ \frac{\lambda^a}{2} \psi(0) \right| P \right\rangle_\mathrm{C}$$

where

$$W(\omega^-, 0) = \mathcal{P} \exp\left[-i g_0 \int_0^{\omega^-} \mathrm{d}y^- A_\alpha^+(0, y^-, \mathbf{0}_\mathrm{T}) T_\alpha \right]$$

Renormalised PDFs

$$f(\xi, \mu) = \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} \mathcal{Z}\left(\frac{\xi}{\zeta}, \mu\right) f^{(0)}(\zeta)$$

Satisfy DGLAP evolution

$$\mu \frac{\mathrm{d}f(\xi, \mu)}{\mathrm{d}\mu} = \frac{\alpha_s(\mu)}{\pi} \int_x^1 \frac{\mathrm{d}\zeta}{\zeta} f(\zeta, \mu) P\left(\frac{\xi}{\zeta}\right)$$

MOMENTS OF PDFs

Mellin moments of PDFs

$$a^{(n)}(\mu) = \int_0^1 \mathrm{d}\xi\, \xi^{n-1} \left[f(\xi,\mu) + (-1)^n \overline{f}(\xi,\mu)\right] = \int_{-1}^1 \mathrm{d}\xi\, \xi^{n-1} f(\xi,\mu)$$

related to matrix elements

$$\langle P|\mathcal{O}^{\{\nu_1 \ldots \nu_n\}}(\mu)|P\rangle = 2a^{(n)}(\mu)\left(P^{\nu_1}\cdots P^{\nu_n} - \text{traces}\right)$$

of local twist-two operators

$$\mathcal{O}^{\{\nu_1 \ldots \nu_n\}}(\mu) = Z_{\mathcal{O}}(\mu)\left[i^{n-1}\overline{\psi}(0)\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_n\}}\frac{\lambda^a}{2}\psi(0) - \text{traces}\right]$$

Moving beyond three moments is very challenging

$$\overline{\psi}\gamma_4\gamma_5 \overleftrightarrow{D}_4 \overleftrightarrow{D}_4 \psi \sim \frac{1}{a^2}\overline{\psi}\gamma_4\gamma_5\psi$$

Cannot reconstruct PDFs from only three moments

Detmold *et al.*, Eur. Phys. J. C 3 (2001) 1

Detmold *et al.*, Phys. Rev. D 68 (2001) 034025

Detmold *et al.*, Mod. Phys. Lett. A 18 (2003) 2681

MOMENTS OF PDFs: AXIAL CHARGE

Nucleon axial charge

$$g_A = \langle 1 \rangle_{\Delta u^+ - \Delta d^+} = \int_{-1}^{1} \mathrm{d}x \, [\Delta u(x) - \Delta d(x)]$$

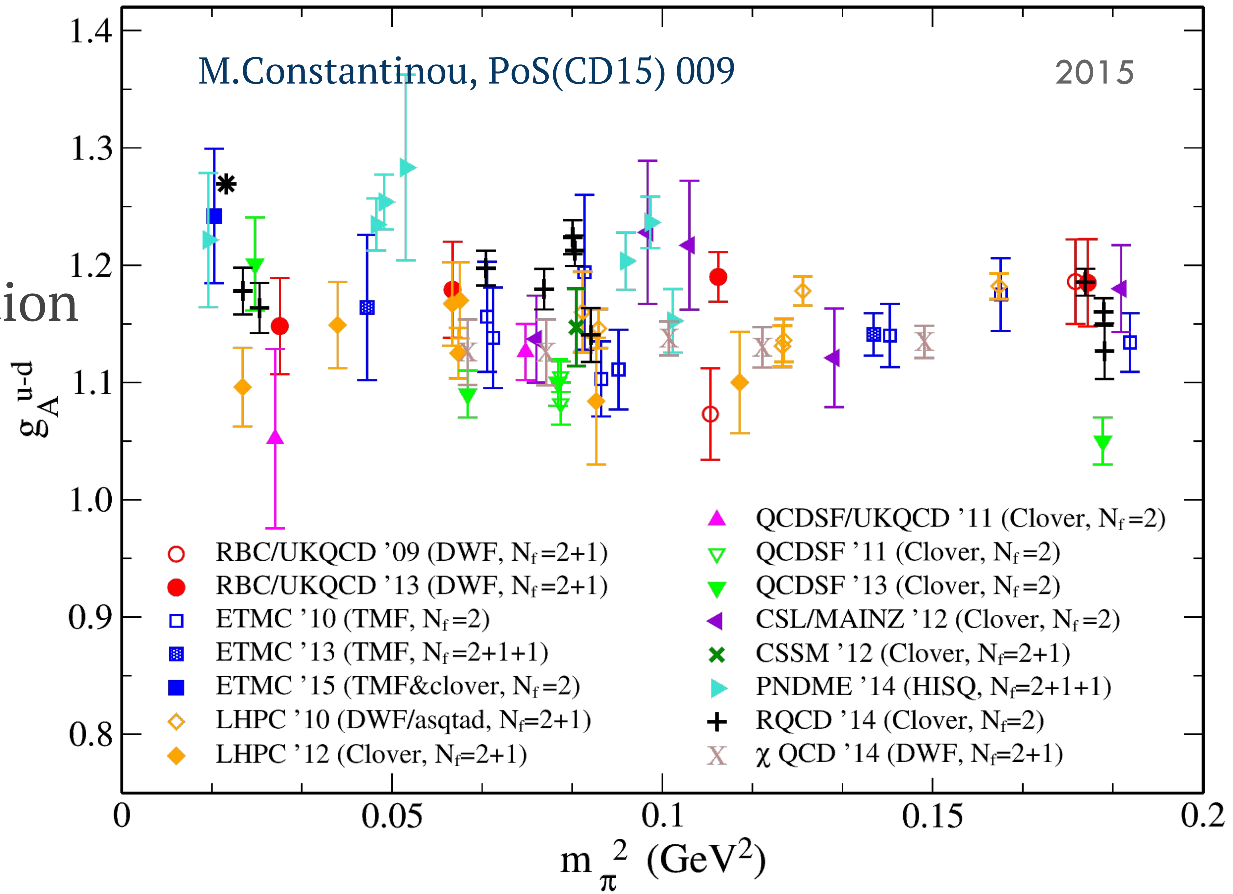
Controls:

- nucleon-nucleon force
- free neutron β-decay
- early Universe composition

Experimental value

- cold neutron decay

$$g_A^{\mathrm{exp}} = 1.2723(23)$$



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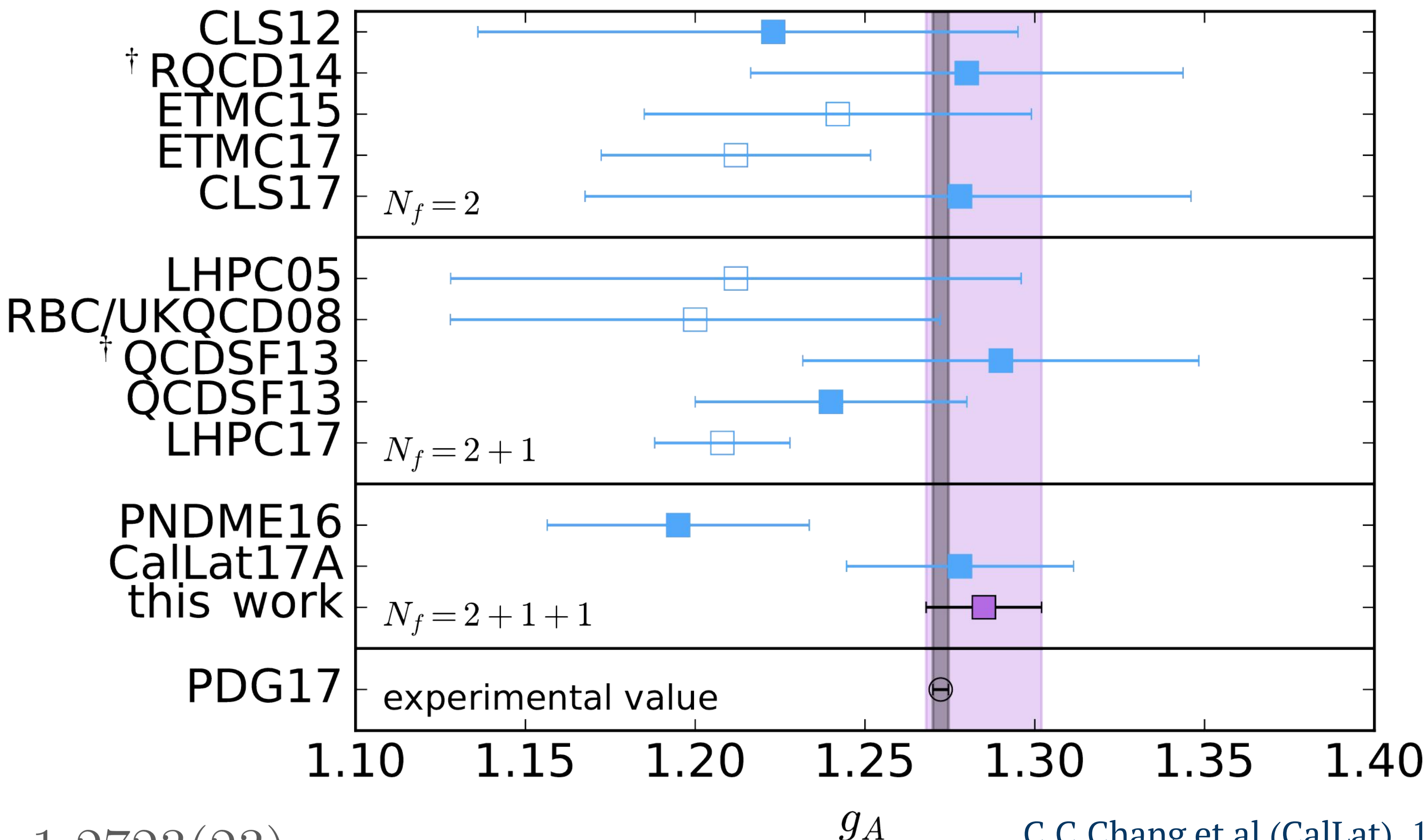
$$g_A^{\mathrm{CalLat}} = 1.285(17)$$



$$g_A^{\mathrm{exp}} = 1.2723(23)$$

C.C.Chang et al (CalLat), 1710.06523
E.Berkowitz et al (CalLat), 1704.01114

SPACELIKE DISTRIBUTIONS

Matrix elements of spacelike nonlocal operators

SPACELIKE DISTRIBUTIONS

- Quasi distributions

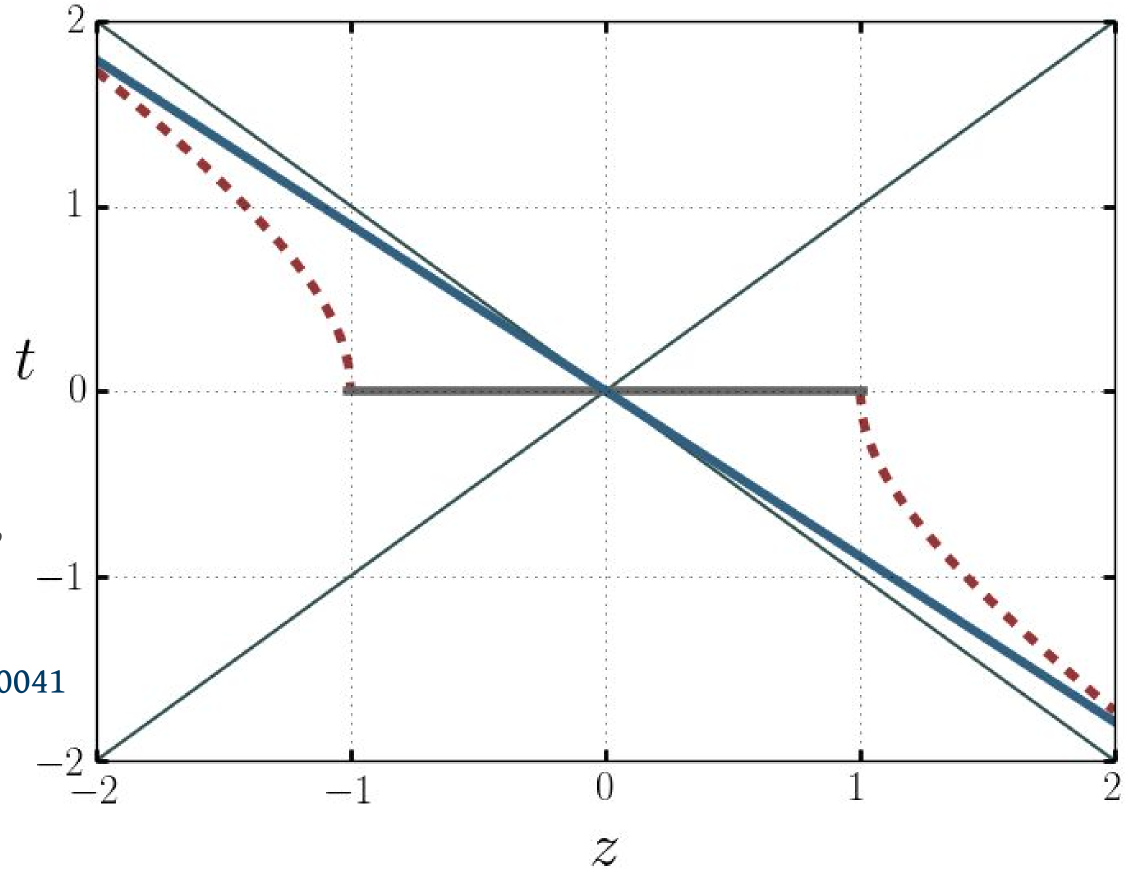
X. Ji, PRL 110 (2013) 262002
X. Ji, Sci.Ch. PMA 57 (2014) 1407

- Pseudo distributions

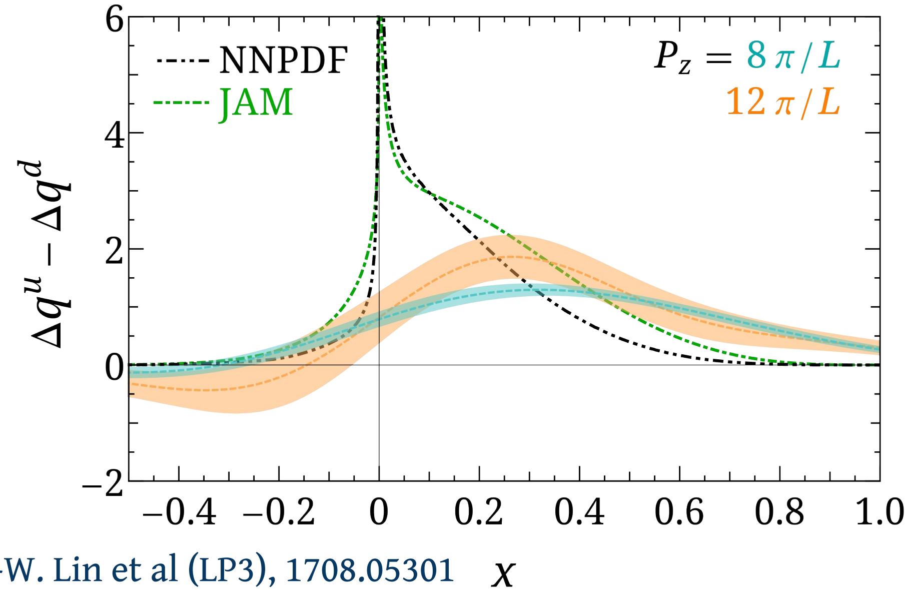
A.Radyushkin, PLB 767 (2017) 314
A.Radyushkin, PRD 96 (2017) 034025

- Lattice "cross-sections"

Y.-Q. Ma & J.-W. Qiu, 1404.6860
Y.-Q. Ma & J.-W. Qiu, IJMP 37 (2015) 1560041
Y.-Q. Ma & J.-W. Qiu, 1709.03018

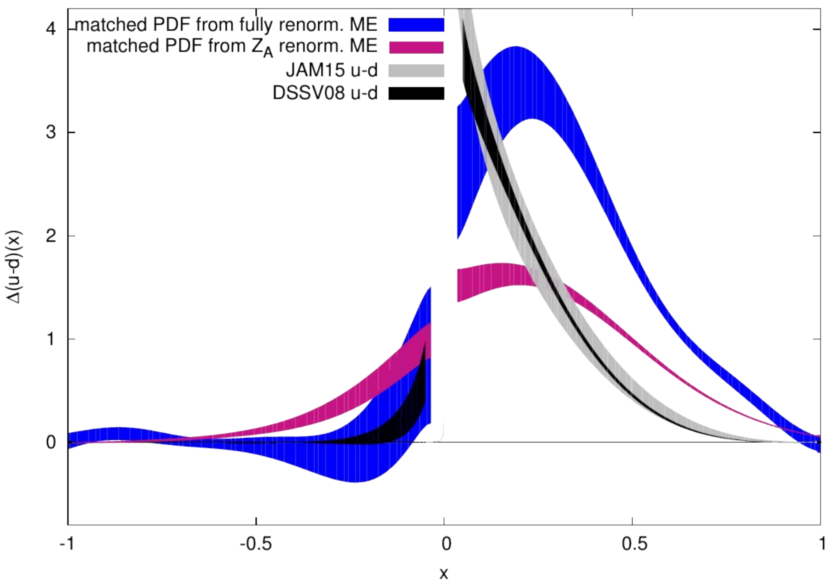


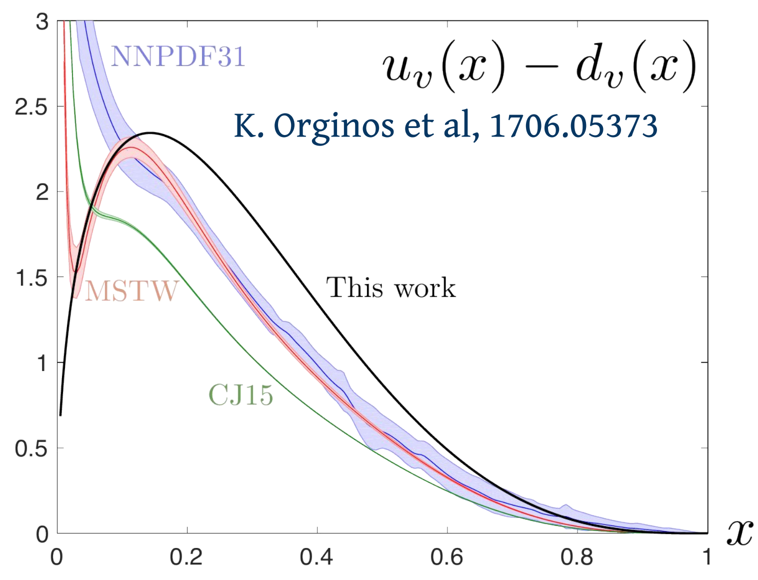
SPACELIKE DISTRIBUTIONS



H.-W. Lin et al (LP3), 1708.05301

C. Alexandrou et al (ETMC), NPB 923 (2017) 394





K. Orginos et al, 1706.05373

See also:

H.-W. Lin et al, PRD 91 (2015) 054510
C. Alexandrou et al., PRD 92 (2015) 014502
J.-H. Zhang et al., arXiv:1702.00008
J.-W. Chen et al., NPB 911 (2016) 246

QUASI DISTRIBUTIONS

X. Ji, PRL 110 (2013) 262002
X. Ji, Sci.Ch. PMA 57 (2014) 1407

Defined as

$$q(x, \mu^2, P^z) = \int \frac{\mathrm{d}z}{4\pi} e^{i\,x\,z\,k^z} \langle P | \overline{\psi}(z) \gamma^z e^{-ig \int_0^z \mathrm{d}z' A^z(z')} \psi(0) | P \rangle_C$$

Recall

$$f^{(0)}(\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega^-}{4\pi} e^{-i\xi P^+ \omega^-} \left\langle P \left| T\, \overline{\psi}(0, \omega^-, \mathbf{0}_\mathrm{T}) W(\omega^-, 0) \gamma^+ \frac{\lambda^a}{2} \psi(0) \right| P \right\rangle_\mathrm{C}$$

Related to light-front PDFs via

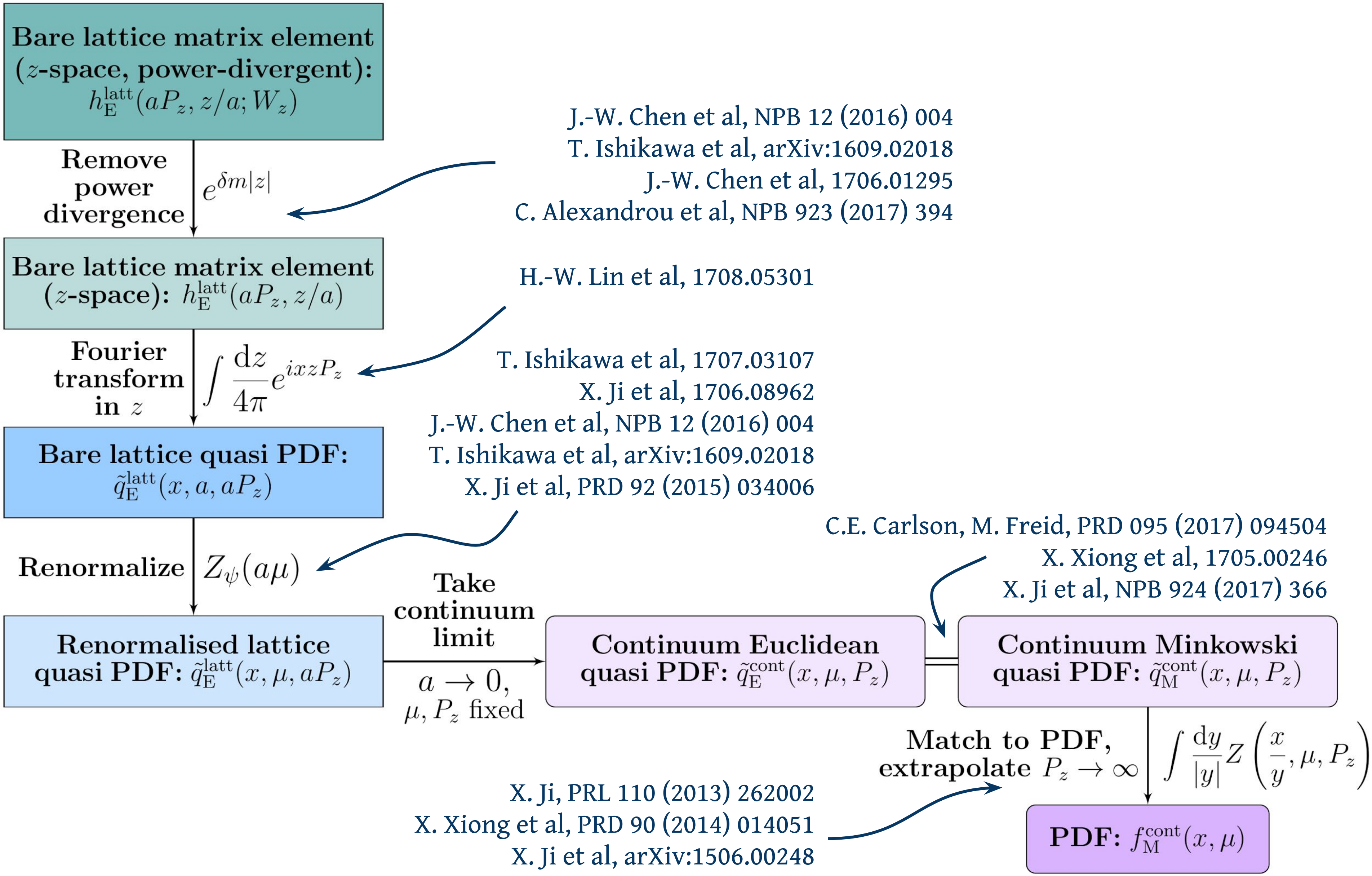
$$q(x, \mu^2, P^z) = \int_x^1 \frac{\mathrm{d}y}{y} Z\left(\frac{x}{y}, \frac{\mu}{P^z}\right) f(y, \mu^2) + \mathcal{O}\left(\frac{\Lambda^2_\mathrm{QCD}}{(P^z)^2}, \frac{M^2}{(P^z)^2}\right)$$

GENERAL PROCEDURE

Bare lattice matrix element
(z-space, power-divergent):
$h^{\mathrm{latt}}_{\mathrm{E}}(aP_z, z/a; W_z)$

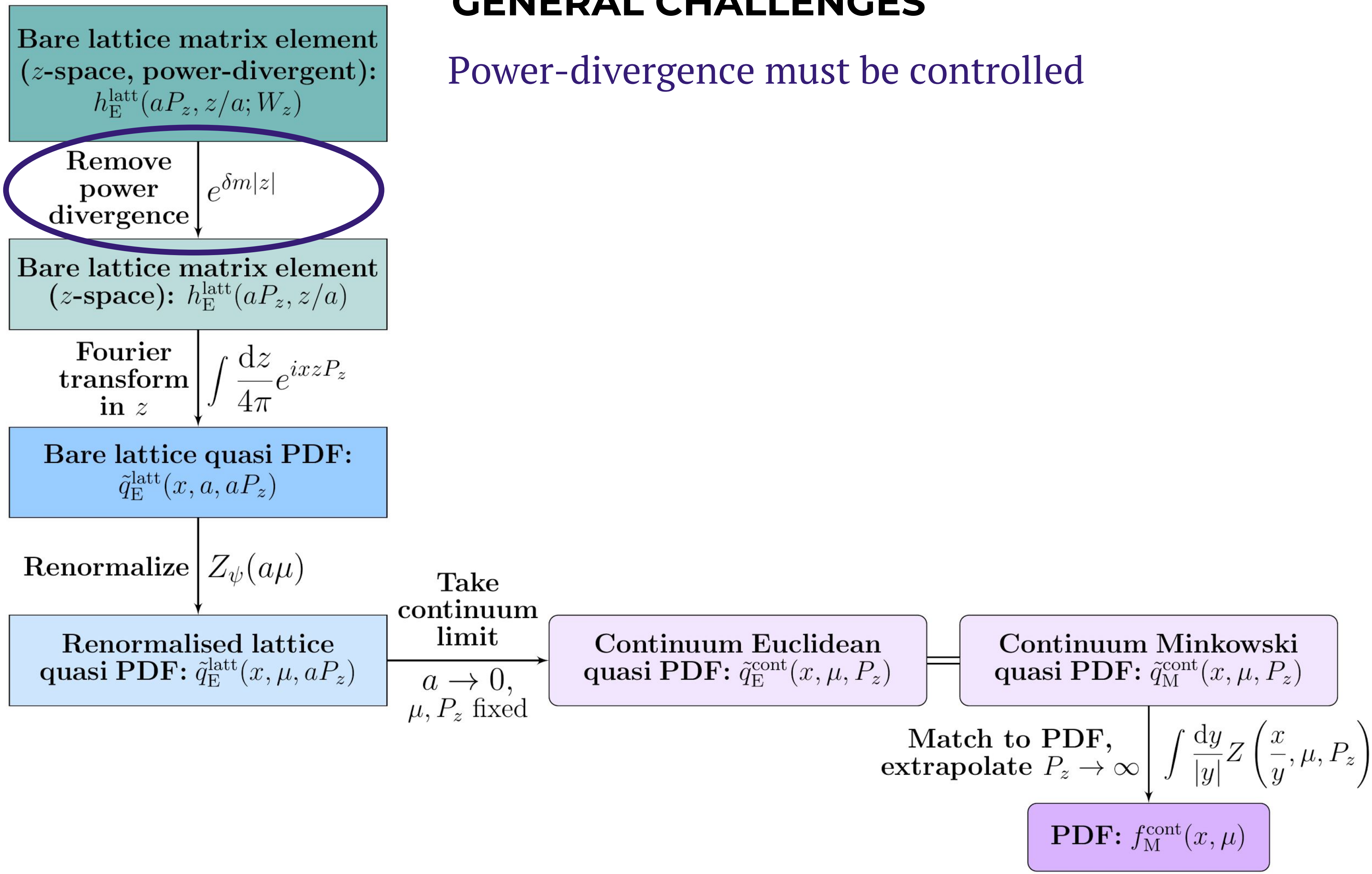
PDF: $f^{\mathrm{cont}}_{\mathrm{M}}(x, \mu)$

GENERAL PROCEDURE



GENERAL PROCEDURE: GENERAL CHALLENGES

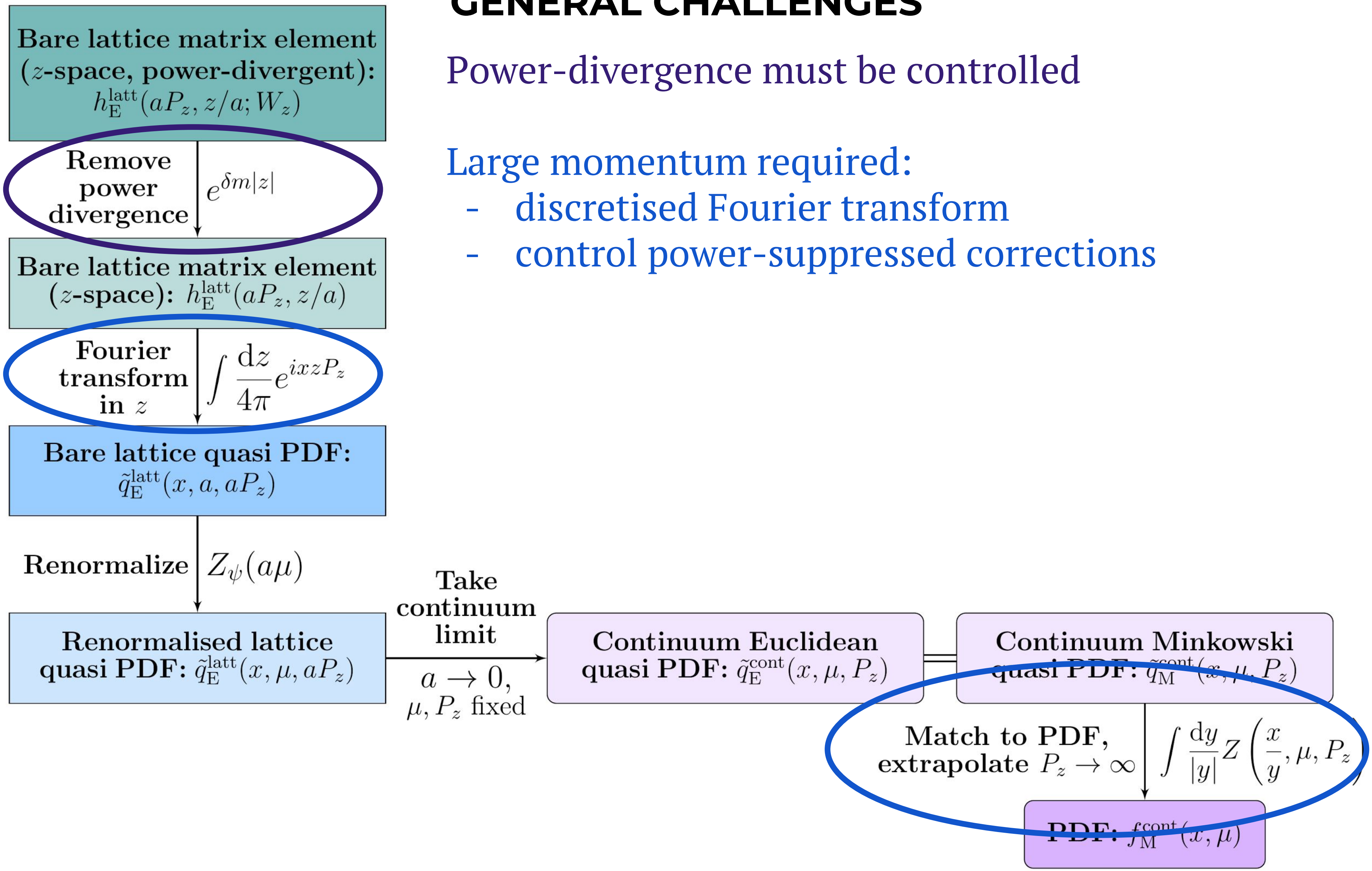
Power-divergence must be controlled



GENERAL PROCEDURE: GENERAL CHALLENGES

Power-divergence must be controlled

Large momentum required:
- discretised Fourier transform
- control power-suppressed corrections



GENERAL PROCEDURE: GENERAL CHALLENGES

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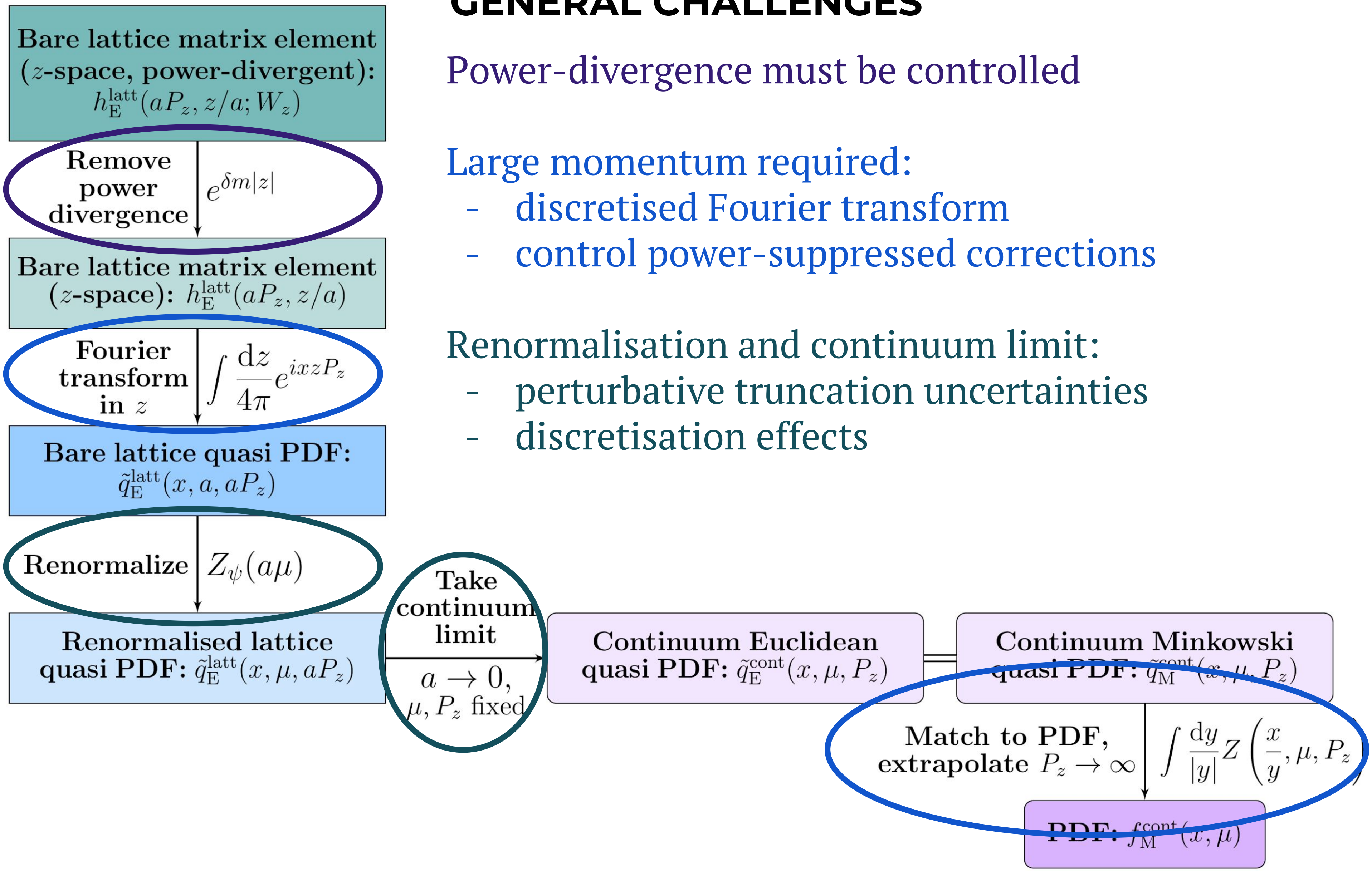
Renormalisation and continuum limit:
- perturbative truncation uncertainties
- discretisation effects



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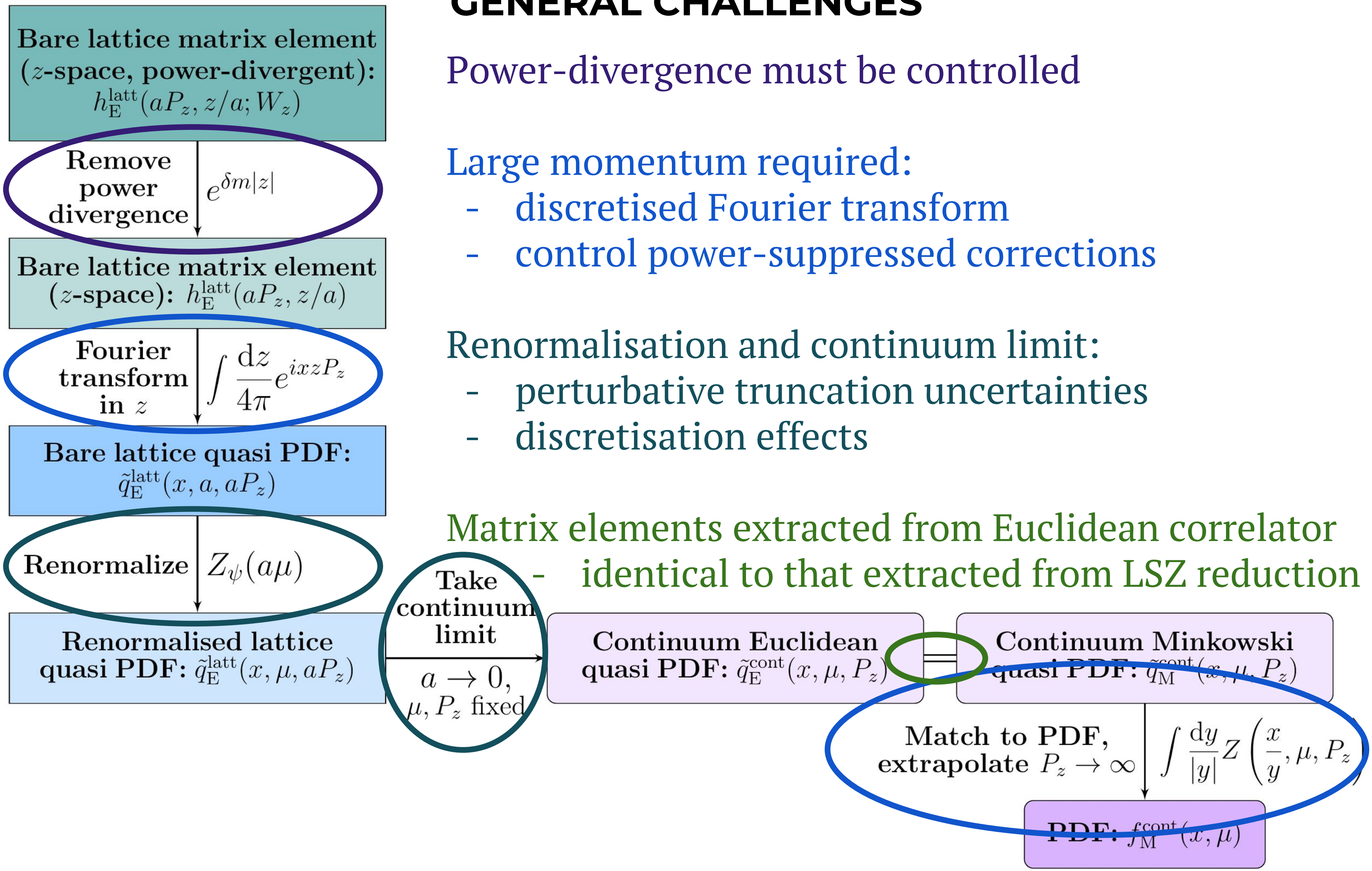

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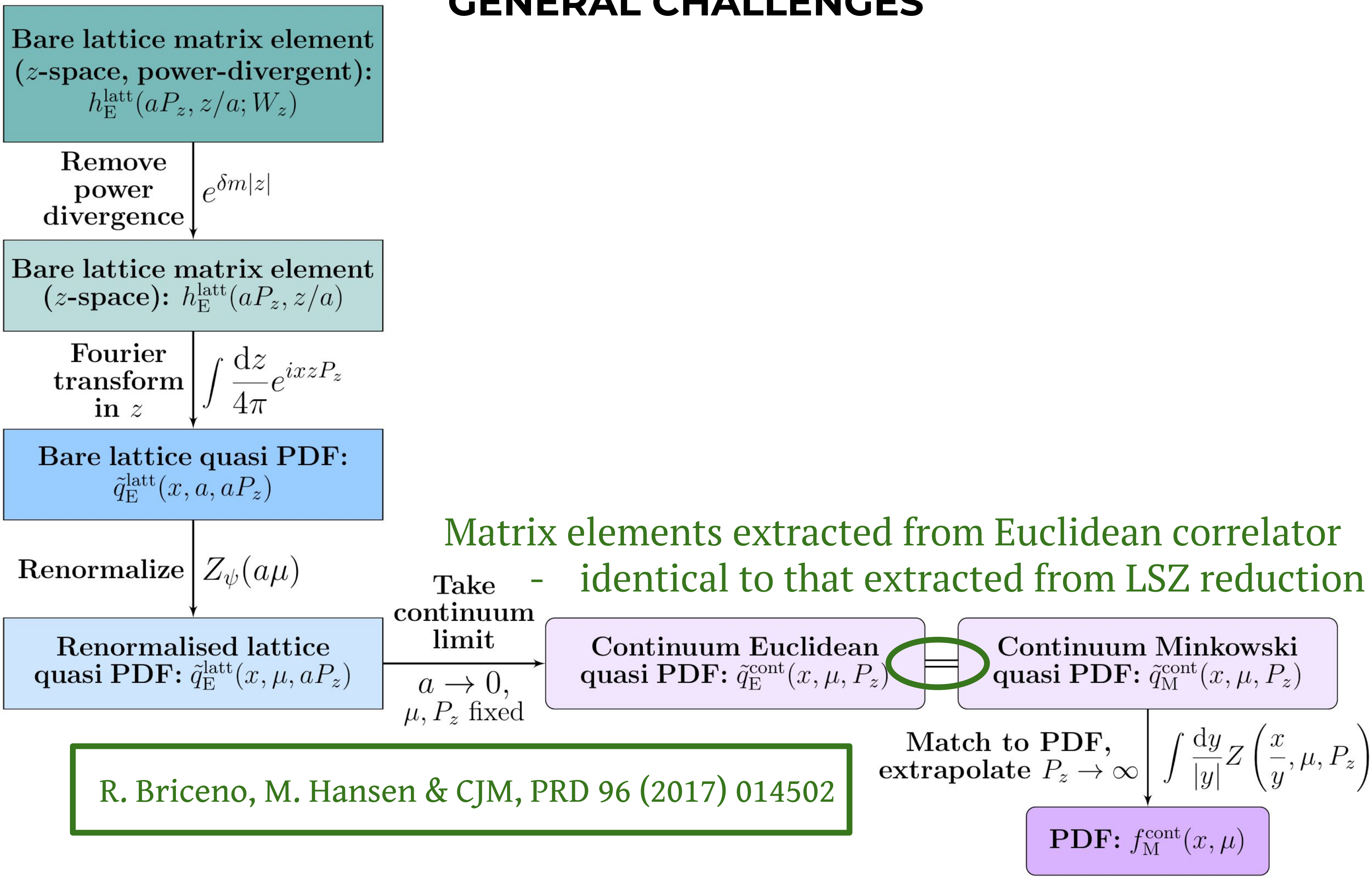
Matrix elements extracted from Euclidean correlator

- identical to that extracted from LSZ reduction

GENERAL PROCEDURE: GENERAL CHALLENGES



Matrix elements extracted from Euclidean correlator - identical to that extracted from LSZ reduction

R. Briceno, M. Hansen & CJM, PRD 96 (2017) 014502

EUCLIDEAN CORRELATORS

Agnostic matrix elements

C.E. Carlson, M. Freid, PRD 095 (2017) 094504

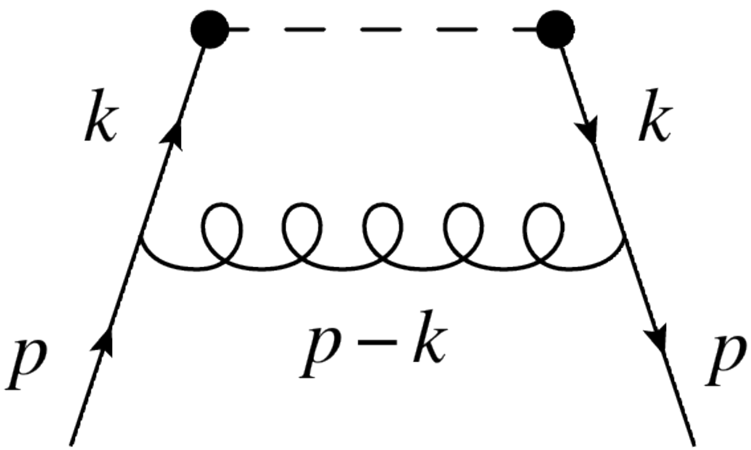
THE WORRY

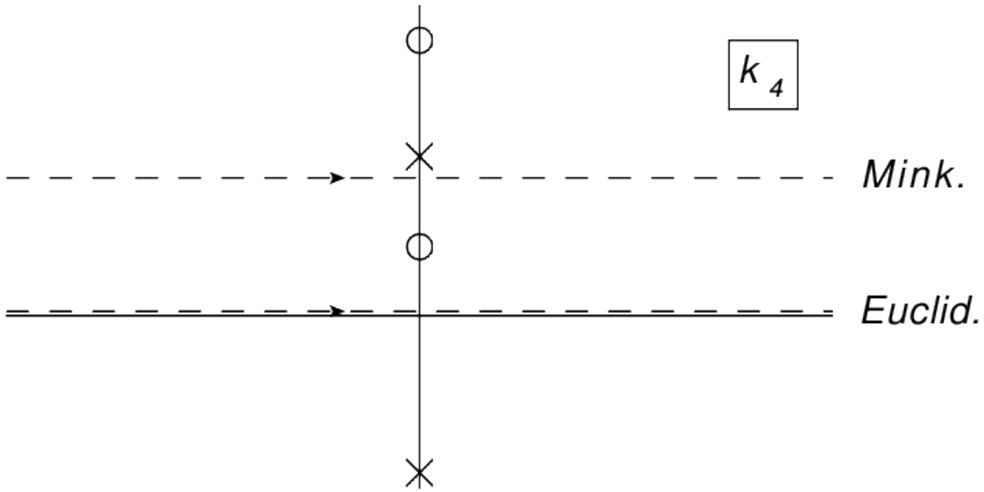
Spacelike distributions assumed identical in Euclidean and Minkowski space

First calculation to work strictly in Euclidean space found no IR divergence!

$$\widetilde{q}^{(1)}(x, P_z) = g^2 I_M(x, P_z)$$

$$\widetilde{q}_E^{(1)}(x, P_z) = g^2 I_E(x, P_z)$$





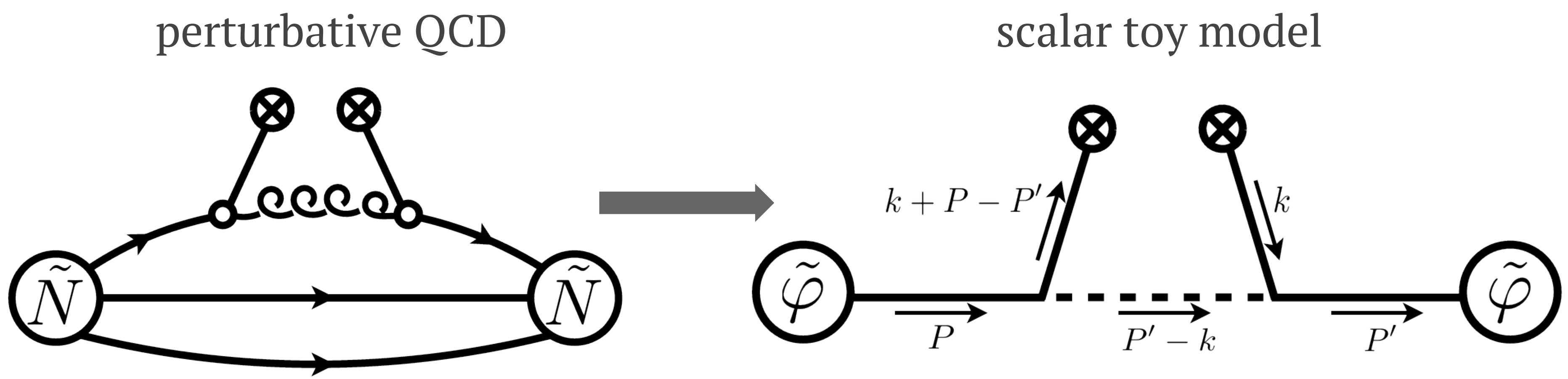
SCALAR TOY MODEL: SPACELIKE DISTRIBUTION

R. Briceno, M. Hansen & CJM, PRD 96 (2017) 014502

Introduce a scalar, toy-model spacelike distribution

$$q(x, P_z) \equiv \int d\xi_z e^{i\xi_z x P_z} \langle \mathbf{P}|\varphi(\xi)\varphi(0)|\mathbf{P}\rangle$$

Momentum space correlation function:



Consider and compare:

1. LSZ reduction in Minkowski spacetime
2. Long time behaviour in Euclidean space

C.E. Carlson, M. Freid, PRD 095 (2017) 094504

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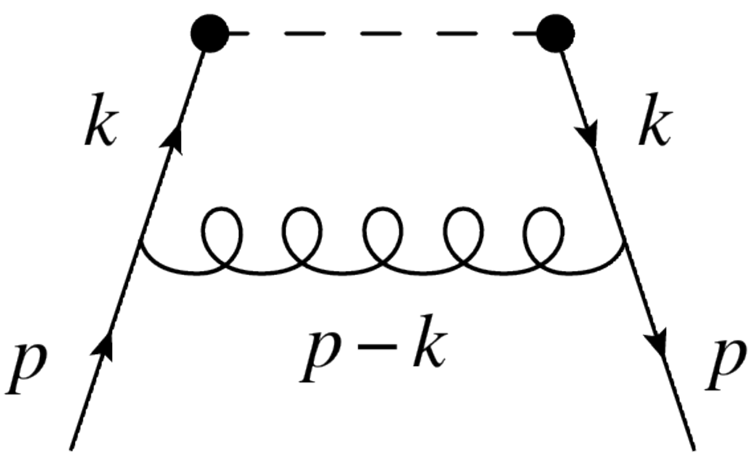
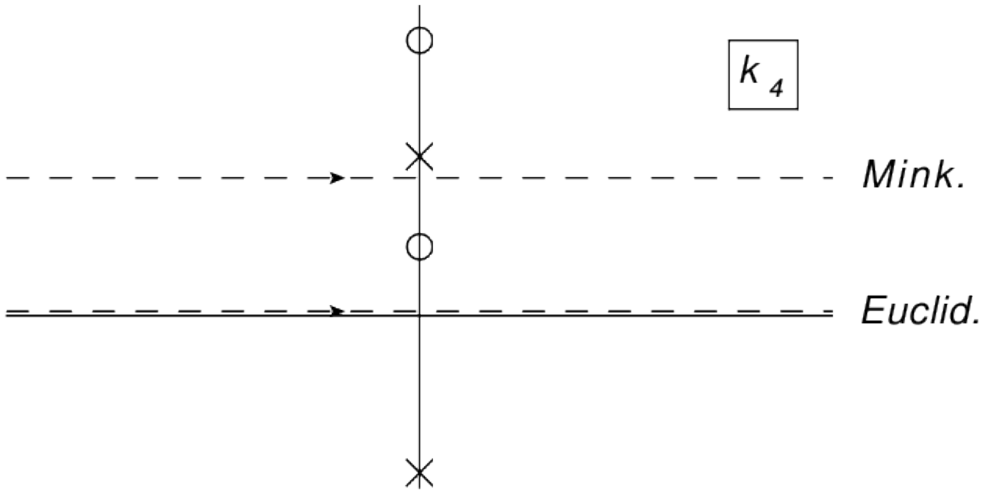


$$\widetilde{q}_E^{(1)}(x, P_z) = g^2 I_E(x, P_z)$$



$$C_E^{(1)}(\tau', \tau, x, P_z) \equiv g^2 \frac{e^{-\omega_{\mathbf{P}}(\tau'-\tau)}}{4\omega_{\mathbf{P}}^2} [I_E(x, P_z) + \Delta I(x, P_z)] + \cdots$$

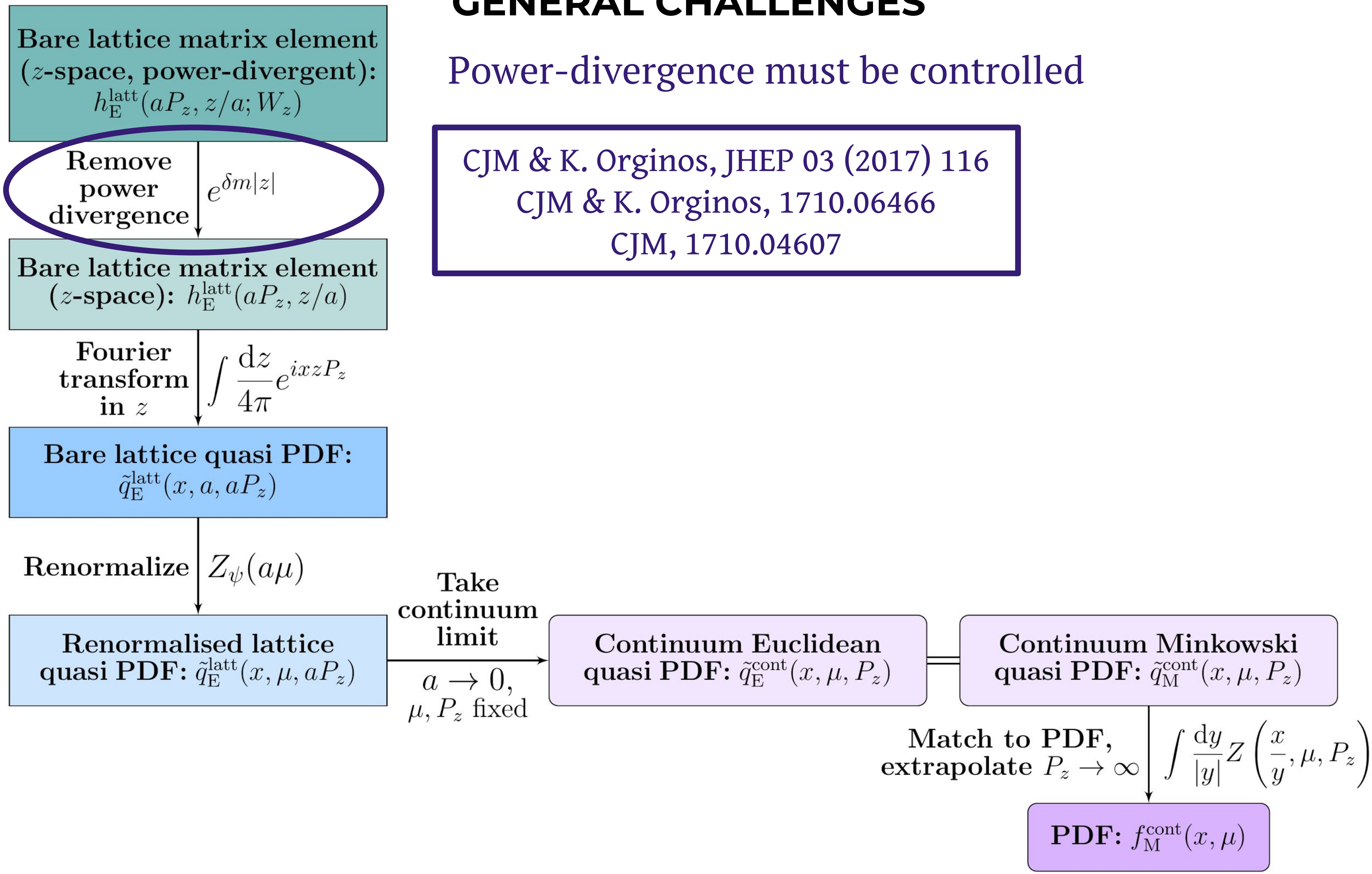
No fundamental challenge to, or problem with, this whole approach

R. Briceno, M. Hansen & CJM, PRD 96 (2017) 014502

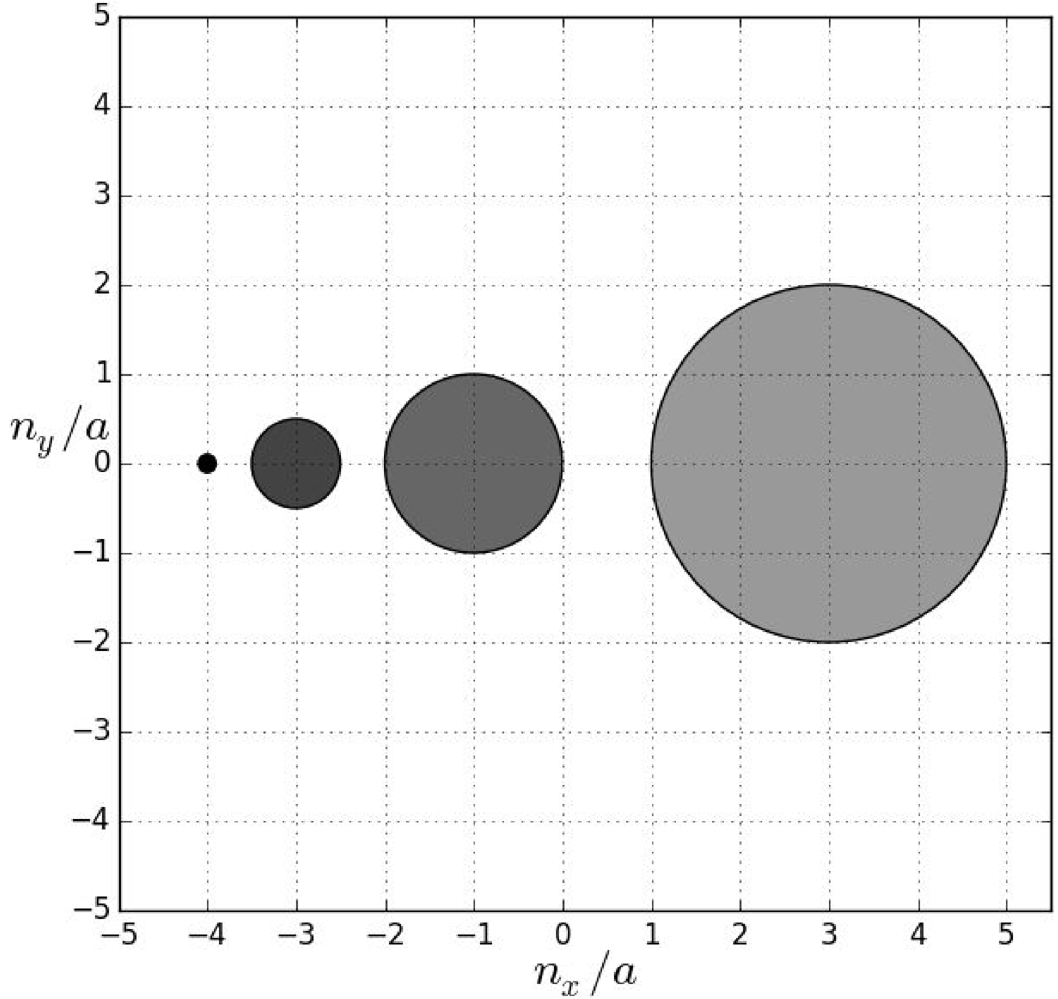
GENERAL PROCEDURE: GENERAL CHALLENGES

Power-divergence must be controlled

CJM & K. Orginos, JHEP 03 (2017) 116
CJM & K. Orginos, 1710.06466
CJM, 1710.04607



SMEARING

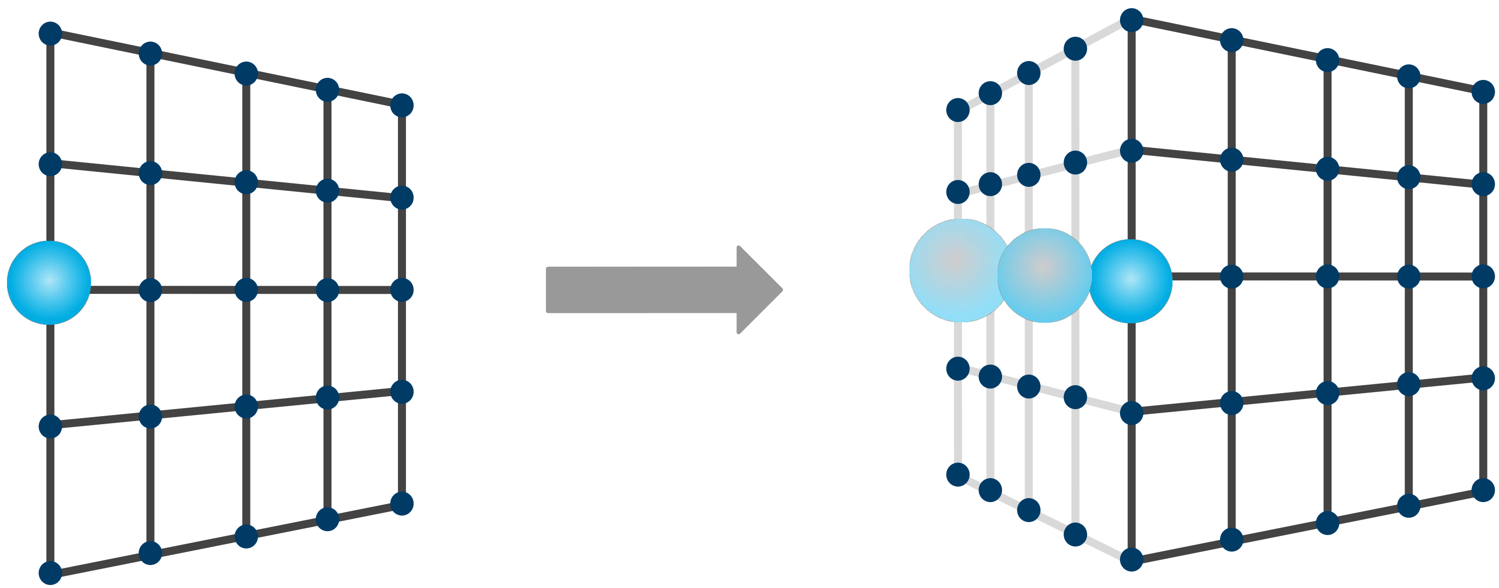


Narayanan & Neuberger, JHEP 0603 (2006) 064
Lüscher, Commun. Math. Phys. 293 (2010) 899

GRADIENT FLOW

Deterministic evolution in new parameter - flow time

- one-parameter mapping
- five-dimensional theory



Drives fields to minimise action - removes UV fluctuations

Lüscher & Weisz, JHEP 1102 (2011) 51
Luscher, JHEP 04 (2013) 123
Makino & Suzuki, arXiv:1410.7538

Finite correlation functions remain finite

Correlation functions of “bulk” fields provide probe of underlying field theory

Narayanan & Neuberger, JHEP 0603 (2006) 064
Lüscher, Commun. Math. Phys. 293 (2010) 899

GRADIENT FLOW

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CJM, PoS(Lattice2015) 052

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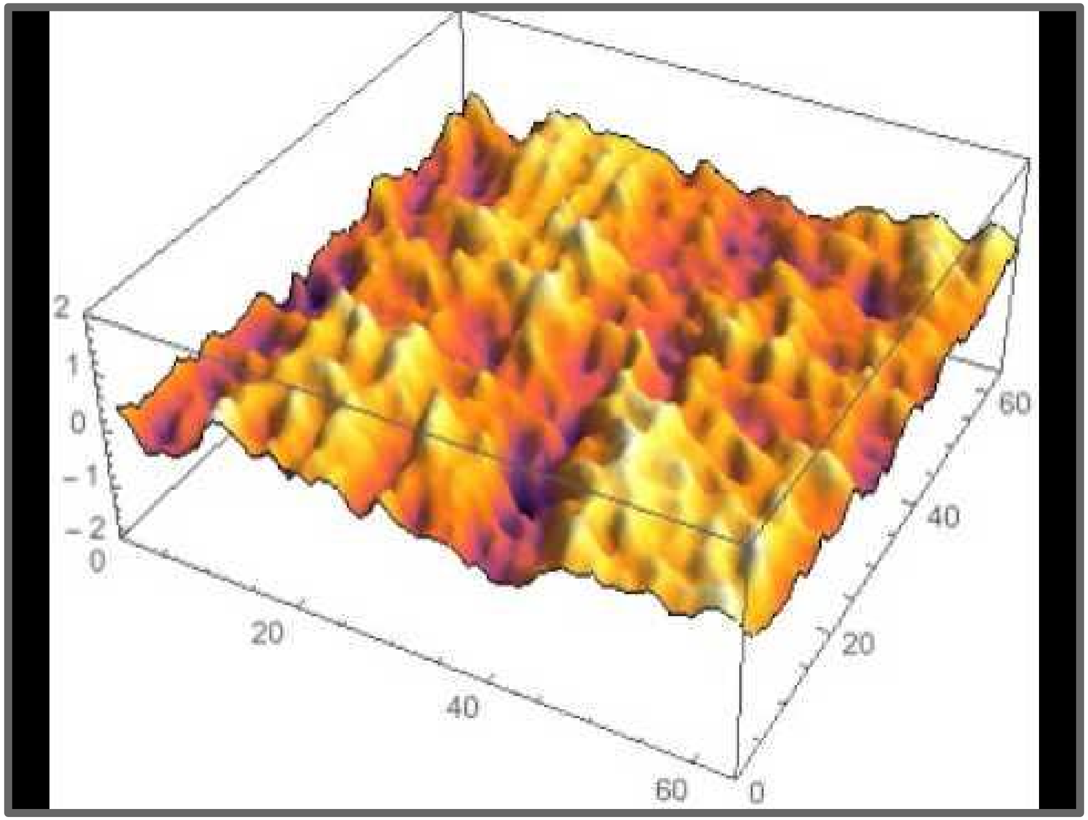

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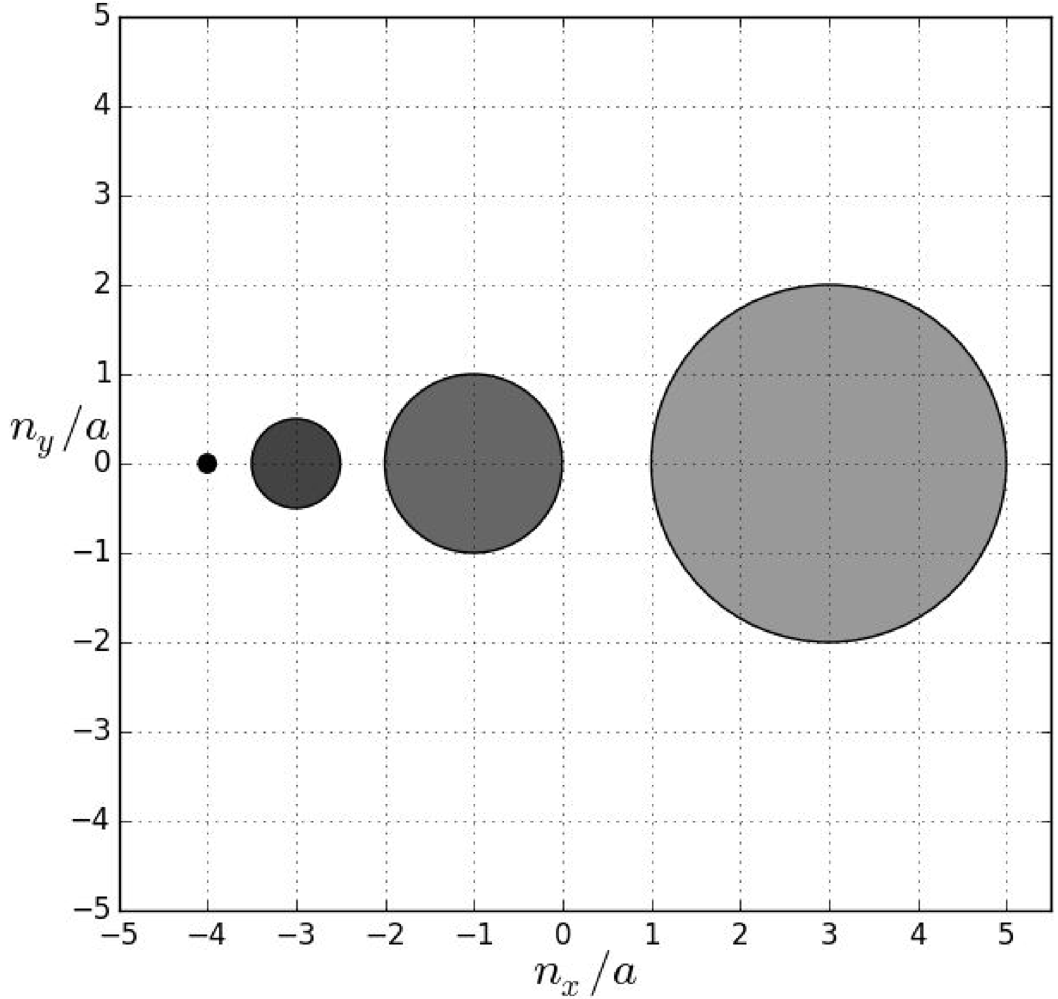


GRADIENT FLOW

Gradient flow is a smearing (smoothing) tool that:

- generates more continuum-like operators
- provides a method to fix smearing length scale

Flow time serves as a nonperturbative, rotationally-invariant cutoff

Matrix elements of operators at fixed flow time are finite

Fixing the flow time (physical units) allows a continuum limit

In essence: exchange lattice regulator for gradient flow regulator

CJM & K. Orginos, PRD 91 (2015) 074513

SMEARED QUASI DISTRIBUTIONS

Provides continuum limit

SMEARED QUASI DISTRIBUTIONS

CJM & K. Orginos, JHEP 03 (2017) 116
CJM, 1710.04607

Defined as

$$q(x, \sqrt{\tau}P^z, \sqrt{\tau}\Lambda_{\rm QCD}, \sqrt{\tau}M_N) = \int \frac{{\rm d}z}{4\pi} e^{ixz\,k^z} \langle P|\overline{\chi}(z,\tau)\gamma^z e^{-ig\int_0^z {\rm d}z' B^z(z',\tau)}\chi(0,\tau)|P\rangle_C$$

Related to light-front PDFs via

$$q(x, \sqrt{\tau}\Lambda_{\rm QCD}, \sqrt{\tau}P^z) = \int_{-1}^{1} \frac{{\rm d}y}{y} Z\left(\frac{x}{y}, \sqrt{\tau}\mu, \sqrt{\tau}P^z\right) f(y,\mu^2) + \mathcal{O}\left(\sqrt{\tau}\Lambda_{\rm QCD}, \frac{\Lambda_{\rm QCD}^2}{(P^z)^2}\right)$$

Provided

$$\Lambda_{\rm QCD}, M_N \ll P_z \ll \tau^{-1/2}$$

Matching kernel satisfies

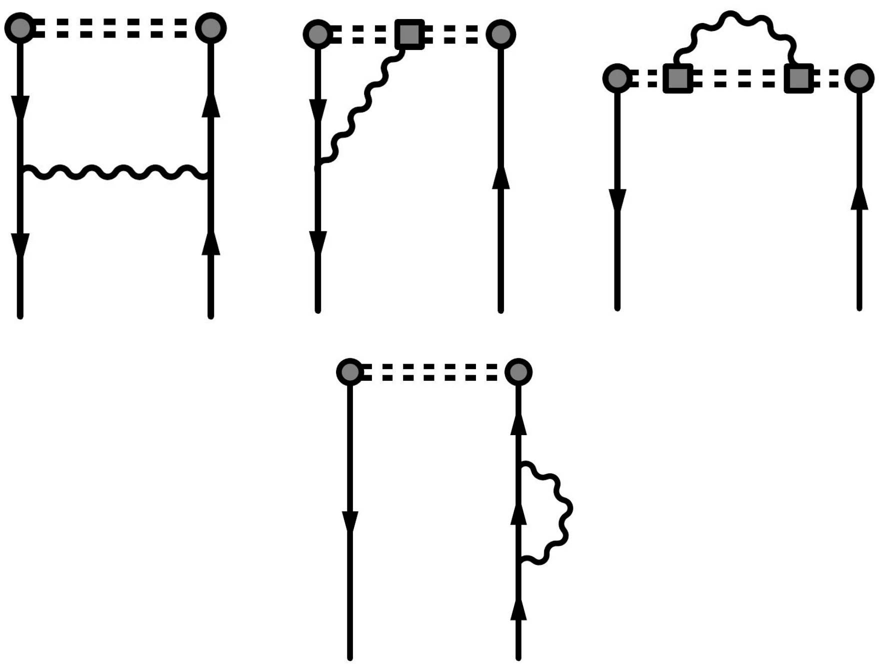
$$\mu\frac{d}{d\mu} Z\left(x, \sqrt{\tau}\mu, \sqrt{\tau}P_z\right) = \frac{\alpha_s(\mu)}{\pi} \int_x^\infty \frac{dy}{y} Z\left(y, \sqrt{\tau}\mu, \sqrt{\tau}P_z\right) P\left(\frac{x}{y}\right)$$

MATRIX ELEMENTS IN PERTURBATION THEORY

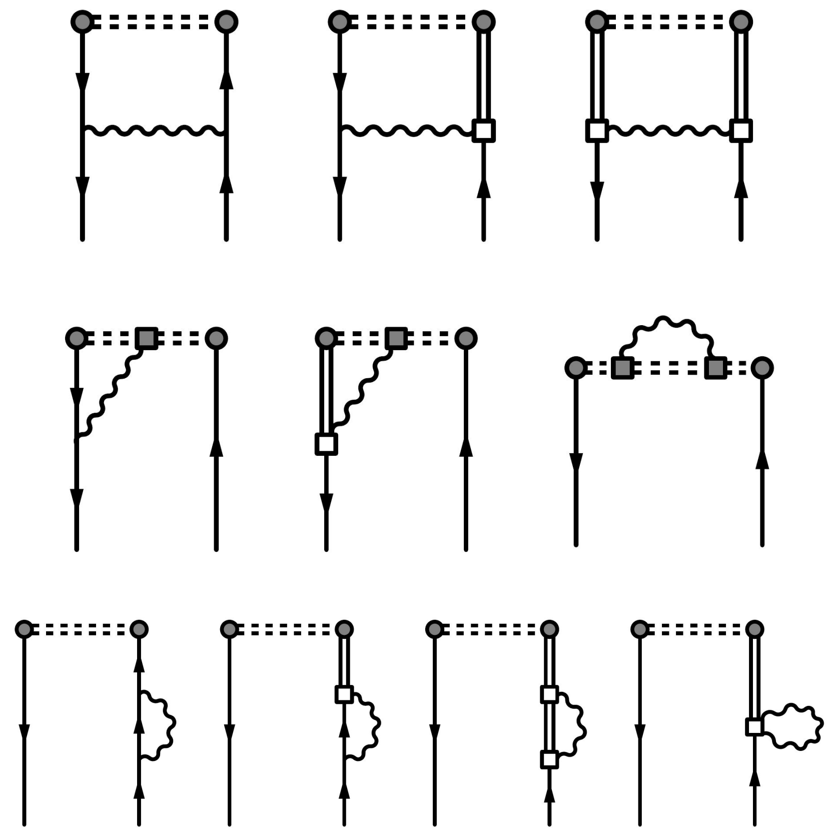
CJM, 1710.04607

Feynman diagrams at one loop in perturbation theory

Quasi distribution



Smeared quasi distribution



MATRIX ELEMENTS IN PERTURBATION THEORY

CJM, 1710.04607

At one loop

$$h_\alpha(\overline{z}) = \mathcal{Z}^{(\alpha)}(\overline{z}) h_\alpha^{(0)}$$

$$\overline{z}^2 = \frac{z^2}{8t}$$

where

$$\mathcal{Z}^{(\alpha)}(\overline{z}) = 1 + \frac{\alpha_s}{3\pi}\left[C^{(\alpha)}(\overline{z}^2) - \gamma_{\mathrm{E}} + \mathrm{Ei}(-\overline{z}^2) - \log(\overline{z}^2) + 2\sqrt{\pi}\,\overline{z}\,\mathrm{erf}(\overline{z})\right]$$

Two regimes:

1. Local vector-current limit

Hieda & Suzuki, MPLA 31 (2016) 1650214

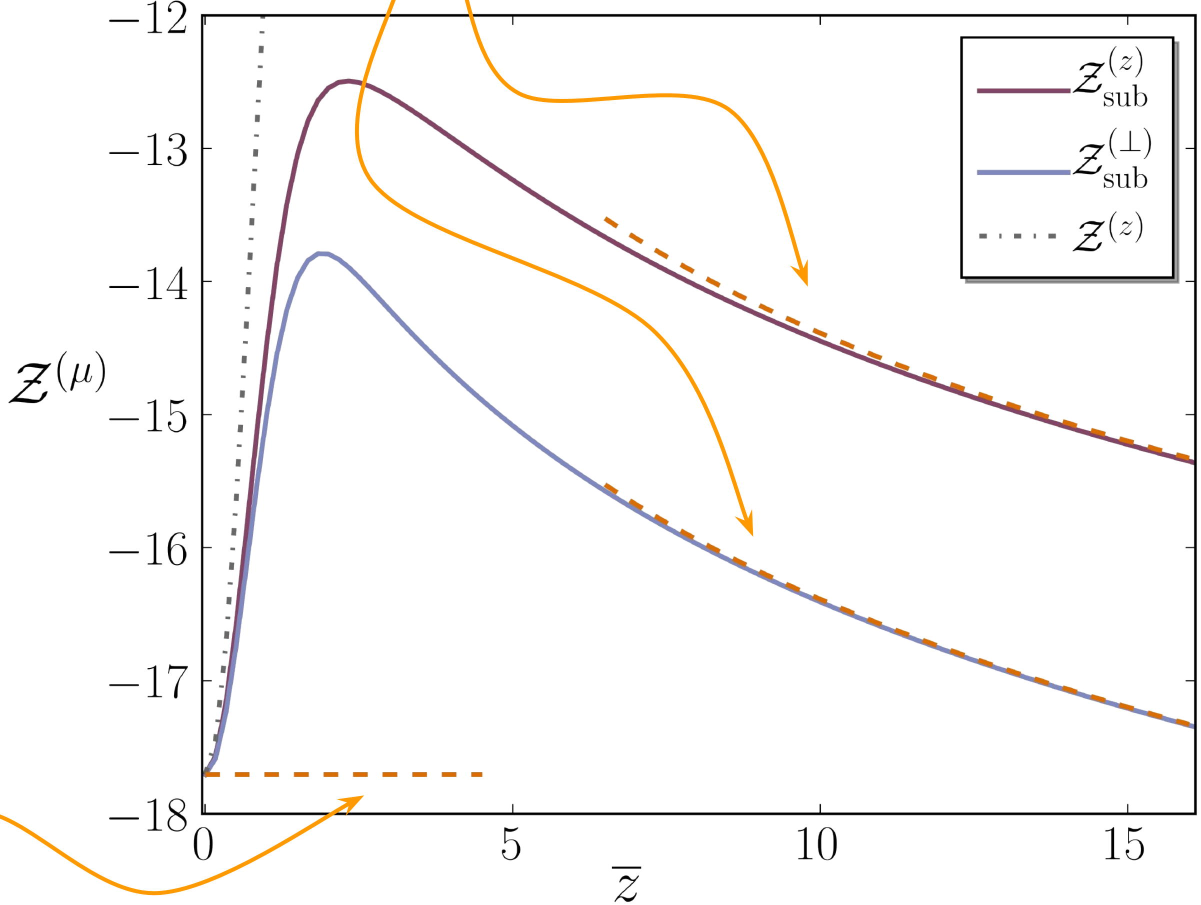
$$\overline{z} \ll 1 \qquad \mathcal{Z}^{(\alpha)}(\overline{z}) \to \mathcal{Z}(\overline{z}) = 1 + \frac{\alpha_s}{3\pi}\left[\frac{1}{2} - \log(432)\right]$$

2. Small flow-time limit

$$\overline{z} \gg 1 \qquad \mathcal{Z}^{(\alpha)}(\overline{z}) \to \mathcal{Z}^{(\alpha)}_{\mathrm{sub}}(\overline{z}) = 1 + \frac{\alpha_s}{3\pi}\left[c^{(\alpha)} - \gamma_{\mathrm{E}} - \log(432) - \log(\overline{z}^2)\right]$$

MATRIX ELEMENTS IN PERTURBATION THEORY

$$\mathcal{Z}^{(\alpha)}_{\text{sub}}(\overline{z}) = 1 + \frac{\alpha_s}{3\pi}\left[c^{(\alpha)} - \gamma_{\text{E}} - \log(432) - \log(\overline{z}^2)\right]$$

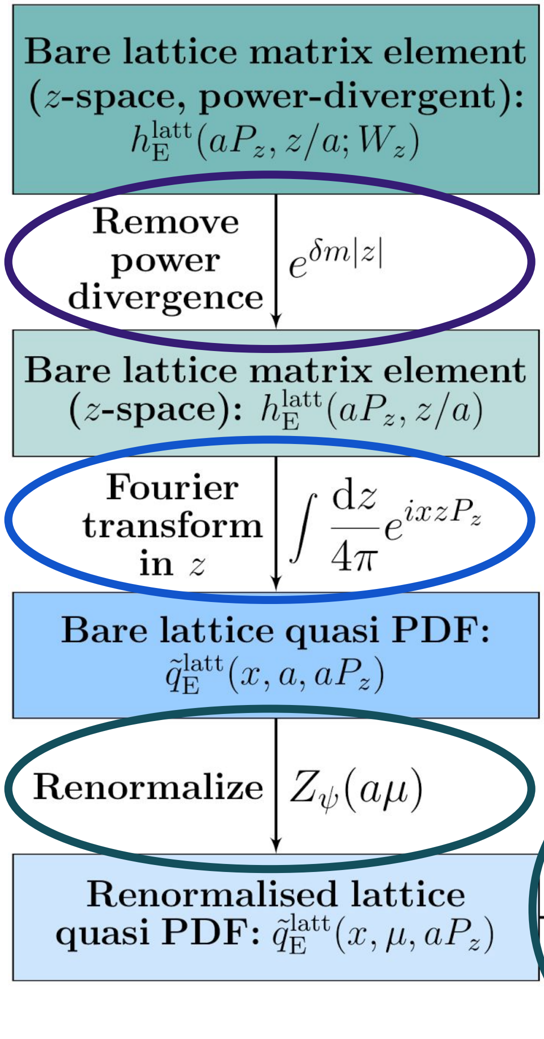


$$\mathcal{Z}(\overline{z}) = 1 + \frac{\alpha_s}{3\pi}\left[\frac{1}{2} - \log(432)\right]$$

CJM, 1710.04607

SUMMARY

GENERAL PROCEDURE: GENERAL CHALLENGES



Power-divergence must be controlled

Large momentum required:

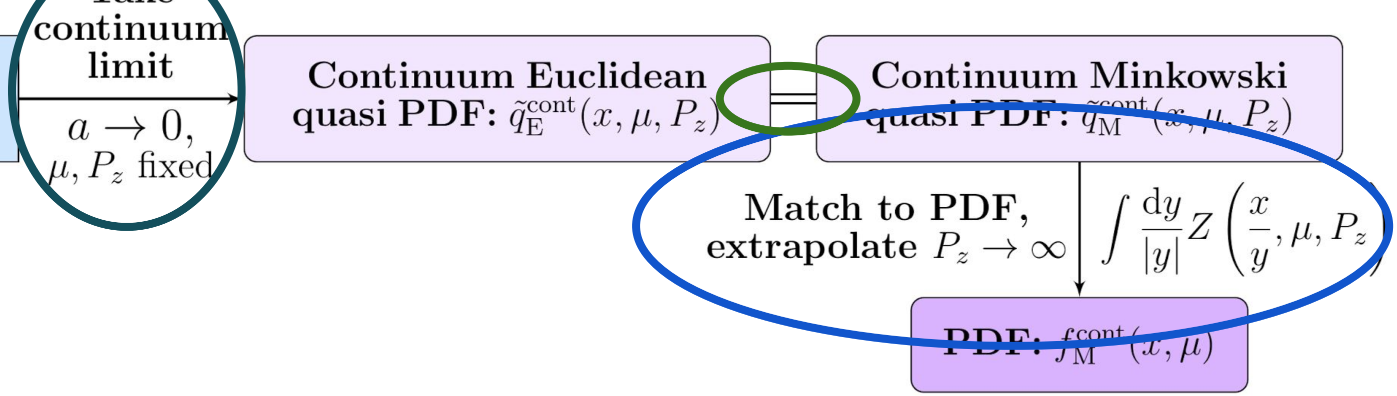
- discretised Fourier transform
- control power-suppressed corrections

Renormalisation and continuum limit:

- perturbative truncation uncertainties
- discretisation effects

Matrix elements extracted from Euclidean correlator

- identical to that extracted from LSZ reduction



PDFs FROM EUCLIDEAN SPACETIME

Quasi distributions

Most theoretical issues generally under control

THE GRADIENT FLOW

Nonperturbative, gauge-invariant regulator

Matrix elements finite at fixed flow time

SMEARED QUASI DISTRIBUTIONS

Finite continuum distributions

Looking forward: study systematics

THANK YOU

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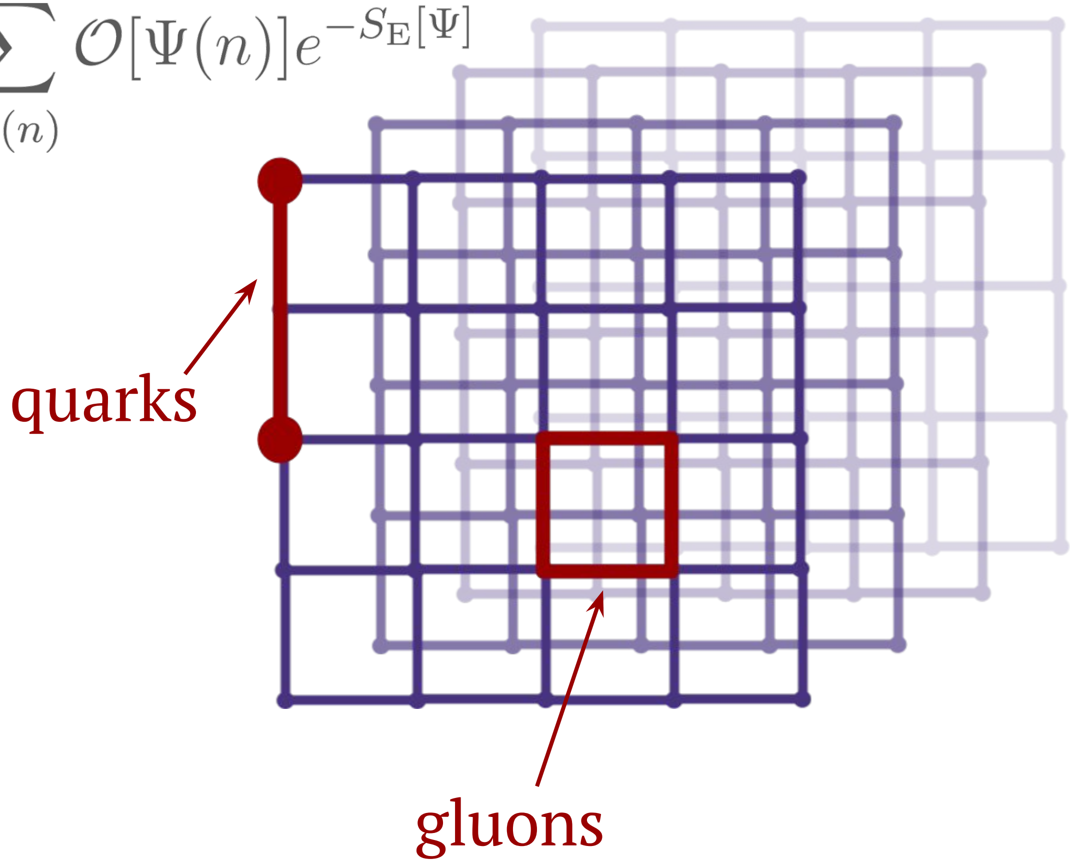
LATTICE QCD

Nonperturbative gauge-invariant regulator

Rigorous definition of the path integral

$$\int \mathcal{D}\Psi\, \mathcal{O}[\Psi(x)]e^{iS_{\mathrm{M}}[\Psi]} \;\Rightarrow\; \sum_{\Psi(n)} \mathcal{O}[\Psi(n)]e^{-S_{\mathrm{E}}[\Psi]}$$



Systematic uncertainties

- finite lattice spacing
- finite volume
- unphysical pion masses
- excited state contamination
- Euclidean spacetime
- nontrivial renormalisation

SCALAR FIELD THEORY

Scalar field theory

$$\frac{\partial}{\partial \tau}\overline{\phi}(\tau,x) = \partial^2\overline{\phi}(\tau,x) \qquad \overline{\phi}(\tau=0,x) = \phi(x) \qquad \tilde{\overline{\phi}}(\tau,p) = e^{-\tau p^2}\tilde{\phi}(p)$$

CJM & K. Orginos, PRD 91 (2015) 074513

Exact solution possible with Dirichlet boundary conditions

$$\overline{\phi}(\tau,x) = \int \mathrm{d}^4y \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)} e^{-\tau p^2}\phi(y) = \frac{1}{16\pi^2\tau^2}\int \mathrm{d}^4y\, e^{-(x-y)^2/(4\tau)}\phi(y)$$

Smearing radius $s_{\mathrm{rms}} = \sqrt{8\tau}$

Interactions occur at zero flow time (*i.e.* in the original "boundary" theory): guarantees that renormalised correlation functions remain finite.

QCD

$$\frac{\partial}{\partial \tau} B_\mu(\tau, x) = D_\nu \Big(\partial_\nu B_\mu - \partial_\mu B_\nu + [B_\nu, B_\mu] \Big) \qquad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$

$$\frac{\partial}{\partial \tau} \chi(\tau, x) = D^F_\mu D^F_\mu \chi(\tau, x) \qquad D^F_\mu = \partial_\mu + B_\mu$$

Exact solution no longer possible (even with Dirichlet boundary conditions)

$$B_\mu(\tau, x) = \int \mathrm{d}^4 y \Big\{ K_\tau(x-y)_{\mu\nu} A_\nu(y) + \int_0^\tau \mathrm{d}\sigma \, K_{\tau-\sigma}(x-y)_{\mu\nu} R_\nu(\sigma, y) \Big\}$$

Smearing radius $s_{\mathrm{rms}} = \sqrt{8\tau}$

Interactions occur at non-zero flow time: generalised BRST symmetry guarantees renormalised correlation functions remain finite.

EXPERIMENTAL EXTRACTION

Process	Subprocess	Partons	x range
$\ell^{\pm}\{p,n\} \to \ell^{\pm} X$	$\gamma^* q \to q$	$q, \bar{q}, g$	$x \gtrsim 0.01$
$\ell^{\pm} n/p \to \ell^{\pm} X$	$\gamma^* d/u \to d/u$	d/u	$x \gtrsim 0.01$
$pp \to \mu^+\mu^- X$	$u\bar{u}, d\bar{d} \to \gamma^*$	$\bar{q}$	$0.015 \lesssim x \lesssim 0.35$
$pn/pp \to \mu^+\mu^- X$	$(u\bar{d})/(u\bar{u}) \to \gamma^*$	$\bar{d}/\bar{u}$	$0.015 \lesssim x \lesssim 0.35$
$\nu(\bar{\nu}) N \to \mu^-(\mu^+) X$	$W^* q \to q'$	$q, \bar{q}$	$0.01 \lesssim x \lesssim 0.5$
$\nu N \to \mu^-\mu^+ X$	$W^* s \to c$	s	$0.01 \lesssim x \lesssim 0.2$
$\bar{\nu} N \to \mu^+\mu^- X$	$W^* \bar{s} \to \bar{c}$	$\bar{s}$	$0.01 \lesssim x \lesssim 0.2$
$e^{\pm} p \to e^{\pm} X$	$\gamma^* q \to q$	$g, q, \bar{q}$	$10^{-4} \lesssim x \lesssim 0.1$
$e^+ p \to \bar{\nu} X$	$W^+\{d,s\} \to \{u,c\}$	d, s	$x \gtrsim 0.01$
$e^{\pm} p \to e^{\pm} c\bar{c} X, e^{\pm} b\bar{b} X$	$\gamma^* c \to c,\ \gamma^* g \to c\bar{c}$	c, b, g	$10^{-4} \lesssim x \lesssim 0.01$
$e^{\pm} p \to$ jet$+X$	$\gamma^* g \to q\bar{q}$	g	$0.01 \lesssim x \lesssim 0.1$
$p\bar{p}, pp \to$ jet$+X$	$gg, qg, qq \to 2j$	g, q	$0.00005 \lesssim x \lesssim 0.5$
$p\bar{p} \to (W^{\pm} \to \ell^{\pm}\nu) X$	$ud \to W^+, \bar{u}\bar{d} \to W^-$	$u, d, \bar{u}, \bar{d}$	$x \gtrsim 0.05$
$pp \to (W^{\pm} \to \ell^{\pm}\nu) X$	$u\bar{d} \to W^+, d\bar{u} \to W^-$	$u, d, \bar{u}, \bar{d}, g$	$x \gtrsim 0.001$
$p\bar{p}(pp) \to (Z \to \ell^+\ell^-) X$	$uu, dd, ..(u\bar{u}, ..) \to Z$	$u, d, ..(g)$	$x \gtrsim 0.001$
$pp \to W^- c,\ W^+\bar{c}$	$gs \to W^- c$	$s, \bar{s}$	$x \sim 0.01$
$pp \to (\gamma^* \to \ell^+\ell^-) X$	$u\bar{u}, d\bar{d}, .. \to \gamma^*$	$\bar{q}, g$	$x \gtrsim 10^{-5}$
$pp \to b\bar{b} X,\quad t\bar{t} X$	$gg \to b\bar{b},\quad t\bar{t}$	g	$x \gtrsim 10^{-5}, 10^{-2}$
$pp \to$ exclusive $J/\psi,\ \Upsilon$	$\gamma^*(gg) \to J/\psi,\ \Upsilon$	g	$x \gtrsim 10^{-5}, 10^{-4}$
$pp \to \gamma X$	$gq \to \gamma q,\ g\bar{q} \to \gamma\bar{q}$	g	$x \gtrsim 0.005$

From PDG 2016

ON THE LATTICE

Implemented nonperturbatively via discretised diffusion equation

$$\partial_\tau V_\mu(x,\tau) = -g_0^2 \left\{\partial_{V_\mu(x,\tau)} S_{\mathrm{latt}}[V_\mu(x,\tau)]\right\} V_\mu(x,\tau)$$

lattice gauge action

and

$$\partial_\tau \chi(x,\tau) = \overrightarrow{\Delta}\chi(x,\tau) \qquad \partial_\tau \overline{\chi}(x,\tau) = \overline{\chi}(x,\tau)\overleftarrow{\Delta}$$

covariant lattice Laplacian

Lüscher & Weisz, JHEP 1102 (2011) 51
Luscher, JHEP 04 (2013) 123