# Matrix Product States and 1+1 dimensional Thirring model

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# Outline

- Motivation.
- Thirring model as a quantum spin chains.
- Methods for spin chains:
  1. Density matrix renormalisation group.
- Methods for spin chains:
  2. Matrix product states.
- Exploratory results for Kosterlitz-Thouless phase transition.
- Remarks and outlook.

# Motivation

# Interests in tensor-network methods

- Hamiltonian formalism, real-time dynamics.
- No sign problem.
- Future quantum simulations?
- New for lattice practirioners.
- Investigating topological aspects of QFT.

**A forward-looking method for computational QFT.**

# The 1+1 dimensional Thirring model and its representations as scalar models

$$S_{\mathrm{Th}}\left[\psi,\bar{\psi}\right] = \int d^2x \left[\bar{\psi} i \gamma^\mu \partial_\mu \psi - m_0 \bar{\psi}\psi - \frac{g}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2\right]$$

strong-weak duality $g \leftrightarrow \kappa$

$$S_{\mathrm{SG}}\left[\phi\right] = \int d^2x \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \frac{\alpha_0}{\kappa^2}\cos\left(\kappa\phi(x)\right)\right]$$

$$\xrightarrow[\text{Euclidean}]{\phi\to\phi/\kappa,\ \text{and}\ \kappa^2=t} \frac{1}{t}\int d^2x \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \alpha_0\cos\left(\phi(x)\right)\right]$$

high-low temperature duality

$$\frac{H_{\mathrm{XY}}}{T} = \frac{-K}{T}\sum_{\langle i,j\rangle}\cos\left(\theta_i - \theta_j\right)$$

Topological phase transition, critical phase,…

# Summary of the dualities

| Thirring | sine-Gordon | XY |
|---|---|---|
| $g$ | $\frac{4\pi^2}{t} - \pi$ | $\frac{T}{K} - \pi$ |

★ The K-T phase transition at $T \sim K\pi/2$ in the XY model.

★ The phase boundary at $t \sim 8\pi$ in the sine-Gordon theory.

→ The cosine term becomes relevant or irrelevant.

| Thirring | sine-Gordon |
|---|---|
| $\bar{\psi}\gamma_\mu\psi$ | $\frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi$ |
| $\bar{\psi}\psi$ | $\frac{\Lambda}{\pi}cos\phi$ |

# Thirring model as a spin chain

# Staggering the Thirring model

- The continuum Hamiltonian

$$\gamma_0 = \sigma_3,\ \gamma_1 = i\sigma_2\ ,\ \ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$H_{\rm Th} = \int dx \left\{ -i\left(\psi_2^* \partial_x \psi_1 + \psi_1^* \partial_x \psi_2\right) + m_0\left(\psi_1^*\psi_1 - \psi_2^*\psi_2\right) + 2g\left(\psi_1^*\psi_1\psi_2^*\psi_2\right)\right\}$$

- Staggered regularisation *a'la* Kogut and Susskind

$$\psi_u \to \frac{1}{\sqrt{a}} c_{2n},\ \psi_d \to \frac{1}{\sqrt{a}} c_{2n+1}$$

$$H_{\rm Th}^{\rm (latt)} = -\frac{i}{2a}\sum_{n=0}^{N-2}\left(c_n^\dagger c_{n+1} - c_{n+1}^\dagger c_n\right) + m_0 \sum_{n=0}^{N-1}(-1)^n c_n^\dagger c_n + \frac{2g}{a}\sum_{n=0}^{\frac{N}{2}-1} c_{2n}^\dagger c_{2n} c_{2n+1}^\dagger c_{2n+1}$$

$\longrightarrow$ $\boxed{\text{No doubler}}$

# The Jordan-Wigner transformation

- The fermion fields satisfy

$$\{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0, \ \ \{c_n, c_m^\dagger\} = \delta_{n,m} \ .$$

- The Jordan-Wigner transformation

$$c_n = \exp\left(i\pi \sum_{j=1}^{n-1} S_j^z\right) S_n^-, \ \ c_n^\dagger = S_n^+ \exp\left(-i\pi \sum_{j=1}^{n-1} S_j^z\right)$$

expresses the the fermions fields in spins,

$$S_j^\pm = S_j^x \pm i S_j^y, \ \ \left[S_i^a, S_j^b\right] = i\delta_{i,j}\epsilon^{abc} S_i^c \ .$$

# Thirring model as a quantum spin chain

- JW transformation on the Thirring model gives

$$\begin{aligned} H_{spin} \;&= -\frac{1}{2a}\sum_n \left(S_n^+ S_{n+1}^- + S_{n+1}^+ S_n^-\right) + m_0 \sum_n (-1)^n \left(S_n^z + \frac{1}{2}\right) \\ &+ \frac{2g}{a}\sum_n \left(S_{2n}^z + \frac{1}{2}\right)\left(S_{2n+1}^z + \frac{1}{2}\right). \end{aligned}$$

XXZ model with two external fields

- The “penalty term”

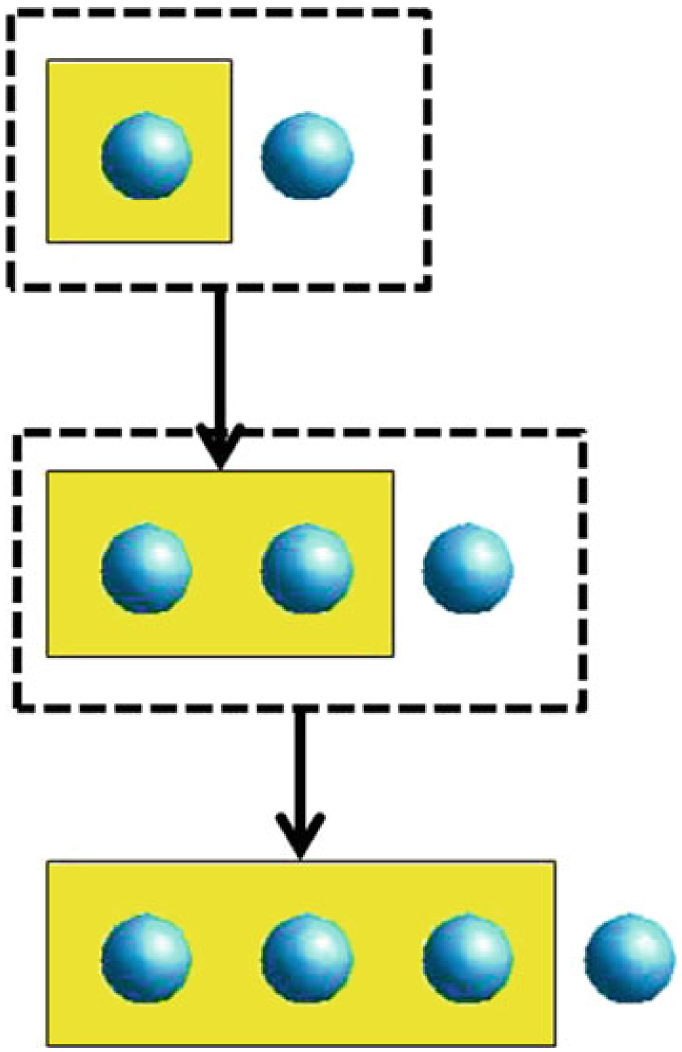
$$H_{spin}^{(penalty)} = H_{spin} + \boxed{\lambda \left(\sum_{n=0}^{N-1} S_n^z - S_{target}\right)^2}$$

projected to a sector of total spin

JW-trans of the total fermion number,
Bosonise to topological index in the SG theory.

# Density matrix RG

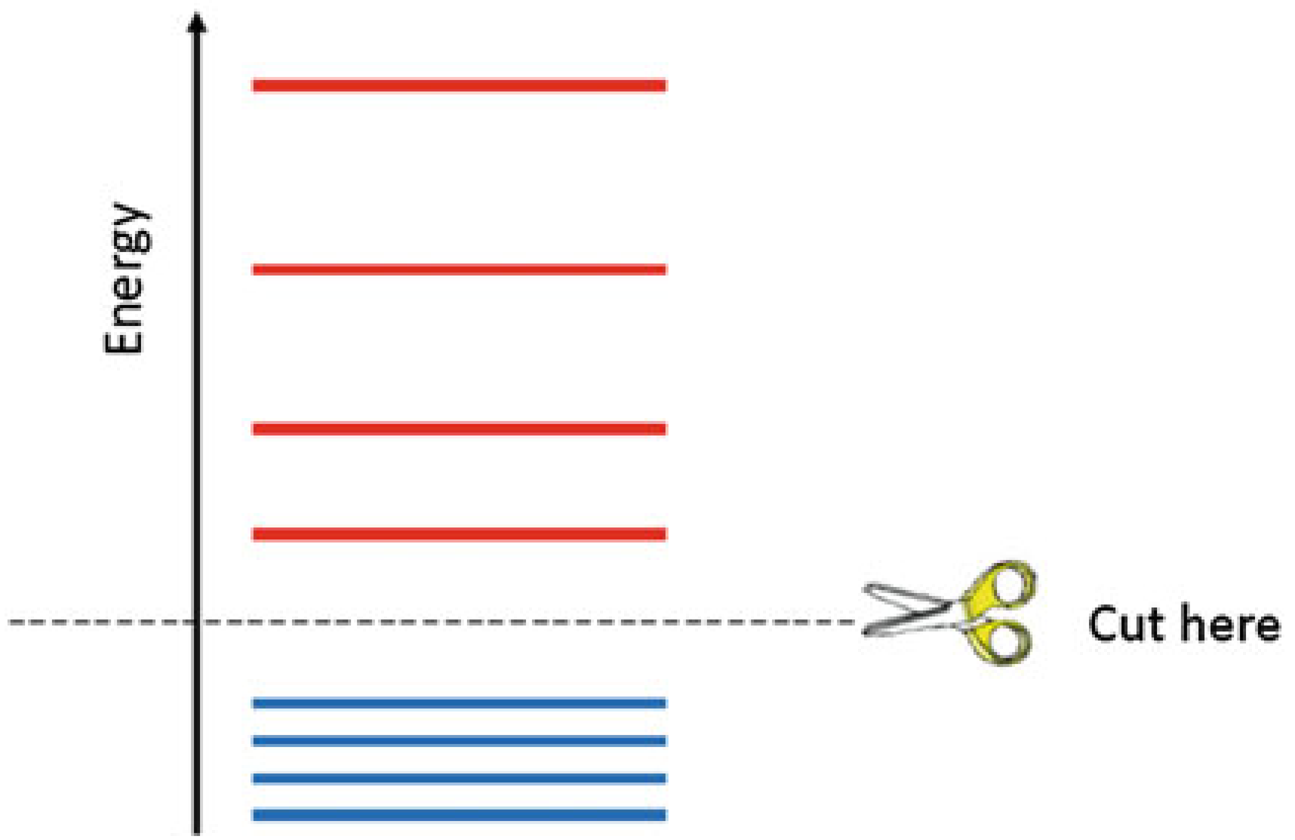
# The large Hilbert space



Size of the Hilbert space increases exponentially when the chain grows.

Challenging to diagonalise the Hamiltonian and look for the ground state.

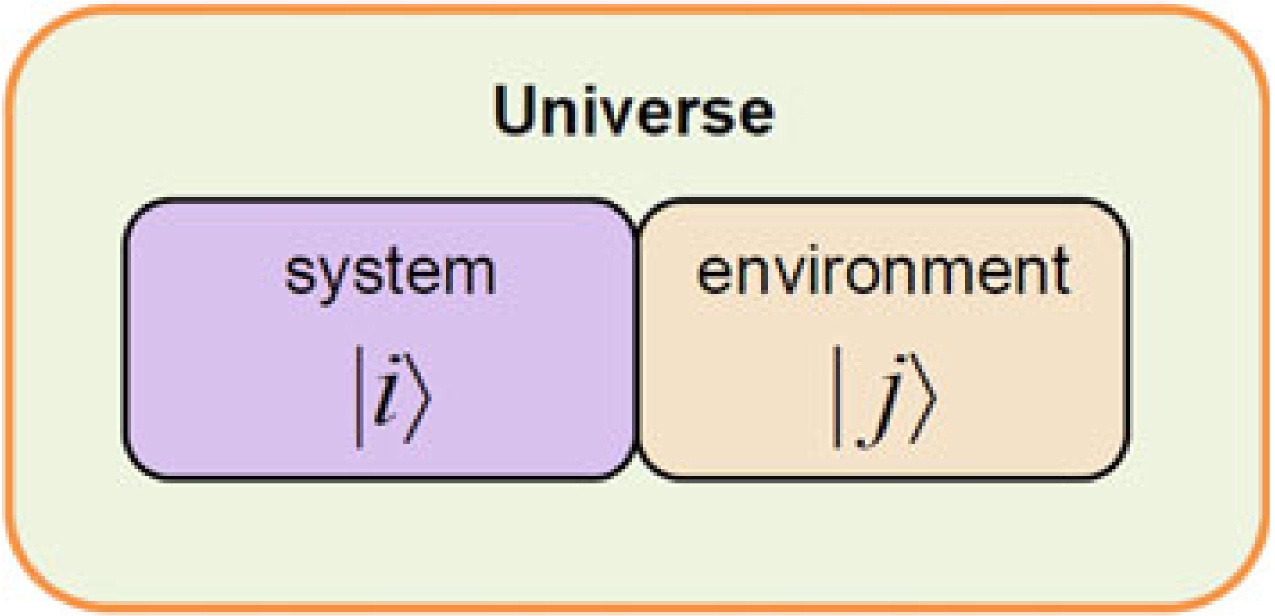
# Issue with numerical RG




Does not work well for strongly-correlated systems

entanglement in quantum systems

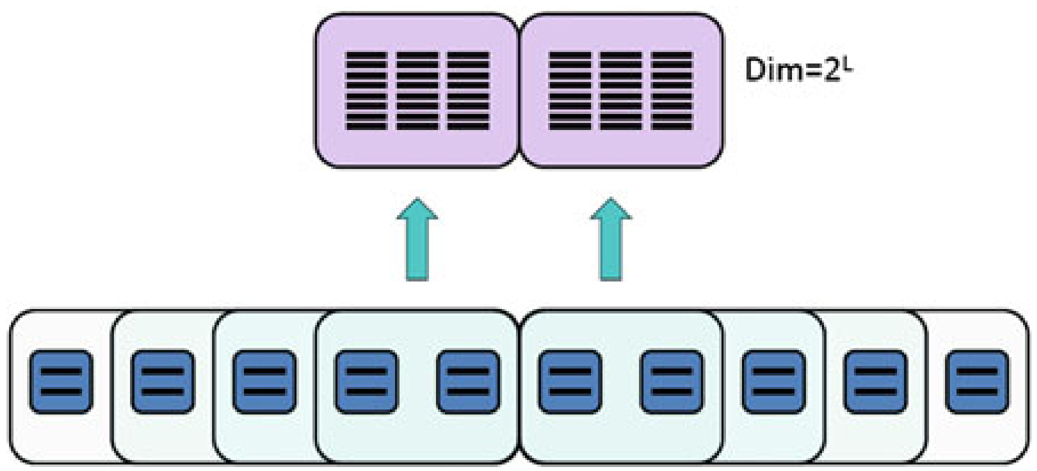
# The key idea behind DMRG




Redistribute the entanglement between the system and the environment, and minimise the loss of information when truncating the Hilbert space.

# The singular value decomposition

$$|\Psi\rangle = \sum_{i=1}^{D_A}\sum_{j=1}^{D_B}\Psi_{i,j}|i\rangle\otimes|j\rangle$$




$\Psi_{i,j}$ can be regarded as elements of a $D_A \times D_B$ (assuming $(D_A \geq D_B)$ matrix.

SVD

$$\Psi_{i,j} = \sum_{\alpha}^{D_B} U_{i,\alpha}\lambda_\alpha \left(V^\dagger\right)_{\alpha,j}$$

$$U^\dagger U = 1,\ VV^\dagger = 1$$

Discard small singular values

$$\Psi_{i,j} = \sum_{\alpha}^{D'_B < D_B} U_{i,\alpha}\lambda_\alpha \left(V^\dagger\right)_{\alpha,j}$$

# Schmidt decomposition and entanglement

$$|\Psi\rangle = \sum_{i=1}^{D_A}\sum_{j=1}^{D_B}\sum_{\alpha}^{D_B} U_{i,\alpha}\lambda_\alpha V^*_{\alpha,j}|i\rangle\otimes|j\rangle = \sum_{\alpha}^{D_B}\lambda_\alpha\left(\sum_{i=1}^{D_A}U_{i,\alpha}|i\rangle\right)\otimes\left(\sum_{j}^{D_B}V^*_{\alpha,j}|j\rangle\right) = \sum_{\alpha}^{D_B}\lambda_\alpha|\alpha\rangle_A\otimes|\alpha\rangle_B$$

Reduced density matrices

$$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi| = \sum_\alpha \lambda_\alpha^2|\alpha\rangle_A\ {}_A\langle\alpha|\ ,\quad \rho_B = \mathrm{Tr}_A|\Psi\rangle\langle\Psi| = \sum_\alpha \lambda_\alpha^2|\alpha\rangle_B\ {}_B\langle\alpha|$$

von Neumann entanglement entropy

$$S = -\mathrm{Tr}\left[\rho_A\ \log\left(\rho_A\right)\right] = -\mathrm{Tr}\left[\rho_B\ \log\left(\rho_B\right)\right] = -\sum_\alpha \lambda_\alpha^2\log\lambda_\alpha^2$$

★ Truncating the Hilbert space by omitting small singular values

→ Throwing away small-entanglement states

# DMRG practical

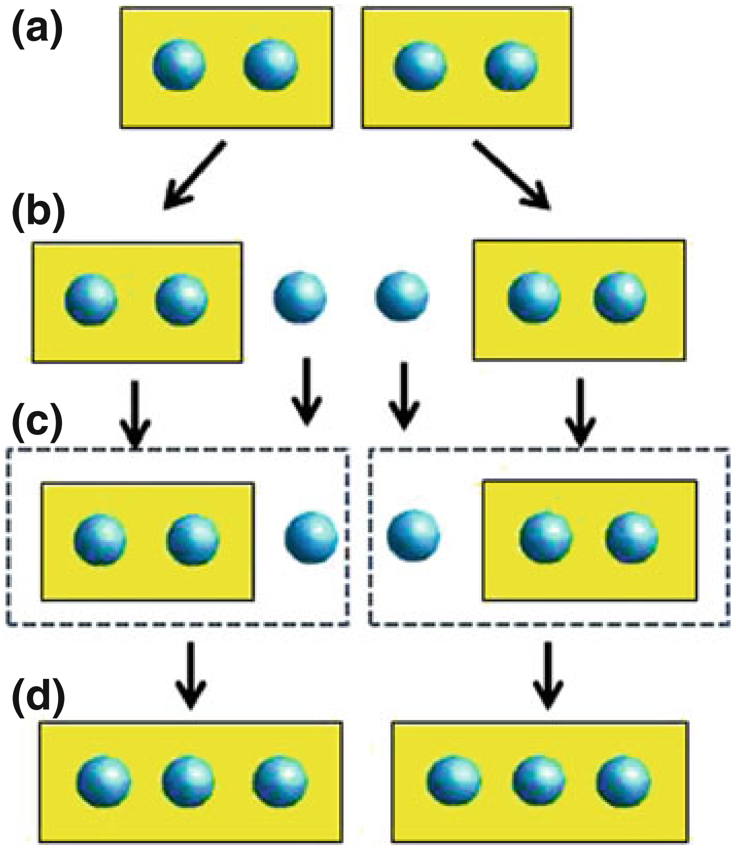
- Grow the chain until the Hilbert space size exceeds the imposed upper limit.

- Diagonalise the full Hamiltonian and obtain the ground state $|\Psi\rangle = \sum_{i=1}^{D_A}\sum_{j=1}^{D_B} \Psi_{i,j}|i\rangle \otimes |j\rangle$.

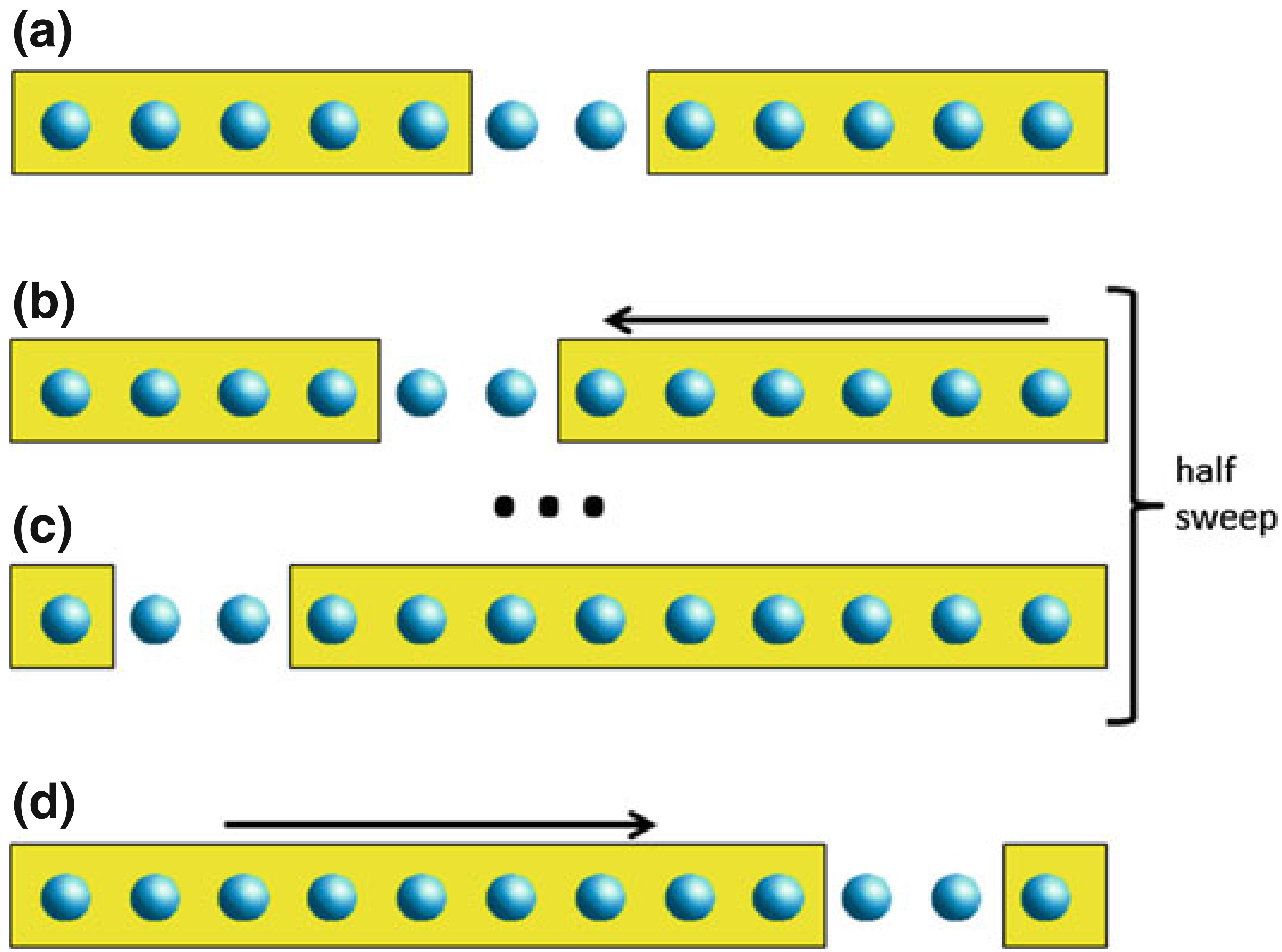
- Compute the reduced density matrix of one of the blocks, and truncate.

- Express the Hamiltonian in the new basis.

- Add two more sites and keep growing.



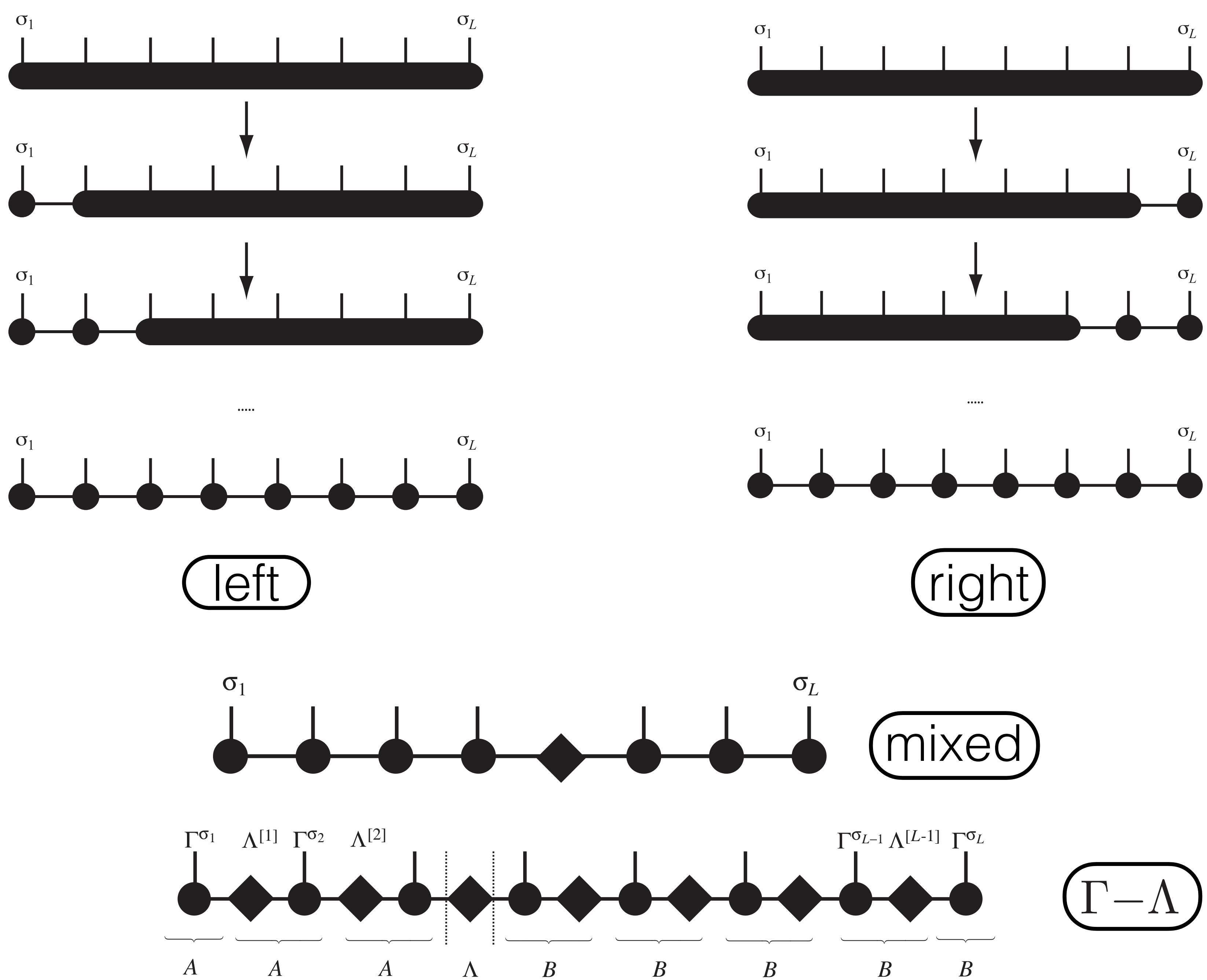
# Finite-size DMRG



# Matrix product states

# Constructing canonical states: pictures

$$|\psi\rangle = \sum_{\sigma_1,\ldots,\sigma_L} c_{\sigma_1,\ldots,\sigma_L} |\sigma_1,\ldots,\sigma_L\rangle$$



# Constructing canonical states: details

$$|\psi\rangle = \sum_{\sigma_1,\ldots,\sigma_L} c_{\sigma_1,\ldots,\sigma_L} |\sigma_1,\ldots,\sigma_L\rangle$$

Reshape, SVD, then reshape.

Choose to keep only the largest SV's

Bond dimension

$$c_{\sigma_1,\ldots,\sigma_L} = \Psi_{\sigma_1,(\sigma_2,\ldots,\sigma_L)} = \sum_{a_1}^{r_1} U_{\sigma_1,a_1} S_{a_1,a_1} (V^\dagger)_{a_1,(\sigma_2,\ldots,\sigma_L)} \equiv \sum_{a_1}^{r_1} U_{\sigma_1,a_1} c_{a_1\sigma_2,\ldots,\sigma_L}$$

Rename and reshape $U_{\sigma_1,a_1} = A^{\sigma_1}_{a_1}$

$$c_{\sigma_1,\ldots,\sigma_L} = \sum_{a_1}^{r_1} A^{\sigma_1}_{a_1} \Psi_{(a_1\sigma_2),(\sigma_3,\ldots,\sigma_L)}$$

SVD on $\Psi_{(a_1\sigma_2),(\sigma_3,\ldots,\sigma_L)}$

$$\sum_{\sigma_\ell} A^{\sigma_\ell\dagger} A^{\sigma_\ell} = I.$$

$$c_{\sigma_1,\ldots,\sigma_L} = \sum_{a_1}^{r_1}\sum_{a_2}^{r_2} A^{\sigma_1}_{a_1} U_{(a_1\sigma_2),a_2} S_{a_2,a_2} (V^\dagger)_{a_2,(\sigma_3,\ldots,\sigma_L)} = \sum_{a_1}^{r_1}\sum_{a_2}^{r_2} A^{\sigma_1}_{a_1} A^{\sigma_2}_{a_1,a_2} \Psi_{(a_2\sigma_3),(\sigma_4,\ldots,\sigma_L)}$$

Keep going

Rename and reshape $U_{(a_1\sigma_2),a_2} = A^{\sigma_2}_{a_1,a_2}$

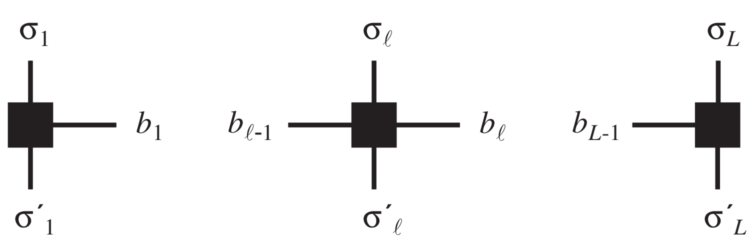
$$c_{\sigma_1,\ldots,\sigma_L} = \sum_{a_1,\ldots,a_{L-1}} A^{\sigma_1}_{a_1} A^{\sigma_2}_{a_1,a_2} \cdots A^{\sigma_{L_1}}_{a_{L-2},a_{L-1}} A^{\sigma_L}_{a_{L-1}} \quad \text{and} \quad |\psi\rangle = \sum_{\sigma_1,\ldots,\sigma_L} A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_{L-1}} A^{\sigma_L} |\sigma_1,\ldots,\sigma_L\rangle$$

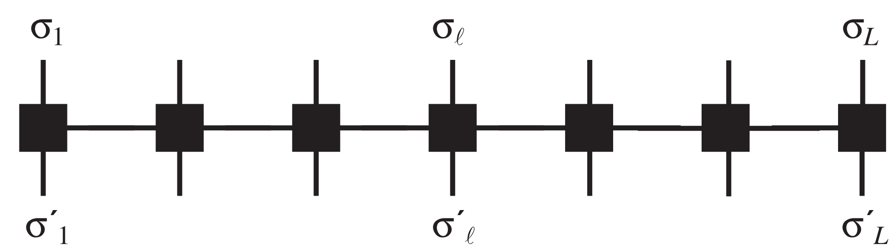
# Matrix Product Operator

$$
\begin{aligned}
\hat{O} &= \sum_i \left( \hat{A}_i \hat{B}_{i+1} + \hat{B}_i \hat{A}_{i+1} \right) \\
&= \hat{A} \otimes \hat{B} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\
&\quad + \mathbb{1} \otimes \hat{A} \otimes \hat{B} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots \\
&\quad + \hat{B} \otimes \hat{A} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\
&\quad + \mathbb{1} \otimes \hat{B} \otimes \hat{A} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots
\end{aligned}
$$

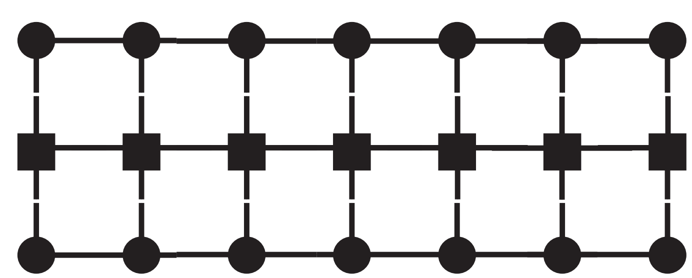
$$
M = \begin{array}{cc} \xrightarrow{\quad b_{l-1} \quad} & \\ \begin{pmatrix} \mathbb{1} & \hat{A} & \hat{B} & 0 \\ 0 & 0 & 0 & \hat{B} \\ 0 & 0 & 0 & \hat{A} \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} & \Big\downarrow b_{l-1} \end{array}
$$

$$
\longrightarrow \quad \hat{O} = \sum_{b_1, \ldots, b_{L-1}} M^{\sigma_1, \sigma'_1}_{1, b_1} M^{\sigma_2, \sigma'_2}_{b_1, b_2} M^{\sigma_3, \sigma'_3}_{b_2, b_3} \ldots M^{\sigma_{L-1}, \sigma'_{L-1}}_{b_{L-3}, b_{L-1}} M^{\sigma_L, \sigma'_L}_{b_{L-1}, 1}
$$




→ matrix elements



It is simple to compute local operator matrix elements with canonical states.

# MPO for the Thirring model Hamiltonian

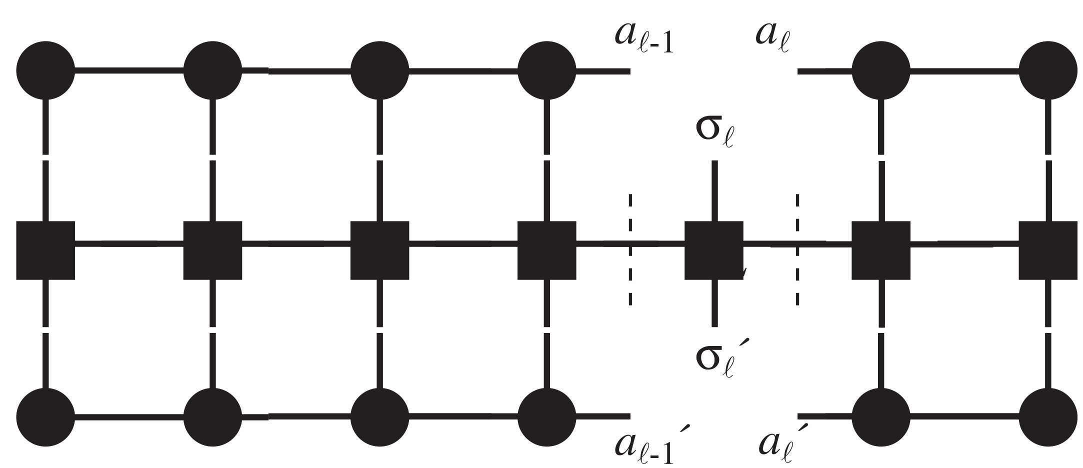
$$M_n = \begin{pmatrix} \mathbb{1} & -\frac{1}{2a} S^+ & -\frac{1}{2a} S- & 2\lambda S^z & \frac{2g}{a} S^z \, \delta_{0,n\%2} & \beta_n S^z + \gamma \mathbb{1} \\ 0 & 0 & 0 & 0 & 0 & S^- \\ 0 & 0 & 0 & 0 & 0 & S^+ \\ 0 & 0 & 0 & \mathbb{1} & 0 & S^z \\ 0 & 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & 0 & \mathbb{1} \end{pmatrix},$$

where

$$\beta_n = \frac{g}{a} + \left( (-1)^n \, m_0 + \mu \right) - 2\lambda \, S_{target}$$

$$\gamma = \lambda \left( \frac{1}{4} + \frac{S_{target}^2}{N} \right) + \frac{g}{4a} + \frac{\mu}{2} \, .$$

# Practice of MPS for DMRG




One step in a sweep of finite-size DMRG

- Reshape the free indices of this tensor to form a matrix

$$(T)^{\sigma_l,\sigma_l'}_{a_{l-1},a_l,a_{l-1}',a_l'} \longrightarrow (T)_{\sigma_l a_{l-1} a_l,\sigma_l' a_{l-1}' a_l'}$$

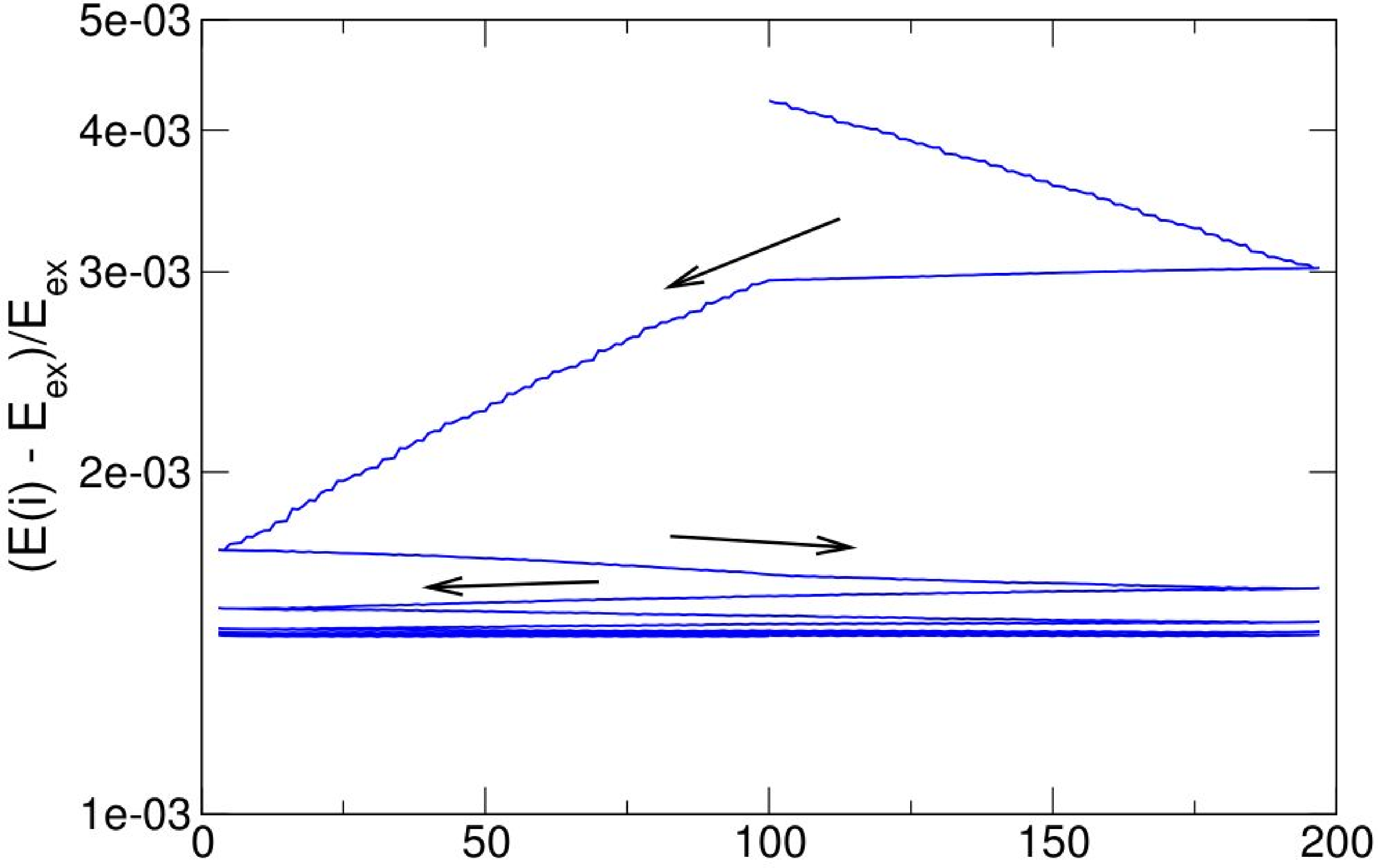
- Obtain the lowest-lying eigenvector $v_{\sigma_l a_{l-1} a_l}$ .

- Reshape $v_{\sigma_l a_{l-1} a_l} \longrightarrow M_{\sigma_l a_{l-1}, a_l}$

SVD on $M_{\sigma_l a_{l-1}, a_l}$ , and then reshape.

# Practice of MPS for DMRG

Michael Sekania (2011)




Heisenberg spin chain

# Spin-spin correlators in the XY model

# Recap: what do we expect?

- Compute $G(|a-b|) = \langle e^{i(\theta(a)-\theta(b))}\rangle$ in the XY model.
- High-T: $G(r) = A'\ e^{-r/\xi}$.
- Low-T: $G(r) = A\ r^{-T/2\pi K}$.

| Thirring | sine-Gordon | XY |
|---|---|---|
| $g$ | $\frac{4\pi^2}{t} - \pi$ | $\frac{T}{K} - \pi$ |

- Phase transition at $T_c \sim K\pi/2,\ g_c \sim -\pi/2$.

# The operator and the correlator

XY model → Sine-Gordon theory

$$\langle e^{i(\theta(a)-\theta(b))}\rangle = \langle e^{-\frac{2\pi}{t}\int_{\Gamma_{a,b}} dx_\mu \epsilon_{\mu\nu}\left(\frac{\partial}{\partial x_\nu}\phi\right)(x)}\rangle$$

↓ Thirring model *via* duality

$$e^{i(g+\pi)\int_a^b dx_1 \bar{\psi}\gamma^1\psi} = e^{i(g+\pi)\int_a^b dx_1 (\psi_d^*\psi_u+\psi_u^*\psi_d)}$$

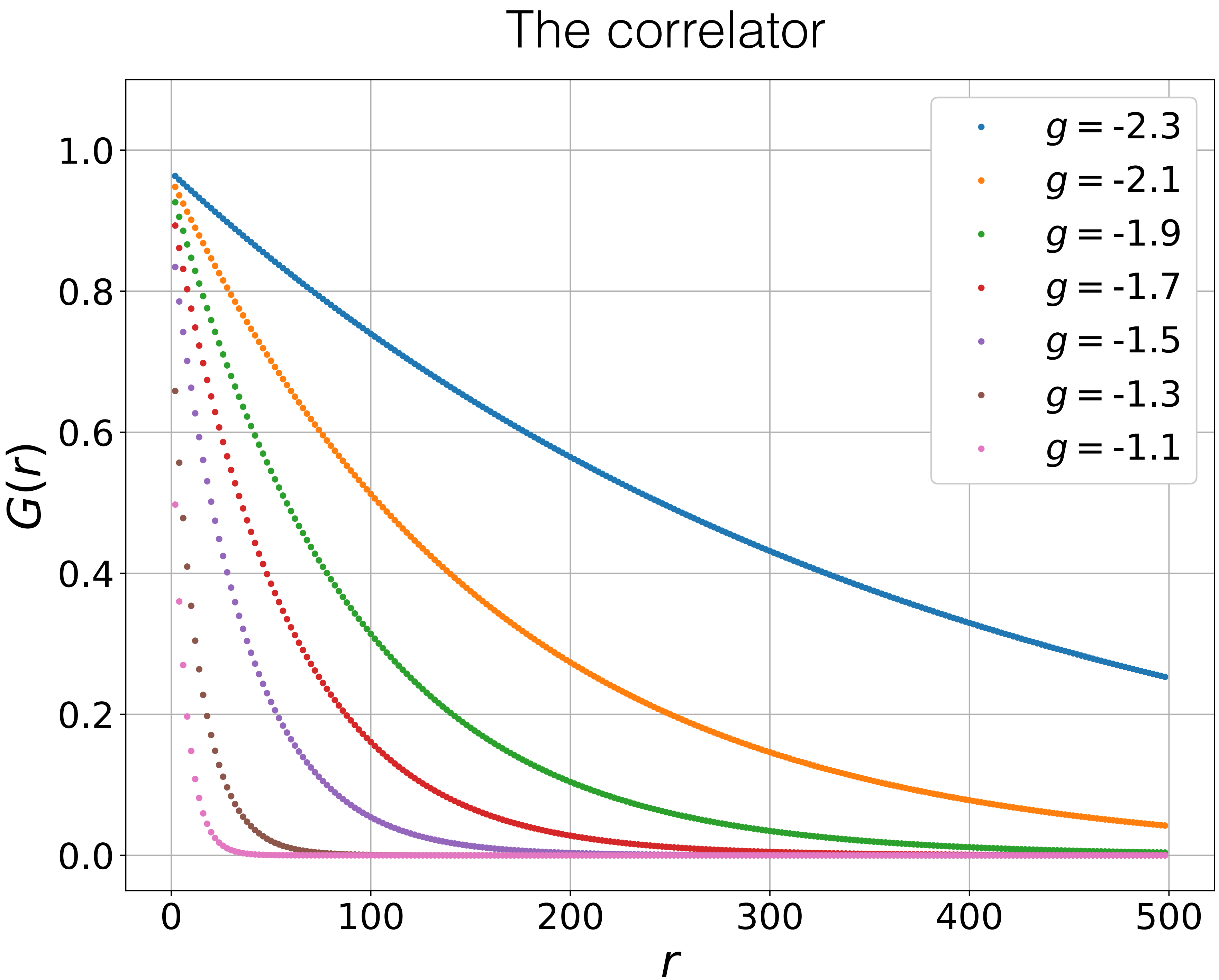
↓ Latticising through staggering

$$e^{i(g+\pi)\int_a^b dx_0 (\psi_u^*\psi_d+\psi_d^*\psi_u)} \rightarrow e^{i(g+\pi)\sum_{j=m}^n \left(c_{2j+1}^\dagger c_{2j}+c_{2j}^\dagger c_{2j+1}\right)}$$

↓ Spin model through the JW trans.

$$\prod_{j=m}^n e^{(g+\pi)\left(S_{2j}^+ S_{2j+1}^- - S_{2j+1}^+ S_{2j}^-\right)}$$

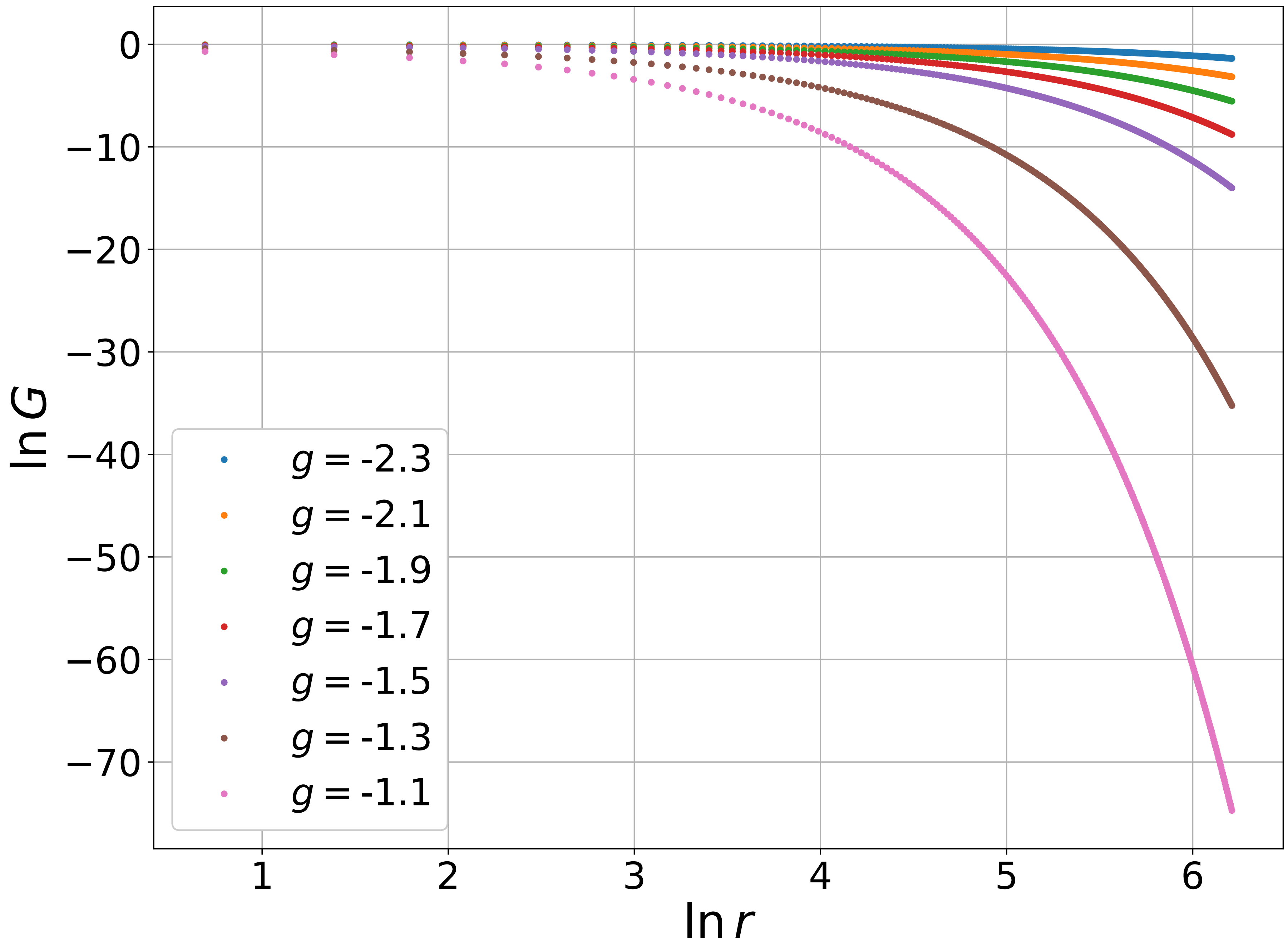
# Results




$$N = 1000, D = 100, m_0 a = 0.5$$

# Results

$$G(r) = A\ r^{-\eta} \text{ or } G(r) = A'\ e^{-r/\xi}$$

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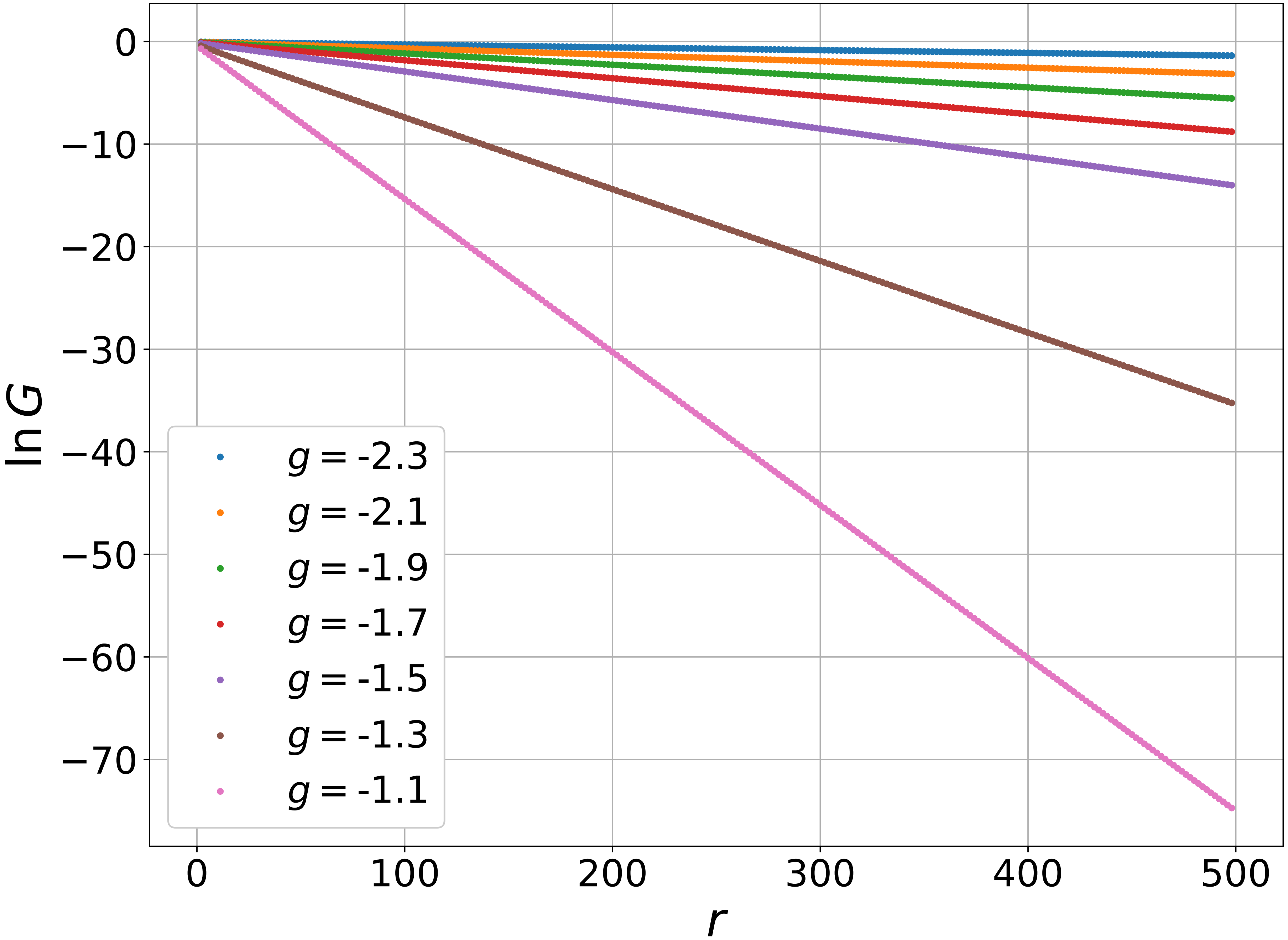


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# Remarks and outlook

- We constructed MPO and MPS for the Thirring model.

- Promising exploratory results on the KT-phase transition.
  → Finite-size scaling analysis is desirable.

- Better simulations: larger system size & bond dimension.
  → Precise determination of the K-T critical temperature.

- Future: other aspects of topological phase transitions.
  → Real-time dynamics?