Geometric interpretation of the partition function of 2D gravity ☆

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We construct explicitly the subspace in the infinite dimensional grassmannian corresponding to the τ-function of the 2D topological gravity. This allows us to give a simple proof of some conjectures on the equations defining this function.

It has been discovered recently that the reduction to $N\times N$ matrix models, $N\to\infty$, can be used to obtain the complete non-perturbative solution of 2D gravity and of string theory coupled with matter fields having central charge $c<1$ [1–3]. The answer for the partition function was expressed in terms of τ-functions. Later is was shown that the same partition function arises in 2D topological gravity [4,5]. Namely the generating function of amplitudes in topological gravity $\tau(t_1, t_3, ...)$ and the partition function $Z(t_1, t_3, ...)$ of the one-matrix model at the $k=1$ critical point are connected by the formula

$$Z(t_1, t_3, ...)=\tau^2(t_1, t_3, ...)\ . \qquad (1)$$

The function $\tau(t_1, t_3, ..., t_{2n+1}, ...)$ satisfies the so called string equation and the following equations of the KdV hierarchy:

$$\frac{\partial u}{\partial t_{2k+1}}=(-2)^k\frac{\partial}{\partial t_1}R_k(u)\ ,$$

$$u=-2\frac{\partial^2}{\partial t_1^2}\ln\tau\ .$$

Here the $R_k(u)$ are defined by the formula

$$\langle x|\exp\{-t[-\partial^2/\partial x^2+u(x)]\}|x\rangle \sim\frac{1}{\sqrt{4\pi t}}\sum_{k=0}^{\infty}t^kR_k(u) \qquad (2)$$

for $t\to0$ (i.e. $R_k(u)$ are Seeley coefficients for the operator $-\partial^2/\partial x^2+u(x)$). Recall that the τ-function of the KP hierarchy can be defined as a function of an infinite number of variables t_1, t_2, ... satisfying bilinear Hirota equations. Such a function can be considered as a τ-function of the KdV hierarchy if it does not depend on the even variables t_2, t_4, ... Every τ-function of the KP hierarchy corresponds to a point of the Sato infinite-dimensional grassmannian Gr [6]. (The points of Gr are linear subspaces of the space H consisting of the formal Laurent series $\sum a_nz^n$, $a_n=0$ for $n\gg0$. The subspace $\mathrm{V}\subset\mathrm{H}$ belongs to Gr if the natural projection π_+ of V into the space H_+ spanned by z^n, $n\geqslant0$, is a Fredholm operator. The big cell of Gr consists of those V for which π_+ is an isomorphism; we denote it by Gr^0.) The τ-functions of the KdV hierarchy correspond to the points of Gr obeying $z^2\mathrm{V}\subset\mathrm{V}$; we denote this subset of Gr by $\mathrm{Gr}_{(2)}$ and let $\mathrm{Gr}^0_{(2)}=\mathrm{Gr}_{(2)}\cap\mathrm{Gr}^0$.

Our aim is to describe the point of Gr corresponding to the τ-function arising in 2D gravity. The description will be based on refs. [7,8]. It is proved in

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these papers that the τ-function in (1) obeys

$$(\tfrac{3}{2}J_{2n+1}+L_{n-1})\tau=0, \quad n\geqslant 0, \tag{3}$$

where the L_n are Virasoro generators given by the formula

$$L_n=\tfrac{1}{4}\sum_{p=2k+1} J_pJ_{2n-p}+\tfrac{1}{16}\delta_{n0}. \tag{4}$$

Here J_p acts as multiplication by $(-p)t_{-p}$ for $p\leqslant 0$ and as $\partial/\partial t_p$ for $p>0$. Our formula (3) differs slightly from the formulas in refs. [7,8] because the τ-function (1) is obtained from the τ-function studied in these papers by the shift $t_3\to t_3+1$. Such a shift is necessary for our aim because we want to consider the τ-function as a formal power series. (The τ-function of refs. [7,8] is singular at $t_1=t_3=...=0$.) Using (3) we will probe below that the τ-function under consideration corresponds to the subspace $V_{(2)}$ from the big cell $Gr^0_{(2)}$ satisfying

$$AV_{(2)}\subset V_{(2)}, \tag{5}$$

where

$$A=\tfrac{3}{2}z+\frac{1}{2z}\frac{d}{dz}-\tfrac{1}{4}z^{-2}. \tag{6}$$

The condition (5) singles out a unique point in the big cell $Gr^0_{(2)}$. We use the notation $V_{(2)}$ having in mind the generalization to the case of W_h-gravity. It is very plausible [7,8] that in this case the partition function is connected with a τ-function as well. The point of the grassmannian corresponding to this τ-function will be denoted by $V_{(h)}$.

Let us sketch the proof that the element $V_{(2)}\in Gr^0_{(2)}$ satisfying (5) is unique. Since the projection of $V_{(2)}$ on H_+ is an isomorphism, there exists a $\varphi\in V_{(2)}$ of the form $\varphi=1+\sum_{i\geqslant 1}c_iz^{-i}$. Furthermore $A^n\varphi=z^n+$lower degree terms, hence these functions with $n\geqslant 0$ form a basis of V. Therefore $z^2\varphi$ is a linear combination of the $A^n\varphi$; it is now easy to check that $z^2\varphi$ is proportional to $A^2\varphi$. This determines φ uniquely.

The equation $A^2\varphi=\text{const.}\times z^2\varphi$ can be reduced to the Airy equation by means of the substitution

$$y=z^{-1/2}\exp(-\tfrac{2}{3}z^3)\varphi, \quad x=\mu^2z^2, \quad \mu^3=-\tfrac{3}{2}.$$

This permits us to describe the subspace $V_{(2)}\subset H$ as follows. It is well known [9] that there exists a solution $\sqrt{x}K_{1/3}(\tfrac{2}{3}x^{3/2})$ of the Airy equation $y''=xy$ having for $x\to+\infty$ the asymptotic expansion

$$y=\sqrt{\tfrac{1}{2}\pi}x^{-1/4}\exp(-\tfrac{2}{3}x^{3/2})\left(\sum_{m=0}^{\infty}a_mx^{-3m/2}\right), \tag{7}$$

where the a_m are found from the recurrent formula

$$48ma_m=(-1)^m(6m-1)(6m-5)a_{m-1}. \tag{8}$$

Then

$$\varphi(z)=1+a_1(\mu z)^{-3}+a_2(\mu z)^{-6}+... . \tag{9}$$

Thus, the space $V_{(2)}$ is the subspace of H spanned by

$$\varphi, A\varphi, z^2\varphi, z^2A\varphi, ..., z^{2n}\varphi, z^{2n}A\varphi, ... \tag{10}$$

It is easy to check that $z^{2n}\varphi=z^{2n}+$lower order terms, $z^{2n}A\varphi=z^{2n+1}+$lower order terms. Therefore the natural projection π_+ of $V_{(2)}$ into H_+ is an isomorphism. We see that $V_{(2)}$ belongs to $Gr^0_{(2)}$. The invariance of $V_{(2)}$ with respect to the operator A follows from the equation $A^2\varphi=\text{const.}\times z^2\varphi$.

Our arguments give a rigorous proof that there exists a τ-function of the KdV hierarchy satisfying (3). Therefore the combination of our results and the results of refs. [7,8] gives a new derivation of the connection between the partition function of the matrix model and the τ-function.

It is important to note that the space $V_{(2)}$ belongs to the Sato grassmannian Gr [6] but does not belong to the Segal–Wilson modification of the Sato grassmannian [10]. (Recall that in the Segal–Wilson definition the space H is replaced by the space $L^2(S^1)$ of all square integrable functions.)

It remains to prove the characterization of the τ-function under consideration by (5) (see the appendix for an alternative proof). We will use in the proof the fermionic representation of Gr. Let us consider the representation of canonical anticommutation relations $[\psi_n, \psi^+_m]_+=\delta_{nm}$, $[\psi_n, \psi_m]_+=[\psi^+_n, \psi^+_m]_+=0$ in the Fock space $\mathscr{F}$ with vacuum vector Φ satisfying $\psi_n\Phi=0$, $n<0$, $\psi^+_m\Phi=0$, $m\geqslant 0$. Let us define $GL(\infty)$ as a group with the Lie algebra consisting of operators $a+\sum c_{mn}:m_m\psi^+_n:$. Here a is a complex number, $c_{mn}=0$ for $|m-n|\gg 0$, dots denote the normal product with respect to the vacuum Φ:

$$\psi_m\psi^+_n=:\psi_m\psi^+_n:+\langle\psi_m\psi^+_n\rangle,$$

where $\langle\psi_m\psi^+_n\rangle=1$ if $m=n<0$, $\langle\psi_m\psi^+_n\rangle=0$ in all other cases.

In $\mathscr{F}$ one can introduce the operators $A_n = \sum :\psi_k \psi^+_{k+n}:$ satisfying the canonical commutation relations

$$[A_m, A_n] = m\delta_{m,-n} . \quad (11)$$

This permits us to perform the bosonization procedure in the following way. Let us associate with every vector $\Psi \in \mathscr{F}$ a function

$$\varphi^{\Psi}(t_1, t_2, ...) = \left\langle \exp\left(\sum_{n=1}^{\infty} t_n A_n\right)\Psi, \Phi \right\rangle . \quad (12)$$

It is easy to check that by this correspondence the operators A_n transform into the operators J_n (i.e. $\varphi^{A_n \Psi} = J_n \varphi^{\Psi}$).

If the vector Ψ belongs to $\mathrm{GL}(\infty)\cdot\Phi$ (to the $\mathrm{GL}(\infty)$-orbit of the vacuum vector) the function φ^{Ψ} is a τ-function of the KP-hierarchy. Conversely every τ-function can be represented in this form (recall that we consider the τ-function as a formal power series). There exists a natural map ρ of the orbit $\mathrm{GL}(\infty)\cdot\Phi$ onto Gr; the τ-function τ^{Ψ} coincides (up to multipliers) with the τ-function corresponding to $\rho(\Psi) \in \mathrm{Gr}$. It is easy to check that in the case when the vector Ψ satisfies the condition $(a + \sum c_{mn} \times :\psi_m \psi^+_n:)\Psi = 0$ the subspace $\rho(\Psi) \subset \mathrm{H}$ is invariant with respect to the operator $\hat{C}$ acting in H. (Here $\hat{C}$ denotes the linear operator in H having the matrix $\hat{c}_{mn} = c_{nm}$ in the basis z^n, $n = 0, \pm 1, ...$). This assertion is evident in the case $\Psi = \Phi$, $\rho(\Psi) = \mathrm{H}_+$. One can reduce the general case to this particular case using the fact that the map ρ is compatible with the action of the group $\mathrm{GL}(\infty)$.

Let us introduce the operators

$$L_n = \tfrac{1}{4} \sum J_{2n-k} J_k + \tfrac{1}{16}\delta_{n0}$$
$$= \tfrac{1}{2} \sum_{k>n} J_{2n-k} J_k + \tfrac{1}{4} J_n J_n + \tfrac{1}{16}\delta_{n0} . \quad (13)$$

Note that the two expressions for L_n in (13) are formally equivalent; however only the second form leads to a well defined operator. In the fermionic representation the operators (13) correspond to the operators

$$L'_n = \tfrac{1}{2} \sum_{k>n} A_{2n-k} A_k + \tfrac{1}{4} A_n^2 + \tfrac{1}{16}\delta_{n0}$$
$$= \tfrac{1}{2} \sum_{\substack{\beta-\alpha+\delta-\gamma=2n \\ \beta-\alpha<\delta-\gamma}} :\psi_\alpha \psi^+_\beta: :\psi_\gamma \psi^+_\delta:$$
$$+ \tfrac{1}{4} \sum_{\beta-\alpha=\delta-\gamma=n} :\psi_\alpha \psi^+_\beta: :\psi_\gamma \psi^+_\delta: + \tfrac{1}{16}\delta_{n0} . \quad (14)$$

Using Wick's theorem one can represent L_n in the normal form. We obtain in this way a sum of three terms: $L_n = L_n^{(1)} + L_n^{(2)} + L_n^{(3)}$ where $L_n^{(1)}$ is a quartic expression with respect to ψ, ψ^+, $L_n^{(2)}$ is a quadratic expression and $L_n^{(3)}$ is a scalar. It is important that all quartic terms cancel: $L_n^{(1)} = 0$. To prove the cancellation we have to note that $\tfrac{1}{2} :\psi_\alpha \psi^+_\beta \psi_\gamma \psi^+_\delta:$ cancels with $\tfrac{1}{2} :\psi_\alpha \psi^+_\delta \psi_\gamma \psi^+_\beta:$ if $\gamma - \beta < \alpha - \delta$, both $\tfrac{1}{2} :\psi_\gamma \psi^+_\beta \times \psi_\alpha \psi^+_\delta:$ if $\alpha - \delta < \gamma - \beta$ and with $\tfrac{1}{4} :\psi_\gamma \psi^+_\beta \psi_\alpha \psi^+_\delta: + \tfrac{1}{4} :\psi_\alpha \psi^+_\delta \psi_\gamma \psi^+_\beta:$ if $\gamma - \beta = \alpha - \delta$. Wick's theorem gives the following expression for the quadratic term: $L_n^{(2)} = \sum c_{\alpha\delta} :\psi_\alpha \psi^+_\delta:$ where $c_{\alpha\delta} = 0$ if $\delta - \alpha \neq 2n$,

$$c_{\alpha,\alpha+2n} = \tfrac{1}{2} \sum_{2\beta<2\alpha+2n} \langle \psi^+_\beta \psi_\beta \rangle + \tfrac{1}{4} \langle \psi^+_{\alpha-n} \psi_{\alpha-n} \rangle$$
$$- \tfrac{1}{2} \sum_{2\beta>2\alpha+2n} \langle \psi_\beta \psi^+_\beta \rangle - \tfrac{1}{4} \langle \psi_{\alpha-n} \psi^+_{\alpha-n} \rangle . \quad (15)$$

It follows from (15) that $c_{\alpha,\alpha+2n} = \tfrac{1}{2}(\alpha - n) + \tfrac{1}{4}$. For $n \neq 0$ the scalar term $L_n^{(3)}$ vanishes; this follows from Wick's theorem. Namely $\langle \psi_\alpha \psi^+_\delta \rangle \langle \psi^+_\beta \psi_\gamma \rangle$ can be non-zero only in the case $\alpha = \beta$, $\beta = \gamma$, and therefore $\beta - \alpha + \delta - \gamma = 2n = 0$. We see that for $n \neq 0$

$$L'_n = \sum \tfrac{1}{2}\alpha :\psi_\alpha \psi^+_{\alpha+2n}: + (\tfrac{1}{2}n + \tfrac{1}{4}) \sum :\psi_\alpha \psi^+_{\alpha+2n}: . \quad (16)$$

In a slightly different language this result is well known.

One can give another proof of this formula. Namely, it is easy to check that the operators (16) satisfy the relations

$$[A_k, L'_n] = \tfrac{1}{2} k A_{k+2n} .$$

The same relations are fulfilled for the operators (14). (This follows from the commutation relation (11).) Therefore the operators (14) and (16) coincide up to an additive constant. Formula (16) permits us to assert that the operator $:\psi_\alpha c_{\alpha\beta} \psi^+_\beta:$ where

$$c_{\alpha\beta} = \tfrac{3}{2}\delta_{\beta-\alpha-1} + (\tfrac{1}{2}\alpha - \tfrac{1}{4})\delta_{\beta-\alpha+2} , \quad (17)$$

annihilate the vector Ψ corresponding to the τ-function satisfying (3) with $n = 0$. The matrix of the operator A in the basis z^n, $n=0, \pm, ...$, can be obtained from (17) by transposition, therefore the space $V_{(2)}=\rho(\Psi)$ satisfies (5).

Note that as a byproduct of our consideration we obtain a proof of conjecture 1 of ref. [8]. Namely we see that the τ-function of the KdV hierarchy satisfying (3) for $n=0$ obeys also (3) for every $n\geqslant 0$. Indeed, eq. (3) for the τ-function of the KdV hierarchy is equivalent to the relation $A^{(n)}V\subset V$ for the corresponding point V of Gr for a certain linear operator $A^{(n)}$ in H. It is easy to check that $A^{(n)}=z^{2n}A$ (see appendix) and therefore the relation $A^{(n)}V\subset V$ follows from $AV\subset V$ and $z^2V\subset V$. In particular, the point $V_{(2)}$ constructed above satisfies $A^{(n)}V_{(2)}\subset V_{(2)}$ and therefore the corresponding τ-function obeys (3).

The results above can be generalized in the following way. Let us consider the operators

$$L_n^{(h)}=\frac{1}{h}\sum_{hn<2l}J_{hn-l}J_l+\frac{1}{2h}J_{hn/2}^2+\frac{h^2-1}{24h}\delta_{n0}\,, \quad (18)$$

acting on functions $\varphi(t_1, t_2, ...)$. (Of course one has to omit the term containing $J_{hn/2}$ if hn is odd.) These operators generate a Virasoro algebra. For $n=2$ they coincide with the operators (13); the methods used for $n=2$ can be applied in the general case too. We obtain that in the fermionic representation the operators (18) can be written as

$$\tilde{L}_n^{(h)}=\frac{1}{h}\bigg(\sum_\alpha \alpha :\psi_\alpha\psi^+_{\alpha+nh}: + \tfrac{1}{2}(nh+1)\sum_\alpha :\psi_\alpha\psi^+_{\alpha+nh}:\bigg)\,. \quad (19)$$

The simplest way to prove this fact (up to an additive constant) is to check that the operators (18) and (19) satisfy the commutation relations

$$[J_l, L_k^{(h)}]=\frac{l}{h}J_{l+kh}\,,\quad [A_l, \tilde{L}_k^{(h)}]=\frac{l}{h}A_{l+kh}\,. \quad (20)$$

It was conjectured in refs. [7,8] that the partition function of W_h gravity is connected with the τ-function $\tau_{(h)}(t_1, t_2, ...)$ that does not depend on $t_h, t_{2h}, ...$ and satisfies the condition

$$\left(\frac{h+1}{h}J_{hn+1}+L_{n-1}^{(h)}\right)\tau=0 \quad \text{for } n\geqslant 0\,. \quad (21)$$

(More precisely, in the definition of operators (18) given in refs. [7,8] l runs over only integers that are not multiples of h, however in (21) this restriction is irrelevant.) It follows from (21) that $V_{(h)}\in$Gr corresponding to the τ-function $\tau_{(h)}$ satisfies the conditions

$$A_{(h)}V_{(h)}\subset V_{(h)}\,,\quad z^hV_{(h)}\subset V_{(h)}\,,$$

where

$$A_{(h)}=\frac{h+1}{h}z+\frac{1}{h}\left(z^{-h+1}\frac{d}{dz}-\tfrac{1}{2}(h-1)z^{-h}\right).$$

The space $V_{(h)}$ is spanned by the elements $A^n_{(h)}\varphi\in$H, $n=0, 1, 2, ...$, where the formal series $\varphi=1+\sum_{i\geqslant 1}b_iz^{-i}$ satisfies the equation

$$A^h_{(h)}\varphi=\text{const.}\times z^h\varphi\,. \quad (22)$$

The solution of this equation can be reduced to the solution of the "generalized" Airy equation $y^{(h)}=xy$. Consider its solution with the asymptotic expansion

$$y=x^{-(h-1)/2h}\times\exp\left(\frac{h}{h+1}\right)x^{(h+1)/h}\sum_{n=0}^\infty a_n^{(h)}x^{-[(h+1)/h]n}\,.$$

Then the power series $\varphi(z)=\sum_{n=0}^\infty a_n^{(h)}\mu^{-n}z^{-(h+1)n}$, where $\mu=(h+1)/h$ and the $a_n^{(h)}$ may be found from the recurrent formula $na_n^{(h)}=\sum_{k=1}^{h-1}c_k^{(h)}(n)a_{n-k}^{(h)}$, is a unique power series (in z^{-1}) solution of (22) with $a_0^{(h)}=1$. (For example, $48c_1^{(2)}(n)=(6n-1)(6n-5)$.) Indeed, (22) with $\varphi(z)=\sum_{n=0}^\infty b_nz^{-n}$ is equivalent to a recurrent formula of the form $nb_n=\sum_{k=1}^{h-1}c_kb_{h-k(h+1)}$.

We are deeply indebted to M. Sato for the explanation that this definition of grassmannian is appropriate for our aims.

Appendix. We show here how the above construction fits in the representation theory of arbitrary simply laced affine algebras [11,12].

(A1) Let E be a subset of the set of non-zero integers and let h be a positive integer such that $E=-E$ and $E=h+E$. Let $E_+=\{m\in E\,|\,m>0\}$, $E_0=\{m\in E\,|\,0<m<h\}$, $l=|E_0|$.

We construct twisted Virasoro operators L_n in terms of the oscillator algebra $[J_m, J_n]=m\delta_{m,-n}$ $(m, n\in E)$:

$$L_0=\frac{1}{h}\sum_{m\in E_+}J_{-m}J_m+\frac{1}{4h^2}\sum_{j\in E_0}j(h-j)\,,$$

$$L_n=\frac{1}{2h}\sum_{m\in E} J_{hn-m}J_m \quad \text{if } n\neq 0 .$$

We have

$$[J_m, L_n]=\frac{m}{h}J_{m+nh} ,$$

$$[L_m, L_n]=(m-n)L_{m+n}+\tfrac{1}{12}(m^3-m)l\delta_{m,-n} .$$

Examples. (a) let $E=\mathbb{Z}\backslash h\mathbb{Z}$. Then the operators L_n are the ones appearing in the equations of refs. [6,7]. When $h=2$, these are the operators (13). On the other hand, they are the twisted Virasoro operators in the basic representation of the affine algebra $\widehat{\mathrm{sl}}_h$ [11,12].

(b) Let $E=\{m_1\leqslant...\leqslant m_l\}$ be the exponents of a simply laced simple Lie algebra $\mathfrak{g}$, and let $h=m_l+1$ be the Coxeter number. Let $E=\{E_0+jh, j\in\mathbb{Z}\}$. Then the L_n are the twisted Virasoro operators in the basic representation of $\hat{\mathfrak{g}}$ [11,12].

(A2) Recall that the basic representation of the affine $\hat{\mathfrak{g}}=(H\otimes\mathfrak{g})\oplus\mathbb{C}K$ is its unique irreducible representation π in a vector space V such that $k=1$ and there exists a non-zero vector Φ such that

$$\pi(H_+\otimes\mathfrak{g})\Phi=0 .$$

Turning to $\mathfrak{g}=\mathrm{sl}_h$, consider the following matrices in $\hat{\mathfrak{g}}$ (for all other $\mathfrak{g}$ the construction is similar, see ref. [11]):

$$s_1=\sum_{i=1}^{h-1} e_{i,i+1}+ze_{h,1}, \quad s_n=s_1^n ,$$

$$\rho=\tfrac{1}{2}\sum_{i=1}^{h}(h-2i+1)e_{ii} .$$

Note that $s_n\in\hat{\mathfrak{g}}$ if $n\in h\mathbb{Z}$ and that $[s_m, s_n]=m\delta_{m,-n}$ (m, $n\in E$). This is called the principal subalgebra of $\hat{\mathfrak{g}}$.

The basic representation π of $\hat{\mathfrak{g}}$ is constructed in the space of formal power series in indeterminates t_n, $n\in E_+$ such that $s_n=J_n$ and $1=\Phi$. The representation of the rest of the generators of $\hat{\mathfrak{g}}$ is achieved by use of twisted vertex operators (ref. [11], ch. 14).

Let G be the Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{G}=G(H)$. Here and further for any ring R, $SL_n(R)=\{(a_{ij})\,|\,a_{ij}\in R, \det(a_{ij})=1\}$; the definition of G(R) for arbitrary G is similar. The formal power series from the orbit $\pi(\tilde{G})\cdot 1$ are called (formal) τ-functions. This condition can be rewritten as an infinite hierarchy of partial differential equations in the bilinear form of Hirota (see ref. [12]).

Introduce the following operators on $\hat{\mathfrak{g}}$:

$$D_n(a(z))=z^{n+1}\frac{\mathrm{d}a(z)}{\mathrm{d}z}+\frac{1}{h}[z^n\rho, a(z)],$$

$$a(z)\in\hat{\mathfrak{g}} .$$

Then we have

$$D_n(s_m)=\frac{m}{h}s_{m+nh} ,$$

and comparing with (A1) we have

$$[\pi(a(z)), L_n]=\pi(D_n(a(z))), \quad a(z)\in\hat{\mathfrak{g}} .$$

(A3) Recall yet another construction of π for $\hat{\mathfrak{g}}=\widehat{\mathrm{sl}}_n$. Consider the natural representation of $\hat{\mathfrak{g}}$ (with $k=0$) and of $\tilde{G}$ on the space of h-vectors H^h. Consider the associated infinite wedge representation $\mathscr{F}$ of $\hat{\mathfrak{g}}$ (ref. [11], ch. 14). (This is actually the same Fock space $\mathscr{F}$ considered earlier.) Then $k=1$ and the vacuum vector $\Phi\in\mathscr{F}$ corresponding to the subspace H_+ generates the representation π in $V\subset\mathscr{F}$. Under the boson–fermion correspondence $\mathscr{F}^{(0)}$ (respectively V) gets identified with the space of formal power series in t_j; $j=1, 2, 3, ...$ (respectively in t_j, $j\in E_+$); here $\mathscr{F}^{(0)}$ is the subspace of zero charge vectors in $\mathscr{F}$.

For each subspace $W\in Gr_{(h)}=\tilde{G}\cdot H_+$ we thus get a τ-function $\tau_W\in\tilde{G}\cdot 1$ and the key remark is that for an operator B, $B\tau_W=0$ if and only if $BW\subset W$, $BW\neq W$. (Note that the big cell is $SL_h(H_-)\cdot H_+$.)

Finally consider the operator (cf. (21))

$$A_{(h)}^{(n)}=z^n\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{h}z^{n-1}\rho+\mu s_{nh+1} ,$$

where μ is a free parameter. It follows from (A2) that

$$[A_{(h)}^{(n)}, \pi(a(z))]=-L_{n-1}+\mu s_{nh+1} .$$

Considering the map $H^h\to H$ defined by

$$(f_1, ..., f_h)\to z^{h-1}f_1(z^h)+z^{h-2}f_2(z^h)+...$$
$$+f_h(z^h) ,$$

the operator $A_{(h)}^{(0)}$ gets transformed to our basic operator

$$A_{(h)}=\frac{1}{h}z^{-h+1}\frac{\mathrm{d}}{\mathrm{d}z}-\frac{h-1}{2h}z^{-h}+\mu z ,$$

and the operators $A^{(n)}_{(h)}$ to the $z^{nh}A_{(h)}$.

Note added. The description of a basis of the element of the grassmannian corresponding to the τ-function in question is equivalent to certain information about the Baker function. We are grateful to the referee who pointed out that this information in the case $h=2$ can be extracted from a recent preprint by Moore [13].

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