FROM LOOPS TO STATES IN TWO-DIMENSIONAL QUANTUM GRAVITY*

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We investigate macroscopic loop amplitudes (at genus zero) using the matrix model and the Liouville theories of two-dimensional quantum gravity. Some loop amplitudes, interpreted as wave functions of local operators, satisfy a linear differential equation known as the Wheeler-deWitt equation. Moreover, from the properties of the loop amplitudes an inner-product space structure on the space of wave functions emerges naturally. In the course of our analysis we resolve several apparent discrepancies between the matrix model and Liouville theory. Macroscopic loops provide a natural ultraviolet cutoff on the theory, rendering universal analytic terms in the coupling constants. They contain more information than the local operators and should be regarded as fundamental.

1. Introduction and conclusion

The study of two-dimensional random surfaces has many applications. One application is to quantum gravity. Major issues of principle in quantum gravity (e.g. the nature of the Hilbert space and the factorization of amplitudes, measurement theory, the importance of the signature of space-time, etc.) are not understood. The theory of random surfaces is a useful toy model for quantum gravity since, in

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some cases, it is solvable and is not plagued by the technical difficulties of its higher-dimensional counterparts. (Of course, it is possible that it consequently misses some crucial aspects of these counterparts.)

One source of the difficulties in the interpretation of quantum gravity is the ill-definedness of time, a result of general covariance. When we attempt to quantize gravity in a canonical formalism, general covariance leads to a constraint on physical wave functions: They should be annihilated by the generator of time reparametrization. This condition is known as the Wheeler–deWitt equation. Unlike ordinary constraints in field theory (e.g. Gauss' law in $A_0 = 0$ gauge), the WdW constraint is implemented weakly. Consider the path-integral representation of the propagator. The Lagrange multiplier, $A_0$, which implements Gauss' law is integrated from $-\infty$ to $+\infty$. The Lagrange multiplier for the WdW constraint – the elapsed time – is integrated only from $-\infty$ to 0. (This point has been stressed in ref. [1].) Therefore, we can specify space coordinate invariant boundary conditions on the functional integral which do not necessarily satisfy the WdW constraint and find non-zero answers. This phenomenon is familiar in (0 + 1)-dimensional gravity and we will meet it below in 1 + 1 dimensions. Interpreting ordinary field theory as (0 + 1)-dimensional gravity on the world line, this is the origin of off-shell physics. Thus, we may hope that our understanding in two dimensions will help clarify the subject of off-shell string theory.

With the above issues in mind we will examine macroscopic loop amplitudes in two-dimensional gravity. In particular we will show that, when appropriate, they satisfy the WdW constraints. Moreover, we will show how one can introduce an inner-product space structure on the "space of wave functions" of the theory. The introduction of this structure allows us to express matrix model macroscopic loop amplitudes as transitions between states, and paves the way for an understanding of factorization properties in (1 + 1)-dimensional gravity.

Some readers might find these results surprising since the WdW equation is a linear differential equation. On the other hand, in matrix model theory, it is well known that loop amplitudes obey non-linear equations [2–4]. This fact has even led to suggestions that in quantum gravity the superposition principle must be abandoned. We disagree with these proposals. Once the wave functions of the theory are properly identified, there is no need to abandon the superposition principle in two-dimensional quantum gravity (at least, at genus zero). The Liouville interpretation of the nonlinear equation of the matrix model is an interesting question we do not address. It is clearly related to string interactions, and thus takes us out of the single string Hilbert space.

We will need several technical results in order to address the issues discussed above. The tools we will use involve both the matrix model [5–9] and the continuum formulation [10–12] which, in conformal gauge, reduces to the Liouville theory [13–18]. Despite recent progress [19] exact calculations in Liouville are still very complicated. On the other hand, the matrix model provides us with an
extremely efficient calculational tool. Therefore we need a complete and precise
dictionary between Liouville and matrix model correlation functions. This is the
first technical issue addressed in this paper. Some difficulties of the matrix
model/Liouville translation are noted in sect. 2. The difficulties can, in part, be
traced to an issue of contact terms. This allows us to find a partial resolution of the
difficulties. Using the WdW equation as a guide we unravel a tangle of mistaken
identities and complete the resolution of the difficulties in subsects. 4.3 and 4.4. It
is a nontrivial fact that, when correctly understood, Liouville theory and the matrix
model are in complete agreement.

We also will need to develop some technology for handling macroscopic loop
amplitudes. We study some general properties of these amplitudes in sect. 3.
Guided by semiclassical reasoning we investigate in detail the relation between
macroscopic loop operators and sums of local operators. In particular we will see
that macroscopic loop amplitudes contain more information than correlation
functions of local operators. With these insights, we obtain general formulae for
loop amplitudes in subsect. 4.1. Then, after we have understood better the relation
to Liouville theory in subsects. 4.3 and 4.4, we are in a position to note some
intriguing factorization properties in subsect. 4.5. Having settled these technical
problems we will finally be in a position to discuss the inner product space
structure in sect. 5. Most of our analysis is phrased in terms of the one hermitian
matrix model in its one-cut phase, or, from the continuum point of view, in terms
of Liouville coupled to perturbations of the \( (2,2m-1) \) minimal conformal field
theories. In sect. 6 we show that the ideas discussed in the paper can be applied to
a wide variety of models of two-dimensional gravity.

Another application of the theory of random surfaces we have not yet men-
tioned is to string theory. In subsect. 5.4 we remark on some string-theoretic
implications of our work.

2. Contact terms and the Liouville/matrix model dictionary

According to the standard dictionary, the \( m \)th multicritical point of the matrix
model corresponds to the coupling of the Liouville theory to the conformal field
theory \( (2,2m-1) \) \([20–23]\). Moreover, it is generally accepted that we have the
correspondence

\[
\alpha_j \rightarrow \int_X e^{\alpha_j \Phi_{1,m-1-j}}, \quad j = 0, \ldots, m - 2, \tag{2.1}
\]

where \( \alpha_j = \frac{1}{2} \gamma(m - j) \), \( \Phi_{n,n'} \) are the primary fields of the conformal field theory in
the standard parametrization of the Kac table, and \( \sigma_j \) is the matrix model operator
whose correlation functions are described by KdV flow. Eq. (2.1) means that the
Liouville path integral is related to the matrix-model free energy at the \( m \)th
multicritical point by

\[ Z_{m}^{\text{mm}}(t_{j}) = Z_{m}^{L}(\tau_{j}) \], \quad t_{j} = \tau_{j}, \quad (2.2) \]

where “L” stands for “Liouville”, the \( t_{j} \) couple to \( \sigma_{j} \),

\[ Z_{m}^{L}(\tau_{j}) = \int d\varphi e^{-S[\varphi]}\langle e^{S_{\text{int}}(2, 2m-1)} \rangle, \quad (2.3) \]

\( \varphi \) is the Liouville field, \( S[\varphi] \) is the Liouville action, and

\[ S_{\text{int}} = \sum_{j=2}^{m-2} t_{j} \int_{\Sigma} e^{\alpha_{j}/\varphi} \phi_{1, m-1-j} + \ldots \quad (2.4) \]

According to eq. (2.2), if, at the \( m \)th multicritical point we tune all \( t_{j} = 0 \) except for \( t_{m-2} \), the resulting model is expected to be a conformal field theory coupled to gravity. If so, the three-point functions of the theory should vanish in accordance with the fusion rules. In particular this implies that, up to analytic terms in \( \tau_{m-2} = \mu \), the cosmological constant, we have \( \langle \sigma_{j} \rangle = 0 \) for \( j \neq m - 2 \), and \( \langle \sigma_{j} \sigma_{k} \rangle \propto \delta_{jk} \). Explicit computations in the matrix model shows these expectations to be false [24]. Similar paradoxes occur in the computations with the \( q \)-matrix model at its \( p \)th multicritical point if one accepts the dictionary

\[ \sigma_{k}(\varphi) \leftrightarrow \int_{\Sigma} e^{\alpha_{k}/\varphi} \phi_{n, n'}, \quad (2.5) \]

where

\[ \frac{\alpha_{k, r}}{\gamma} = \frac{p + q - (kq + r)}{2q}, \quad k = \left[ \frac{pn'}{q} \right] - n, \quad r = n'p - q \left[ \frac{pn'}{q} \right]. \quad (2.6) \]

In particular, the two-point functions are not diagonal.

It has been conjectured that these paradoxical results are related to contact terms arising when two operators are at coincident points [25] (this is indeed the case in the topological interpretation of these models [4, 26]). Since contact terms play a key role in the following let us pause to recall the basic idea behind them.

In quantum gravity (and string theory) we consider operators like (2.1) and (2.5) which involve integrals over space-time (or the world-sheet). Therefore, to compute correlation functions we must give a definition to the local correlation functions at coincident points: \( \langle \ldots \phi_{1}(z)\phi_{2}(z)\ldots \rangle \). (More generally, we must define correlation functions at the boundaries of moduli space.) Typically such expressions are singular and cannot be obtained as a limit of \( \langle \ldots \phi_{1}(w)\phi_{2}(z)\ldots \rangle \) as \( w \to z \). If we do find prescriptions to define correlation functions at coincident
points, two such prescriptions will, in general, differ by distributions supported at \( w = z \) (more generally, by distributions supported on the boundary of moduli space). These ambiguous \( \delta \)-function-supported changes in the Green functions are known as contact terms. A useful way to think about a change of contact terms is that it is equivalent to an analytic redefinition of coupling constants [27]. Suppose that couplings \( \lambda^i \) multiply operators \( \int \Phi_i \) so that correlation functions are computed according to

\[
\left< \exp \left[ \sum_k \lambda^k \Phi_k \right] \prod_j \int \Phi_j \right> = \prod_j \frac{\partial}{\partial \lambda^j} \left< \exp \left[ \sum_k \lambda^k \Phi_k \right] \right>.
\]

If we make an analytic change of couplings \( \lambda^i \to \mu^i + A^i_{jk} \mu^j \mu^k + \ldots \) and define correlation functions of new operators \( \Psi_j \) by taking derivatives \( \partial / \partial \mu^j \):

\[
\left< \exp \left[ \sum_k \lambda^k \Phi_k \right] \prod_i \int \Psi_j \right> = \prod_j \frac{\partial}{\partial \mu^j} \left< \exp \left[ \sum_k \lambda^k \Phi_k \right] \right>,
\]

then we see that locally the \( \Psi \)-correlators differ from the \( \Phi \)-correlators by delta functions. With a little thought one can see that conversely every modification of the Green functions by additions of delta functions (subject to some criteria of physical reasonableness, e.g., locality) can by summarized by an analytic redefinition of coupling constants*.

In the examples studied in this paper the original frame of operators \( \sigma_n(\Theta_n) \) and couplings \( t_n(\alpha) \), the KdV frame, will be transformed to a new frame of operators \( \hat{\sigma}_n(\Theta_n) \) and couplings \( \tau_{n,\alpha} \), which we refer to as the “conformal field theory frame”. One important property of the KdV frame is that the operators \( \sigma \) are scaling fields. In order to preserve this feature, the change of variables to the CFT frame must be compatible with the dimensions. Together with the analyticity of the transformation the form of the allowed transformations is severely restricted**.

We now consider three examples of analytic operator mixing.

**Example 1: Pure gravity.** As an extreme example of analytic redefinitions of couplings we show that at the \( m = 2 \) fixed point there is, up to analytic redefinition, only one nontrivial coupling. The specific heat for the \( m = 2 \) fixed point is obtained from the string equation

\[
\sum t_i u^i + u^2 = t_0,
\]

* In supersymmetric theories the situation is more intricate [28].
** S. Shenker has pointed out to us that a similar ambiguity in the identification of the scaling field was noticed in Wegner’s original paper in the case where the critical exponents are rational.
and thus has the form

\[ u = t_0^{1/2} \left\{ 1 + \sum_n c_n \prod_{i} \left( t_i t_0^{i/2 - 1} \right)^{n_i} \right\}. \]  (2.10)

Let us split the sum in eq. (2.10) into two terms \( \Sigma' \) and \( \Sigma'' \) defined according to whether \( \sum n_i (i/2 - 1) \) is an integer or a half-integer. It follows that if we define

\[ \tau_i = t_i, \quad i \neq 0, \]
\[ \mu = \tau_0 = t_0 (1 + \Sigma')^2, \]  (2.11)

then the specific heat is of the form

\[ u = \sqrt{\mu} + \text{analytic}. \]  (2.12)

That we can do this is not surprising, although it contrasts sharply with the result of ref. [29] where an infinite set of BRST cohomology classes in the Liouville \( \times \) ghost system was found. We expect infinitely many of the states in ref. [29] to correspond to redundant operators in the Liouville theory. Eq. (2.12) is one way of making that statement precise. This example can probably be considerably generalized: an analytic redefinition of couplings should show that, up to analytic terms in the free energy, the only physical couplings in the \( p, q \) models are those coupling to the matter operators in the Kac table. Moreover, such a redefinition should lead to fusion rules in agreement with conformal field theory*. We will see special cases of this in example 3 and in sect. 4 below.

**Example 2: The boundary operator [30].** Here we briefly summarize ref. [30] as an example of analytic operator mixing. Although we work at genus zero throughout the paper, in this example there is no difficulty in extending our argument to all orders. In the string equation

\[ \sum t_j (j + \frac{1}{2}) R_j = 0, \]  (2.13)

the coupling \( t_{m-1} \) in the \( m \)th multicritical point is redundant, both on the lattice and in the continuum. Using analytic operator redefinitions we may eliminate it using the identity

\[ R_k [u + \rho] = R_k [u] + \sum_{j < k} \alpha_{k,j} \rho^{k-j} R_j [u], \]  (2.14)

* There are two possible directions one might follow to extend this example to other models. First, it could be useful to apply the methods of singularity theory. Second, our change of variables \( t_i \rightarrow \tau_i \) is probably closely related to the Virasoro constraints on the matrix model partition function [3,4]. It would be worth understanding this connection better.
where $\rho$ is a constant. In particular $\alpha_{k,k-1} = -(k - \frac{1}{2})$, so defining

$$\rho = \frac{1}{m - \frac{1}{2}} t_{m-1},$$

we find that the operator coupling to $\rho$ completely decouples beyond the one-point function. That is, computing at fixed $\tau$, we have $(\partial/\partial \rho)^n u = 0$ for $n > 1$. The operator coupling to $\rho$ is the boundary operator, and does nothing but measure the length of macroscopic loops.

**Example 3: The Ising model.** As a third example we show how analytic redefinitions of couplings in the Ising model can resolve some paradoxes. The dimension of the temperature, $t$, is $\frac{1}{3}$ where the dimension of the cosmological constant, $x$, is always taken to be 1. The correlation functions on the sphere are determined by solving [21–23, 31]

$$u^3 + tu^2 = x.$$  

(2.16)

Using our rules above there is a one-parameter family, labeled by $a$, of redefinitions

$$T = t, \quad \mu = x + at^3.$$  

(2.17)

For $a = -\frac{3}{2\pi}$ eq. (2.16) becomes

$$(u + \frac{1}{3}T)^3 - \frac{1}{3}T^2(u + \frac{1}{3}T) = \mu.$$  

(2.18)

The one-point function of the energy operator $\langle \epsilon \rangle = \frac{1}{a\pi^2}$ is analytic in $\mu$ and hence does not correspond to macroscopic surfaces (below we will study such analytic terms in $\mu$ in detail). Moreover, it is clear from eq. (2.18) that $\langle \epsilon^n \rangle = 0$ for $n$ odd precisely as expected from the continuum Liouville approach. Since the magnetic field enters the string equation quadratically, all of the Ising fusion rules (at the level of selection rules) are now satisfied. A similar redefinition in the unitary discrete series coupled to gravity can resolve some paradoxes presented in ref. [32] where the correlation functions in this theory were computed from the KdV [9] point of view. For example, it is easy to check that using analytic (in $\mu$) redefinitions of the operators the two-point function can be made diagonal, as expected from the continuum approach. We suspect (but did not check explicitly) that as in the example of the Ising model, a similar redefinition can make the correlation functions compatible with the fusion rules of ref. [33].
In general, finding the analytic redefinitions of operators with good physical properties is complicated. One of the technical results of this paper is an identification of the matrix model couplings $t_i(\mu)$ corresponding to the Liouville theory coupled to a conformal field theory. That is, the theory defined by $\tau_i = 0$ for $j \neq m - 2, m = 2 = \mu$. This is given in eqs. (4.22) and (4.23) below. Since, from the point of view of string theory the $t_i$ define a background for string propagation we refer to such a choice of couplings as a “conformal background.” We will see in subsect. 4.3 below that the conformal background is most easily derived by examining macroscopic loop amplitudes. In secs. 3 and 4 we discuss macroscopic loop amplitudes, from the point of view of Liouville theory and of the matrix model.

3. Loops in Liouville

3.1. ACTION AND BOUNDARY CONDITIONS

We first discuss Liouville theory on manifolds with boundary along the lines of ref. [18]. The action is

$$S = \int_\Sigma \frac{1}{2\pi} \partial \phi \bar{\partial} \phi$$

$$+ \frac{Q}{8\pi} \left( \int_\Sigma \phi \hat{R} + \phi \phi \hat{k} \, d\hat{s} \right) + \frac{\mu}{8\pi \gamma^2} \int_\Sigma \gamma \phi + \frac{\rho}{4\pi \gamma^2} \phi \gamma / 2 \, d\hat{s},$$

(3.1)

where $\hat{k}$ is the extrinsic curvature of the boundary, $\mu$ and $\rho$ are the volume and boundary cosmological constants. Classically, $Q = 2/\gamma$ where $\gamma$ is the Liouville coupling constant$^*$. 

From $S$ we obtain the usual Liouville equation of motion:

$$\frac{1}{4\pi} \Delta \phi - \frac{\mu}{8\gamma} \gamma \phi + \sum \alpha_i \delta^{(2)}(z - z_i) = 0$$

(3.2)

(we have included sources of curvature for later convenience). We may set the boundary term to zero in the variational principle by choosing Dirichlet boundary conditions $\delta \phi |_{\partial \Sigma} = 0$, or Neumann boundary conditions:

$$\frac{\partial (\gamma \phi)}{\partial n} + \hat{k} + \frac{\rho}{2} \gamma / 2 = 0,$$

(3.3)

where the first term is the normal derivative.

$^*$ We will write the formulas in terms of $Q$ in such a way that the generalization to the quantum case is straightforward.
3.2. CLASSICAL THEORY

At the classical level we should solve the constant negative curvature equation. Recall that in the absence of a boundary a solution exists only when

$$X = \sum_i \alpha_i + Q(2h - 2)/2$$

is positive ($\alpha_i$ are sources of curvature and $h$ is the number of handles). Constraining the area of the surface the equation of motion states that the metric has constant curvature. The nature of the surface and hence the nature of associated quantum states depends crucially on the sign of $X$. The solution has negative curvature for $X > 0$ and positive curvature for $X < 0$. A similar story is true in the presence of boundaries. It is no longer necessary to constrain the area, since, if there is at least one boundary there is always a constant negative curvature metric. Therefore we henceforth restrict attention to the case with a single boundary. The nature of the surface, and hence the nature of the associated quantum states depends crucially on the sign of $Y = X + \frac{1}{2}Q = \sum \alpha_i - \frac{1}{2}QX$. We must consider several cases.

**Case 1**: Fixed $\mu$, $Y > 0$. If $Y > 0$ there exists a classical solution with constant negative curvature as the loop is shrunk to a point. In this case a small loop of length $l$ behaves like a local source of curvature $Q/2$ (this is the origin of this term in the definition of $Y$). In other words, the loop behaves like a puncture, and the surface has the shape shown in fig 1.

**Case 2**: Fixed $\mu$, $Y < 0$. If $Y < 0$ there is no classical solution with constant negative curvature when the loop is replaced by a puncture. As $l$ becomes small the surface looks like fig. 2 and the loop cannot be thought of as a local

![Fig. 1. A large constant negative curvature surface with a small loop. The spikes originate from sources of curvature, e.g. from operator insertions.](image1)

![Fig. 2. A constant negative curvature cone with two loops of decreasing length sliding towards the apex.](image2)
disturbance to the surface. For very small $l$ the cosmological constant in the equation of motion is negligible (of order $\mu l^2$) and therefore the surface is almost flat. We can understand this case better if we constrain the area to be $A$. There are then two relevant domains:

**Case 3**: $Y < 0$, Fixed $A \gg l^2$. If we constrain the area $A$ when $Y < 0$ and examine the case of small $l^2/A$ the classical solution has positive constant curvature and looks like fig 3. In this case the small-$l$ limit is smooth and the loop becomes as a puncture with curvature $Q/2$.

**Case 4**: $Y < 0$, Fixed $A \ll l^2$. When $l^2/A$ is large or of order one the surface has constant negative curvature and the loop cannot be thought of as a local disturbance. We therefore always obtain a surface of constant negative curvature as $A \to 0$.

We end this subsection by illustrating these remarks with some examples of constant curvature metrics. Up to coordinate transformation the solution of (3.2) with a source of curvature of strength $1 - a$ is

$$g = e^{\gamma} = \frac{16a^2}{\mu} \frac{\xi}{(1 - \xi(z\bar{z})^a)^2(z\bar{z})^{1-a}},$$

where

$$\xi = \frac{\mu l^2}{\mu l^2 + 64\pi^2a^2}.$$

As $l \to 0$ the change of variables to the standard constant curvature punctured disk goes like $w \sim \xi^{1/2}z$, thus illustrating fig. 2. Similarly, up to coordinate transformation, the solution with fixed area and boundary length is

$$g = e^{\gamma} = \frac{y}{(1 - x|z|^2)^2},$$

where $y = (2A/l)^2$ and $x = 1 - 4\pi A/l^2$. As an illustration of case 3 note that as $l \to 0$, $x$ changes sign, giving a positive curvature metric. Changing variables to
$w = (\sqrt{A}/l)z$ we see that the surface approaches the Riemann sphere as the boundary $|z| = 1$ approaches the pole at $w = \infty$.

### 3.3. Quantum Theory

Recall that, at the quantum level the expression for $Q$ is renormalized $Q = 2/\gamma + \gamma$. Every matter operator is "dressed" by $e^{a_0 \phi}$ and the coefficients $a_i$ lead to curvature sources as in the classical theory. The probability measure for a surface of area $A$, with $h$ handles and insertions of operators is $Z(A) dA = (dA/A)^{A/X}Z(1) [12, 15]$. Therefore, in the partition function at fixed $\mu$,

$$Z(\mu) = \int_0^\infty e^{-\mu A} Z(A) dA$$  \hspace{1cm} (3.7)

for $X > 0$ the large-$A$ part of the integral dominates and the typical surface has negative curvature. The fixed cosmological constant amplitude behaves like $\mu^{-X/\gamma}$. Thus, when negative curvature surfaces dominate the path integral the result is convergent and gives a negative, nonanalytic power of $\mu$. For $X < 0$ the typical surface has constant positive curvature and the integral over the area diverges at small $A$. Regularizing this integral, the dominant configurations have very small $A$. Their contribution is analytic in $\mu$ and not universal. The large-$A$ surfaces contribute $\mu^{-X/\gamma}$ (times log $\mu$ when $-X$ is an integer). We now generalize this to the case with boundaries.

First, we need the quantum analog of $Y$. Consider an amplitude with one loop of size $l$. The small-$l$ behavior of $Z(\mu, l)$ is controlled by $Y = X + \alpha_m$, where $\alpha_m$ is the curvature associated with the lowest dimension operator $\phi_m$ which can couple to the loop (typically the lowest dimension operator in the theory). The term $Q/2$ in the classical theory is replaced by $\alpha_m$ because this is the maximum curvature that can be localized in a point in the quantum theory. Since there is always a constant negative curvature metric on such a surface, there are no small area divergences in the definition of $Z(\mu, l)$. Thus, the amplitude is well defined and universal. As in the classical analysis we consider several different cases.

#### Case 1: Fixed $\mu$, $Y > 0$.

When $Y > 0$ the small-$l$ limit of the typical surface is smooth. Replacing the loop by the leading local operator $\phi_m$ we find there is still a classical solution and hence there are no divergences in the $l \to 0$ behavior of the path integral. By scaling we then find that the amplitude behaves like $l^{(Q-2\alpha_m)/\gamma} \mu^{-Y/\gamma}$ (we will derive this explicit formula below). Notice that the amplitude is not analytic in $\mu$. Higher-order corrections in the small-$l$ expansion correspond to replacing the loop by other local operators.

If there is more than one loop, the situation is similar. In this case, shrinking all the loops but one is smooth and the shrunk loops can be replaced by local operators. The reason is that there is a classical solution with constant negative curvature.
curvature whenever there is at least one macroscopic loop. Thus, in these cases we can safely replace a loop by a local operator expansion.

**Case 2**: Fixed $\gamma < 0$. When there is only one macroscopic loop and $\gamma < 0$ there is no classical solution without the loop*. Therefore, $Z(\mu, l)$ diverges for small $l$. The typical surface is small and looks like fig. 2. Note that since the amplitude is finite at nonzero $l$, the loop length plays the role of an ultraviolet cutoff on the theory. Indeed, by the shift $\varphi \rightarrow \varphi + (2/\gamma)\log l$, we may scale $l$ out of the Liouville path integral except for the cosmological constant term

$$\int \mu e^{\gamma \varphi} \rightarrow \int \mu l^2 e^{\gamma \varphi}.$$  \hspace{1cm} (3.8)

In Liouville theory it does not make sense to expand in the cosmological constant: there is no sense in which the interaction is small. However, in this case, having used our freedom to shift the zero mode of $\varphi$ we see that an expansion in $\mu l^2$ around free field theory could make sense, and should give the leading contributions in a small-$l$ expansion. In particular, by scaling the leading term is $l(Q + 2X/\gamma)$ and the negative powers of $l$ multiply analytic terms in $\mu$. The expansion in $\mu l^2$ is sensible at least for the first few orders.

Surfaces of the type shown in both fig. 3 and fig. 2 contribute to

$$Z(\mu, l) = \int_0^\infty dA Z(A, l)e^{-\mu A}$$  \hspace{1cm} (3.9)

in the large- and small-$A$ region of integration, respectively. We can understand the nature of these contributions by analyzing each case as follows.

**Case 3**: $\gamma < 0$, Fixed $A > l^2$. If we constrain the area to be large relative to $l^2$ the typical surface looks like fig. 3. In this case the presence of the loop is a small perturbation on the geometry of the surface so we expect that the loop can be replaced by a local operator expansion. As opposed to case 1 the large surface has constant positive curvature because, replacing the loop by the leading local operator $\mathcal{O}_m$, the resulting parameter $X$ is negative. By scaling one finds that the contribution of such surfaces to (3.7) is $l(Q - 2\alpha_m/\gamma)A^{Y/\gamma - 1}$, leading to nonanalytic terms in $\mu$**.

**Case 4**: $\gamma < 0$, Fixed $A \ll l^2$. For small $A$ the loop is not a small perturbation on the geometry of the surface and we cannot expect to replace it by a sum of local operators. Nevertheless, for nonzero loop length there is always a classical solution so the path integral $Z(\mu, l)$ discussed in case 2 is finite and well defined. In particular, in the expression (3.7) the integral over $A$ converges at small $A$. As

* Note, in particular, that this occurs in the situation with one operator inserted on a disk, a situation we study in great detail below.

** If $-\gamma/\gamma \in \mathbb{Z}$ then the amplitude has a factor of $\log \mu$. 
discussed above, the short-distance cutoff is provided by the length \( l \) and the small-\( A \) part gives analytic terms in \( \mu \). The moral of the story is that even the analytic terms in \( \mu \) can be universal since the loop length provides a physical ultraviolet cutoff on the theory*.

We end this subsection by illustrating these remarks with some examples of the behavior of partition functions on the disk. Using the solutions (3.6), (3.5) one can calculate the action (3.1) and verify the behavior we have predicted for \( Z(A,l) \) in the leading semiclassical approximation on the disk. In this approximation one finds that \( Z(A,l) \) has the form \( A^x l^y e^{-l^2/A} \). From the semiclassical expansion in Liouville theory we would expect that the exact disk amplitude with fixed area \( A \) and perimeter \( l \) has the same general form \( Z(A,l) = A^x l^y e^{-l^2/A} \), the exponents \( x, y \) merely being expressed as an expansion in \( 1/c \). Accepting this, we can derive the exact values of \( x, y \) as follows. For simplicity assume the lowest dimension operator which couples to the loop is the unit operator. Applying the scaling arguments used to derive the KPZ formulae \([12,15]\) we find that \( Z(A,l) = A^{-3/2-Q/2} f(l / \sqrt{A}) \). Now, as we have discussed, the limit as \( l \to 0 \) should give the disk with an insertion of the cosmological constant, thus the leading term as \( l \to 0 \) must have the form \( l^x A Z(A) \), where \( Z \) is the partition function for a closed surface. Again by ref. \([15]\) the \( A \)-dependence of \( Z(A) \) is known so we find

\[
Z(A,l) = l^{-3+Q/2} A^{-Q/2} e^{-l^2/A}.
\]

We will verify from matrix model calculations below that (3.10) is indeed exact.

One easily checks that (3.10) is compatible with the previous discussion. At large \( A \) the exponential is negligible, and the amplitude behaves according to the scaling \( A^x l^y \) predicted by the insertion of a local operator. As \( A \) shrinks to zero we pass from fig. 3 to fig. 2. The exponential in eq. (3.10) provides an explicit ultraviolet cutoff, which is present as long as \( l \neq 0 \), and cures the small-\( A \) divergences in the Laplace transform to fixed cosmological constant.

3.4. Liouville wave functions

Understanding the functional integral over a manifold with a boundary is an important step towards a construction of the space of states of a theory. As is standard in quantum field theory, and often used in conformal and topological field theories, the functional integral with Dirichlet boundary conditions on the fields defines a vector in a state space associated to a boundary. This makes the

* This point of view resolves a paradox about topological field theory. Since this is somewhat outside our main line of development we explain this in appendix A. Another application of this remark is that in the Ising model, the one-point function of the energy operator is computable and nonzero. The boundary, which provides the UV cutoff leading to a universal analytic term breaks chirality and leads to a nonzero answer.
factorization and gluing properties of amplitudes obvious. We would like to do something similar in the case of gravity. Here, our wave functionals, $Z[\phi(\sigma), M(\sigma), c(\sigma)]$ depend on the Liouville mode, the matter fields and the ghosts on the boundary.

By general covariance the wave functionals $Z$ of canonical quantization satisfy some constraints. They should be in the BRST cohomology of the left- and right-moving BRST operators $Q$ and $\bar{Q}$, i.e. they should satisfy the momentum constraints associated with space diffeomorphisms and the Hamiltonian constraint associated with time reparametrizations. The latter is known as the WdW equation.

As stressed in ref. [1], the momentum constraints should be implemented as ordinary constraints since their Lagrange multiplier is integrated between $-\infty$ and $+\infty$. This is not the case for the Hamiltonian constraint. The range of integration of its Lagrange multiplier is bounded so $Z$ does not necessarily satisfy this constraint. This phenomenon can equivalently be understood as a contact term arising when an operator, a handle or another loop touches the boundary*. For example, the disk amplitude with one insertion of an operator $\phi$ (the wave function of this operator) should satisfy the WdW equation but the disk with two insertions should not. More generally, whenever our amplitude involves integration over moduli, the region of integration is such that the WdW equation is not implemented on the boundaries. This issue of the range of integration over the moduli makes the factorization and gluing properties of quantum gravity more subtle than in a theory which does not have integrated moduli. In the context of the critical string this phenomenon has been discussed, for example, in ref. [34].

In the matrix model, we do not know how to study $Z$ as a functional of all its arguments. First, the role and the origin of the matter is mysterious. In the one-matrix model at its third critical point the matter can be identified as in ref. [20]. In the p-matrix model we can think of the matter as being the label of the matrix. However, in general we cannot specify $M(\sigma)$ on the boundary. The ghosts are even more elusive and are not easy to identify in a completely gauge invariant description like the matrix model. Furthermore, we do not know how to specify $\phi(\sigma)$. The only observables we know how to compute are macroscopic loops. They are related to $Z$ through

$$\langle \omega(1, \ldots) \rangle = Z(1, \ldots) = \int [D\phi DMDc] \delta \left( e^{\gamma \phi/2} - I \right) f(M, c, \phi) Z[\phi(\sigma), M(\sigma), c(\sigma)].$$

(3.11)

where $f(M, c, \phi)$ is some "wave function" for matter, ghosts and Liouville, $I$ is the

* We thank T. Banks and S. Shenker for stressing this interpretation of the violation of the WdW equation.
length of the boundary, and the ellipsis stands for an unspecified insertion of operators and/or loops. (In the $p$-matrix model we have $p$ different loops and correspondingly $p$ $f$'s.)

This wave function $Z(I, \ldots)$ is similar to the minisuperspace wave function discussed in the quantum gravity literature. By general covariance physical wave functions are functions on superspace, which in pure gravity is the set of spatial metrics modulo spatial diffeomorphisms. In the minisuperspace approximation one simplifies the problem by reducing further to a one-dimensional quotient of superspace defined by the spatial volume. In pure gravity in two dimensions superspace and minisuperspace are identical since the length completely specifies the spatial geometry up to spatial diffeomorphisms. Given this equivalence, one might expect the minisuperspace approximation to be exact. In particular, from our remarks regarding integration over moduli, one might hope that the amplitude for a disk with a single operator $\mathcal{O}$ inserted, i.e. the wave function associated to the operator $\mathcal{O}$, $Z(I, \mathcal{O}) = \psi_{\mathcal{O}}(I)$, will satisfy the minisuperspace WdW equation.

We have not proven from the path integral that minisuperspace is exact. Nevertheless we can already present some evidence in favor of exactness. In the conformal backgrounds of sect. 2 the wave functions factorize between matter and gravitational degrees of freedom:

$$
\psi_{\mathcal{O}} = \psi_{\text{matter}} \otimes \psi_{\text{gravity}}.
$$

Therefore the (minisuperspace) WdW equation simplifies drastically and becomes Bessel's differential equation

$$
\left[ -\left( t \frac{\partial}{\partial t} \right)^2 + 4\mu t^2 + \nu^2 \right] \psi_{\text{gravity}} = 0,
$$

where $\nu^2$ is related to the undressed matter conformal dimension $\Delta^0(\mathcal{O})$ by

$$
\nu^2 = \frac{8}{\gamma^2} \left[ \frac{Q^2}{8} - (1 - \Delta^0(\mathcal{O})) \right] = \frac{4}{\gamma^2} \left( \alpha - \frac{Q}{2} \right)^2
$$

and $\alpha$ is the Liouville charge associated with the dressing. In sect. 2 we argued that the formula (3.10) for $Z(A, I)$ is exact. Given that, one can take the Laplace transform, using eq. (B.7) of appendix B* to obtain a Bessel function,

$$
Z(\mu, I) = t^{-2}(\sqrt{\mu})^{Q/\gamma - 1} K_{Q/\gamma - 1}(2\sqrt{\mu} I).
$$

From eq. (3.15) we can obtain the wave functions of the cosmological constant and

* Several pertinent facts about Bessel functions are gathered in appendix B.
the boundary operator, and we can verify that both of these indeed satisfy (3.13). In sect. 4 we will prove that the minisuperspace approximation is exact using the calculational techniques of the matrix model.

4. Loops in matrix models

4.1. n-LOOP AMPLITUDES AT GENUS ZERO

In this section we compute wave functions and loop amplitudes from the matrix model formalism elaborated in refs. [8, 35].

As is well known, in the representation of matrix model integrals as sums over surfaces, the operator \( w^{mm}(l) = \frac{1}{l} \text{tr} \langle \phi^{l/a} \rangle \) creates a hole of boundary length \( l \) in units of the lattice spacing \( a \). The extra factor of \( 1/l \) is a symmetry factor needed to take into account the fact that we are working with unmarked loops. In the double-scaling limit, \( a \to 0, N \to \infty \), correlation functions of this operator may be expressed in terms of integrals of the kernel

\[
\langle x | e^{-Q}| y \rangle, 
\]

where \( Q = -\kappa^2 d^2/dx^2 + u(x, \kappa) \) is the Schrödinger operator associated to the model, \( \kappa \) is the topological coupling, and \( u(x, \kappa) = x^{1/m} + \sigma(\kappa^2/x^2) \). In the double-scaling limit, the amplitude for one macroscopic loop, to be compared with the Liouville quantity \( Z(\mu, l) \) discussed in sect. 3 is given by [8, 35]

\[
Z^{mm}(l; t_j) = \frac{\sqrt{4\pi}}{l} \int_{t_0}^{\infty} dx \langle x | e^{-Q}| x \rangle. 
\]

In the following we will use the notation \( w(l) = lw^{mm}(l) \).

We now compute \( n \)-loop amplitudes at genus zero. We begin with one loop. Using the Campbell–Baker–Hausdorff formula, and inserting a complete set of eigenfunctions

\[
|p, \mu \rangle \equiv \frac{1}{\sqrt{2\pi \kappa}} e^{ilp x/\kappa}, 
\]

we obtain the genus-zero answer

\[
\langle w(l) \rangle = \frac{1}{\kappa l^{1/2}} \int_{-i_0}^{\infty} dx \ e^{-ln(x; t_i)} 
= \frac{1}{\kappa l^{1/2}} \int_{i_0}^{\infty} dy \left( \sum_{j=1}^{\infty} j t_j y^{-j-1} \right) e^{-iy} 
= \frac{1}{\kappa l} \sum_{j=1}^{\infty} j t_j \mu^{-1/2} \psi_{j-1}(\tilde{l}), 
\]
where
\[ \psi_j(x) \equiv j! x^{-j-1/2}(1 + x + x^2/2! + \ldots + x^j/j!)e^{-x}, \quad (4.5) \]

\[ l = u_l, \] and we have used the string equation
\[ \sum_{j \geq 0} t_j u^j = 0. \quad (4.6) \]

Surprisingly, the genus-zero two-loop formula is much simpler. The expression to all orders is [35]
\[ \langle w(l_1) w(l_2) \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{-l_0} dy \langle x|e^{-l_1 Q}|y\rangle \langle y|e^{-l_2 Q}|x\rangle, \quad (4.7) \]

so the genus-zero approximation is
\[ \langle w(l_1) w(l_2) \rangle_{\kappa=0} = \frac{e^{-l_1 + l_2}}{\sqrt{l_1 l_2 \kappa^2}} \int_{-l_0}^{\infty} dx \int_{-\infty}^{-l_0} dy \exp \left[ -\left( x - y \right)^2/(4\kappa^2 l_1 l_2) \right] \]
\[ = 2\sqrt{l_1 l_2} \frac{e^{-\alpha(l_1 + l_2)}}{l_1 + l_2}. \quad (4.8) \]

Note that the small-\( l \) behavior of these amplitudes is in accord with the predictions of the Liouville theory discussed in sect. 3. In particular, the two-loop amplitude is smooth as \( l \rightarrow 0 \), while the one-loop amplitude is divergent as \( l \rightarrow 0 \). Indeed, using the string equation we have
\[ \langle w(l) \rangle = \frac{1}{\kappa} \sum_{j \geq 0} j! t_j l^{-j-1/2} + O(l^{1/2}). \quad (4.9) \]

(Henceforth we will set \( \kappa = 1 \).) In accordance with the previous discussion we interpret the nonanalytic terms in \( t_i \) in the small-\( l \) expansion as arising from the insertions of local operators in the theory. It is in this sense that we may use the formula
\[ "w(l) = \sum_{n=0}^{\infty} (-1)^n + 1 l^{n+1/2} n!^{-1} \sigma_n". \quad (4.10) \]

The operators* \( \sigma_n \) are the standard operators of two-dimensional gravity, in

* Note that \( \sigma_n \) corresponds to \( \partial/\partial t_n \), with \( t_n \) being defined in eq. (4.6). Our normalization therefore differs from the “Gelfand–Dikii” normalization.
particular their correlation functions are computed from KdV flow [35]. The expansion (4.10) is correct in correlation functions if we ignore divergent terms in \( l \). However, as noted before, if there is at least one other hole on the surface then there is no divergence as \( l \to 0 \) and then (4.10) is exact. This simple observation allows us to compute a formula for the genus-zero \( n \)-macroscopic loop amplitude. Consider, for example the case \( n = 3 \), which we can write as

\[
\langle w(l_1)w(l_2)w(l_3) \rangle = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{l_1^{n+1/2}}{n!} \langle \sigma_n w(l_2)w(l_3) \rangle
\]

\[
= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{l_1^{n+1/2}}{n!} \frac{\partial}{\partial t_n} \langle w(l_2)w(l_3) \rangle
\]

\[
= -2 \sqrt{l_2l_3} \ e^{-tl_2l_3} \sum_{n=0}^{\infty} (-1)^{n-1} \frac{l_1^{n+1/2}}{n!} \frac{\partial u}{\partial t_n}
\]

\[
= 2 \frac{\partial u}{\partial t_0} \sqrt{l_2l_3} \ e^{-ul_1l_2l_3},
\]

where in the last line we may obtain the \( l \)-dependence immediately since the amplitude must be totally symmetric in \( l_1, l_2, l_3 \). The same argument may be applied recursively to obtain the \( n \)-loop formula. The argument giving the last line of (4.11) implies that the \( n \)-loop amplitude is, up to a factor of \( \prod l_i^{1/2} \) a function only of the sum of the loop lengths \( \Sigma \equiv \sum l_i \). Thus we can obtain the general formula immediately by specializing to the case where all but one loop is microscopic*. In this case the amplitude is also proportional to \( \langle \sigma_0 w(l) \rangle \). Hence, defining \( \tilde{w}(l) = w(l) / \sqrt{l} \) we obtain the final result for the genus-zero \( n \)-loop formula:

\[
\left\langle \prod_{i=1}^{n} \tilde{w}(l_i) \right\rangle = \left( \frac{\partial}{\partial t_0} \right)^{n-1} \left( \langle \tilde{w}(\Sigma) \rangle \right).
\]

It is remarkable that this amplitude only depends on the sum of the loop lengths. This fact deserves to be understood better.

4.2. WAVE FUNCTIONS FROM THE MATRIX MODEL

By shrinking one of the loops we obtain the amplitude for one macroscopic loop and one local operator, \( \langle \sigma w(l) \rangle \). In subsect. 3.4 above we identified a similar quantity \( Z(l, \theta) \) in the Liouville theory and argued that it should be thought of as

* A more formal proof of (4.12) proceeds by using KdV flow and induction in \( n \).
the wave function associated to a microscopic operator. Since wave functions are half-densities, what we call the wave function is ambiguous up to a function of \( l \) until we specify the measure. The proper measure for the wave function of the operator \( \sigma \), which is a half-density \( (d\phi)^{1/2} \), is obtained by computing [18]

\[
\psi_{\sigma}(l) = \langle \sigma | w_{mm}(l) \rangle = \langle \sigma | w(l) \rangle .
\]  

By expanding (4.8) we obtain the wave functions in terms of the truncated exponential functions (4.5):

\[
Z_{j}^{nm}(l) = \langle \sigma j w(l) \rangle = u^{j+1/2} \psi_j(lu)
\]

\[= j! l^{-j-1/2} \text{ regular.} \quad (4.14)\]

Using the matrix model, it was shown in ref. [18] that the leading behavior of the wave function of \( \sigma_j \) is \( l^{-j-1/2} \). This was derived by examining the large-\( n \) limit of the coefficients \( a_n(j) \) in \( \sigma_j = \text{tr}(1 - M)^{j+1/2} = \sum_n a_n(j) n^{-1} \text{tr} M^n \), which can be interpreted as the wave function of \( \sigma_j \).

4.3. CONFORMAL BACKGROUNDS

The wave functions \( Z_{j}^{nm} \) of sect. 3 do not satisfy simple linear differential equations. As we emphasized in sect. 1, the KdV basis of operators is not necessarily the best basis for comparing Liouville theory with the matrix model. In particular, we mentioned that the dictionary (2.1) leads to paradoxical results for one-, two-, and three-point functions. The origin of these problems is two-fold. First, we have not correctly identified the matrix model operators \( \hat{\sigma}_j \) coupling to \( \tau_j \). Furthermore, we have not even correctly identified the matrix-model background corresponding to gravity coupled to a conformal field theory.

We will find the correct conformal background using the Wheeler–DeWitt equation as a guide. Recall from subsect. 3.4 that in a conformal background the wave function factorizes \( \psi_\sigma = \psi_{\text{matter}} \otimes \psi_{\text{gravity}}^\text{gravity} \) and the minisuperspace WdW equation reduces to the Bessel equation (3.13). The Bessel equation admits a two-dimensional space of solutions spanned by the modified Bessel functions \( I_{\pm}(2\sqrt{\mu} \ l) \), where

\[
I_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \nu + 1)} \left( \frac{z}{2} \right)^{2k} .
\]

Since \( \psi_{\text{gravity}}^\text{gravity} \) must decay in the infrared \( (l \to \infty) \) we learn that in conformal backgrounds the wave functions of scaling operators are expressed in terms of the
modified Bessel function $K_{\nu}(x)$:

$$K_{\nu} = \frac{\pi}{2 \sin \nu \pi} \left[ I_{-\nu}(z) - I_{\nu}(z) \right],$$  \hspace{1cm} (4.16)

where $\nu \notin \mathbb{Z}$. In sect. 2 we saw that the conformal background is defined in terms of the couplings $\tau_j$ of (2.4) by $\tau_j = 0$ for $i \neq m - 2$ and $\tau_{m - 2} = \mu$. In such a background the insertion of $\hat{\sigma}_{m - 2}$ in correlation functions is the derivative with respect to the physical cosmological constant:

$$\frac{\partial}{\partial \mu} \langle \mathcal{O} \ldots \rangle = \langle \hat{\sigma}_{m - 2} \mathcal{O} \ldots \rangle. \hspace{1cm} (4.17)$$

Applying this relation to the wave function of the area operator in the background with $\tau_0 = \ldots = \tau_{m - 3} = 0$ we expect

$$\langle \hat{\sigma}_{m - 2} w(l) \rangle = \frac{\partial}{\partial \mu} \langle w(l) \rangle = \left( \sqrt{\mu} \right)^{m - 3/2} K_{m - 3/2} \left( \sqrt{\mu} l \right). \hspace{1cm} (4.18)$$

It is possible to integrate (4.18) with respect to $\mu$ with the result that

$$\langle w(l) \rangle = l^{-1} \left( \sqrt{\mu} \right)^{m - 1/2} K_{m - 1/2} \left( \sqrt{\mu} l \right). \hspace{1cm} (4.19)$$

We can now express the conformal background in terms of the KdV coordinates $t_j$, $j = 0, \ldots, m - 2$, at least to first order in $\tau_0, \ldots, \tau_{m - 3}$. The most general analytic change of variables has the form

$$t_i = \tau_i + \sum_n c_n^{(i)} \prod \tau_j^{n_j}, \hspace{1cm} (4.20)$$

where, by dimensional analysis,

$$c_n^{(i)} \neq 0 \iff 1 - i/2 = \sum n_j (1 - j/2). \hspace{1cm} (4.21)$$

In particular, for $m$ even, the most general analytic change of variables, to first order in $\tau_0, \ldots, \tau_{m - 3}$ has the form

$$t_{2j} = \tau_{2j} + c^{(2j)} \mu^{m/2 - j} \quad j = 0, \ldots, \frac{1}{2} m - 1,$$

$$t_{2j+1} = \tau_{2j+1} \quad j = 0, \ldots, \frac{1}{2} m - 1, \hspace{1cm} (4.22)$$

for $m$ odd we modify $t_j$ for $j$ odd by powers of $\mu$. Combining eqs. (4.9), (4.19) and
one may determine the $c^i$ from the first $m$ singular powers of $l$. The result is

$$c^{(m-2p)} = \frac{(-1)^{m+1} \pi}{\sqrt{8}} \frac{2^{m-2p}}{(m-2p)!p!\Gamma(p-m+\frac{3}{2})}.$$  

This completes the determination of the conformal background. Note that we have not yet shown that (4.18) and (4.19) are satisfied. We only arranged the $t$'s such that the singular part of these equations is correct. However, using the explicit expressions (4.4) and (4.5) it is possible to show that the regular part is also correct.

In a conformal background we can also use the WdW equation to identify unambiguously the conformal scaling operators $\hat{\sigma}_j$ since these should diagonalize the action of the WdW operator on the space of wave functions. We expect

$$\langle \hat{\sigma}_j(u) w(l) \rangle = u^{j+1/2} K_{j+1/2}(ul),$$

where $u = 2\sqrt{\mu}$. The transformation coefficients used in passing from $\hat{\sigma}$ to $\sigma$ are most easily found by comparing the singular terms in the wave function for $l \to 0$. Using eq. (4.15) one can compare (4.14) with (4.24) to obtain

$$\hat{\sigma}_j = (-1)^j \frac{\pi}{2} 2^{j+1/2} \sum_{s=0}^{[j/2]} \frac{u^{2s}}{s!(j-2s)!\Gamma(s+\frac{1}{2}-j)} \sigma_{j-2s}.$$  

Note that the transformation is analytic in $\mu^*$. It is a nontrivial fact that once we have fixed the singular powers of $l$ to get (4.25) the remaining powers of $l$ work out to give (4.24). One can prove this by explicit computation of the coefficients of $l^{u+1/2}$. Another proof is given at the end of subsect. 4.4.

### 4.4. Correlation Functions in Conformal Backgrounds

As a check on our identification of the conformal background we now compute the one- and two-point functions of the operators $\hat{\sigma}_j$. These calculations finally resolve the paradoxes pointed out in ref. [24] and discussed in sect. 2.

We begin by noting that the wave functions $\psi_{\mu}(l)$ have a small-$l$ behavior which agrees nicely with the general predictions of sect. 3. Using the defining relation (4.24) and (4.16) we split the wave function into a sum of two terms. From the expansion (4.15) we see that the divergent terms for $l \to 0$ come from $l_{-j-1/2}$. Since $u^2 = 4\mu$ these are analytic in the couplings, as expected. The nonanalytic terms come from the expansion of $I_{j+1/2}$ and, in a small-$l$ expansion, the first nonanalytic term in $\mu$ enters at order $l^{j+1/2}$.

*In a conformal background $\hat{\sigma}_{m-1} = \sigma_B$ is the boundary operator [30], which satisfies $\langle \sigma_B w(l) \rangle = l \langle w(l) \rangle$. 
One-point functions are most efficiently computed by using the operator expansion of \( w(l) \) to compute the nonanalytic terms in \( \mu \) in a small-\( l \) expansion of \( \langle w(l) \rangle \). To obtain the expansion of \( w(l) \) in terms of \( \hat{\sigma} \), we first invert the expansion (4.25) to express \( \sigma \) in terms of \( \hat{\sigma} \) as

\[
\sigma = 2 \frac{j!}{[\nu/2]} \sum_{s=-1}^{1} \frac{(2j+1-4s)}{2^{j+1/2} \Gamma(j-s+\frac{3}{2})} u^2 \hat{\sigma}_{j-2s}.
\]  (4.26)

Substituting eq. (4.26) into eq. (4.10) yields the expansion

\[
\langle w(l) \rangle = 2 \sum_{j=0}^{\infty} \hat{\sigma}_j (-1)^j (2j + 1)^{l^{1/2} + l/2} \left( \frac{l_{j+1/2}(\mu l)}{(\mu l)^{j+1/2}} \right).
\]  (4.27)

Notice that we expand in terms of \( I_{j+1/2} \) and not \( K_{j+1/2} \). This is in accord with the results of sect. 3 where we learned that only the nonanalytic terms in \( \mu \) are related to an operator expansion.

The one-point functions of \( \hat{\sigma}_{j+1/2} \) are now easily computed by combining eq. (4.27) with (4.19). Using properties (B.6) of the Bessel function we may rewrite eq. (4.19) as

\[
\langle w(l) \rangle = \frac{u^{m+1/2}}{2m-1} \left( K_{m+1/2}(2\sqrt{\mu}l) - K_{m-3/2}(2\sqrt{\mu}l) \right).
\]  (4.28)

It follows from eqs. (4.27) and (4.28) that

\[
\langle \hat{\sigma}_j \rangle = 0, \quad j \neq m, m - 2,
\]

\[
\langle \hat{\sigma}_m \rangle = \frac{2\pi}{(2m-1)(2m+1)} u^{2m+1},
\]

\[
\langle \hat{\sigma}_{m-2} \rangle = \frac{-2\pi}{(2m-1)(2m-3)} u^{2m-1}.
\]  (4.29)

These equations are understood to hold up to terms analytic in \( \mu \). We thus may write our matrix model/Liouville dictionary as follows:

\[
\hat{\sigma}_j \leftrightarrow \int_\Sigma e^{\gamma \phi_1} \Phi_{1,m-1-j}, \quad j = 0, \ldots, m - 2
\]

\[
\hat{\sigma}_m \leftrightarrow \oint_{\Sigma} e^{\gamma \phi/2}
\]

\[
\mathcal{E}_m = m + \frac{1}{2} - \frac{1}{4} \hat{\sigma}_m - \mu \hat{\sigma}_{m-2} \leftrightarrow -\partial \bar{\phi} + \frac{\mu}{8\gamma} e^{\gamma \phi} + \frac{1}{4\pi} \hat{R}.
\]  (4.30)
In the last line we have written the matrix-model version of the Liouville equation of motion. The linear combination is determined by demanding that the one-point function of the equation of motion is zero. As a nontrivial test, one can then check that on the sphere \( \langle \mathcal{E}_m(\hat{\sigma}_{m-2}) \rangle = n \langle (\hat{\sigma}_{m-2})^n \rangle \). Furthermore, we can consider an insertion of the equation of motion \( \mathcal{E}_m \) on the disk. From the above formulae we find

\[
\langle \mathcal{E} w(l) \rangle = l \frac{d}{dl} \langle w(l) \rangle ,
\]

(4.31)
as expected from the surface term in the equation of motion.

The two-point functions are also easily computed. Using the expansion (4.27) in (4.24) we learn that

\[
\langle \hat{\sigma}_j \hat{\sigma}_k \rangle = \delta_{j,k} \left( -\frac{\pi}{4} \right) \frac{u^{2j+1}}{2j+1},
\]

(4.32)

up to analytic terms in \( \mu \).

In this subsection we have used analytic redefinitions of couplings, of the type discussed in sect. 2. However, as emphasized in sect. 2, redefinitions involving only the relevant couplings are only a small part of the story. We expect that many properties of the macroscopic loop amplitudes discussed below will become much more transparent when the correct notion of analytic redefinitions of loop couplings (and loop contact terms) has been found.

We conclude this subsection by noting a peculiar fact about the above redefinitions, namely, that one can carry out many of the above manipulations in an arbitrary background \( t_i \). To see this, define the operator

\[
\mathcal{H}_j \equiv \left[ -\left( l \frac{\partial}{\partial l} \right)^2 + u^2 l^2 + \left( j + \frac{1}{2} \right)^2 \right].
\]

A short calculation reveals that

\[
\mathcal{H}_j Z_{jm} = u^2 j(j - 1) Z_{jm},
\]

(4.33)

and hence

\[
\hat{Z}_j(l; u) = Z_j + \sum_{k=1}^{\left[j/2\right]} (-1)^k u^{2k} \binom{j}{2k} \binom{2j}{2k}^{-1} Z_{j-2k}
\]

(4.34)
satisfies eq. (3.13) with \( \mu \to u^2/4 \). In the \( m \)th multicritical theory the space of relevant couplings is \((m - 1)\)-dimensional. For a general point in that space the specific heat cannot be simply related to the cosmological constant \( t_{m-2} \). However,
there is a submanifold in which $u^2 = t_{m-2} + \text{analytic}$. One can check that in this case the coefficients agree with those found in passing from $\sigma$ to $\hat{\sigma}$ in (4.25). Using the properties of Bessel functions one can check that one- and two-point functions vanish modulo $u^2$. Of course, away from a conformal background $u^2$ is not necessarily analytic in $\mu$. Moreover, we can only interpret eq. (3.13) as the Wheeler–DeWitt equation if the background is such that in the continuum theory only the cosmological constant is turned on. Thus the significance of these last observations is unclear.

4.5. OPERATOR FORMALISM

In subsect. 4.4 we have analyzed the operator expansion (4.27) of $w(l)$ in some detail. This allows us to develop an "operator formalism" for constructing correlation functions of the local operators $\hat{\sigma}$. To do this we use the expansion (4.27), subject to the rules of section three, to compute macroscopic loop amplitudes.

We begin with the two-loop amplitude $G(l_1, l_2) \equiv \langle w(l_1)w(l_2) \rangle$. For $l_1 < l_2$ we can replace $w(l_1)$ by a sum of local operators as in (4.27) because one loop remains. Substituting this and using (4.24), we find

$$G(l_1, l_2) = \sum_{j=0}^{\infty} (-1)^j (2j + 1) u^{-(j+1/2)} I_{j+1/2}(l_1\mu) \langle \hat{\sigma} w(l_2) \rangle$$

$$= \sum_{j=0}^{\infty} (-1)^j (2j + 1) I_{j+1/2}(l_1\mu) K_{j+1/2}(l_2\mu).$$

The fact that this infinite sum of Bessel functions is equivalent to (4.8) follows from the Gegenbauer addition formula (see eq. (B.13) of appendix B) which in our case reads

$$\sum_{j=0}^{\infty} (-1)^j (2j + 1) I_{j+1/2}(l_1\mu) K_{j+1/2}(l_2\mu) = \frac{e^{-u(l_1+l_2)}}{l_1^{1/2} l_2^{1/2}}. \quad (4.36)$$

Proceeding to amplitudes for $n \geq 3$ loops we first note that from eq. (4.12) it follows that as all $l_i \to 0$ there is no singularity, so we can freely use eq. (4.27), and consequently the $(n \geq 3)$-point functions can be written

$$\langle w(l_1) \ldots w(l_n) \rangle = \sum_{j_1} \prod_{s} \left[ (-1)^{j_s} (2j_s + 1) u^{-j_s-1/2} I_{j_s+1/2}(ul_s) \right] \left\langle \prod_{j} \hat{\sigma}_{j_s} \right\rangle. \quad (4.37)$$

We therefore extract correlation functions of the local operators $\hat{\sigma}$ by extracting the coefficients of the "wave functions" $I_{j+1/2}$ in the expansion (4.37). (Note that, by the reasoning of sect. 2 we could have expected divergent terms in $l$ in
\[ \psi_0 = \begin{array}{c} \includegraphics[width=0.2\textwidth]{fig4} \\ \text{Fig. 4. A cap with an operator inserted.} \end{array} \quad \text{and} \quad \begin{array}{c} \sum_v I_v \ \ \ \ \ \ K_v \\ \text{Fig. 5. A tube with } I \text{ at one end and } K \text{ at the other.} \end{array} \]

\[ \begin{array}{c} \sum_{\mu, v, p > 0} I_\mu \ I_v \ I_p \\ \text{Fig. 6. } n \text{ macroscopic loops with } I \text{'s on the ends.} \end{array} \]

\[ \langle \hat{\sigma}, \hat{\sigma} \ldots w(I) \rangle \text{ for large } j \text{ or } k \text{ since } X \text{ would then become negative. The above calculation shows that with our universal cutoff these potentially divergent terms in fact have zero coefficient.)} \]

Eqs. (4.24), (4.35) and (4.37) strongly suggest the following geometrical picture. We associate (4.24) with fig. 4 depicting the creation of a state by insertion of an operator on a disk. Similarly, eq. (4.35) suggests that we interpret the annulus amplitude as a propagator. We see a sum of states with diagonal propagation, every term in the sum corresponding to the propagation of a different state. The two wave functions \( I_{j+1/2} \) and \( K_{j+1/2} \) are physical wave functions satisfying the WdW equation with two different asymptotics. Since \( I_1 < I_2 \) we should impose good behavior (no divergence) only as \( I_1 \to 0 \) and as \( I_2 \to \infty \). These asymptotics determines the combination \( I_{j+1/2}(l_1u)K_{j+1/2}(l_2u) \). Thus we associate (4.35) with fig. 5*. Moving on to \( n > 3 \) loops, eq. (4.37) can be associated to fig. 6 where we have wavefunctions \( I_{j+1/2} \) on the external legs. Geometrical intuition suggests that the correlation function of \( n \) local operators \( \hat{\sigma} \) should be obtained by “sewing” pictures like fig. 4 into surfaces like fig. 6. We see from the above formulae that this intuition is reproduced if we define a formal “inner product” in the space of modified Bessel functions by the rule

\[ \langle I_v, I_\rho \rangle \propto \delta_{v+\rho}. \quad (4.38) \]

The inner product \( \langle , \rangle \) represents the geometrical operation of sewing.

* Remarkably, although we have different expansions for the propagator for \( I_1 < I_2 \) and for \( I_2 < I_1 \), the total sum (4.35) is not singular at \( I_1 = I_2 \). This follows from eq. (4.36), which also shows that the singularity is at \( I_1 = -I_2 \).
In sect. 5 we return to the Liouville theory and attempt to explain the above formulae, especially (4.38), from the continuum point of view.

5. The space of states of two-dimensional gravity

5.1. MACROSCOPIC AND MICROSCOPIC STATES

In this section we show that the space of states in two-dimensional gravity has an inner product structure. Moreover, we will give a space-time interpretation of the results of sect. 4. In Liouville theory there are two very different kinds of states [18, 24] referred to as macroscopic and microscopic states. We begin by reviewing briefly the analysis which leads to these states. In the minisuperspace approximation* the Liouville hamiltonian is

\[ H_L = \frac{\gamma^2}{8} \left[ -\left( l \frac{\partial}{\partial l} \right)^2 + \mu l^2 + \frac{Q^2}{\gamma^2} \right], \]  

(5.1)

where \( l = e^{\gamma \phi/2} \) and \( H_L \) acts on functions on the real \( \phi \) axis (\( l > 0 \)).

Consider first the macroscopic states. These are (delta-function) normalizable wave functions with the standard \( L^2 \) norm: \( ||\psi||^2 = \int_{-\infty}^{\infty} d\phi |\psi(\phi)|^2 = \int_{0}^{\infty} (dl/l) |\psi_\phi(l)|^2 \). The spectrum of the hamiltonian acting on such a space is then \( \{\Delta: \Delta > Q^2/8\} \). Defining \( p^2 = 8\Delta/\gamma^2 - Q^2/\gamma^2 \) the delta-function-normalized wave functions with energy \( \Delta \) are

\[ \psi^{\text{macro}}_p = (p \sinh \pi p)^{1/2} K_{ip}(2\sqrt{\mu l}) \]  

(5.2)

with \( p \) real. The wave functions are real, and \( \psi_p = \psi_{-p} \) due to the complete reflection off the potential \( V = 4\mu l^2 \), so we can restrict to \( p > 0 \). The asymptotic behavior of the wave function as \( l \to \infty \) (\( \phi \to +\infty \)) is proportional to \( l^{-1/2} e^{-\gamma l} \) and as \( l \to 0 \) (\( \phi \to -\infty \)) it is a plane wave proportional to \( \sin(p \log l + \alpha(p)) \).

In ordinary quantum mechanics we usually restrict attention to \( L^2 \)-normalizable states. However, if we consider Liouville theory as a theory of gravity, the physical metric is \( e^{\gamma \phi} \). Therefore, wave functions associated with local operators should be peaked at short distances in the physical metric, i.e. for \( l \to 0 \). Hence, if we wish to find wave functions associated with local operators we cannot use the macroscopic states since they are not peaked at small distances. We are thus forced to consider eigenfunctions of \( H_L \) which diverge in the ultraviolet (\( l \to 0 \)) [18]. Such eigenfunctions must have energy \( \Delta < Q^2/8 \), hence \( \nu = ip \) is real. A second boundary condition, that \( \psi^{\text{micro}} \) vanishes in the infrared (IR) (\( l \to \infty \)), fixes the microscopic

* This problem of Liouville quantum mechanics was first studied in ref. [36].
wave functions up to a constant:

$$\psi^\text{micro}_\nu = (\nu \sin \pi \nu)^{1/2} K_\nu(2\sqrt{\mu} l).$$  \hspace{1cm} (5.3)

If \( \nu = n \) is an integer the expansion of \( K_{\nu} \) contains \( z^n \log z \). This is the source of logarithms in some correlation functions at \( c = 1 \) [37].

In order to compute transition amplitudes associated with microscopic states, i.e. correlation functions of local operators, we need some notion of inner product. However the divergence of \( \psi^\text{micro}_\nu \) as \( l \to 0 \), its very raison d'etre, means it is not \( L^2 \)-normalizable. Nevertheless, as we show in subsect. 5.2 the space of microscopic wave functions has a natural dual pairing which fits very well with the amplitudes we have computed from the matrix model.

5.2. INNER PRODUCTS FOR MICROSCOPIC WAVE FUNCTIONS

We take our cue from the matrix model calculations of subsect. 4.4. From that discussion we expect to use an inner product to obtain correlation functions from an LSZ-like formula of the form

$$\langle \psi^\text{micro}_\nu \rangle = \int_0^\infty \frac{dl}{l} \psi^\text{micro}_\nu(l) w(l) \ldots \rangle. \hspace{1cm} (5.4)$$

Unfortunately the integral in (5.4) is nonsensical.

Since the divergences in (5.4) arise from the region of small \( l \), we consider integrals of the form \( \int_0^\infty (dl/l)l^a \). Such expressions can be defined by analytically continuing \( l = e^{\gamma/2} \to e^{\gamma t/2} \), and integrating along the real \( t \)-axis*, so

$$\int_0^\infty \frac{dl}{l} l^a \to \int_{-\infty}^{\infty} dt e^{-ia't} = 2\pi \delta(a).$$  \hspace{1cm} (5.5)

The prescription (5.5) was suggested in ref. [18] by examining the semiclassical limit of Liouville correlation functions**. More physically, we can put a cutoff on the \( l \)-integral, drop the power divergences and keep only the log divergence. This log divergence (and similarly \( \delta(0) \)) in the norm of a state is removed by the ghosts in quantum gravity rendering the final norm finite [18]. In any case, our inner product, which is defined on functions having an expansion in (not necessarily

* If \( a \) is rational we can restrict the region of integration to be finite. This is the case in the minimal models coupled to gravity. For example, in the one-matrix model all physical states have \( \nu = j + \frac{1}{2} \). The \( j \)th scaling field in the KdV basis \( \sigma_j \) is associated with a wave function which behaves asymptotically for small \( l \) like \( l^{-j+1/2} \), so the expression \( \sigma_j = (j!(-1)^{j+1/2}/2\pi i) \int (dl/l)^{-(j+1/2)} w(l) \) in ref. [35] can therefore be interpreted as a definition of the inner product in (5.4).

** Thinking of Liouville as euclidean time, as in subsect. 5.4, and thinking of ordinary field theory as quantum gravity on the world-line we can also derive this prescription.
integral) powers of $l$, is given by

$$\int f(l)g(l) = a_0,$$  \hspace{1cm} (5.6)

where $a_0$ is the coefficient of $l^0$ in an expansion of $f(l)g(l)$ in powers of $l$.

Let us now check that (5.6) indeed gives

$$\langle \hat{\sigma}_r \ldots \rangle \propto \left(\frac{\sqrt{\mu}}{\sin \pi \nu}\right)^{\nu} \int l^{-\nu}(2\sqrt{\mu} l)^{\nu} w(l) \ldots.$$  \hspace{1cm} (5.7)

Using the identity (B.14) of appendix B it is easy to check that

$$\int l^{-\nu} I_{\rho} = \delta_{\nu,\rho} \left(\frac{\sin \pi \nu}{\pi \nu}\right),$$  \hspace{1cm} (5.8)

and therefore, given an expansion like (4.27) we obtain (5.7). (Here $\nu$ is not an integer.) In particular, for the one-matrix model we find $\hat{\sigma}_r = \frac{1}{4} \pi (2\sqrt{\mu})^{\nu} l^{\nu + 1/2}$. This is in accord with the intuition (5.4) since $K_{\nu}$ is the wave function of $\hat{\sigma}_r$.

5.3. A LIOUVILLE DERIVATION OF THE PROPAGATOR

As discussed in refs. [16, 18, 24], although we may insert local operators, and hence microscopic states externally on a surface, we expect that in general only macroscopic states propagate in the intermediate channels. We may therefore suspect that it is possible to derive the “propagator” (4.35) using the macroscopic states. In this subsection we show how that can be done.

As a warmup problem we first replace the $l^2$ potential in the Liouville theory with a boundary condition which limits $\phi$ to lie on the semiaxis, $l < \mu^{-1/2}$, i.e. $\phi < -(\log \mu)/\gamma$. This toy model captures many of the essential features of the Liouville theory and is technically easier to analyze. In the toy model the macroscopic wave functions are $\psi_\rho = \sin(p \log \mu^{1/2} l)$ and the microscopic ones are $\psi_\nu = \sinh(\nu \log \mu^{1/2} l) = \frac{1}{2}((\mu^{1/2} l)^{\nu} - (\mu^{1/2} l)^{-\nu})$. Note that as in the original Liouville theory we impose the correct boundary conditions in the IR $(\mu^{1/2} l = 1)$ and find a divergence as $l \to 0$. Combining this system with a matter theory with wave functions $\chi_\nu^m(x)$ ($x$ represents the matter coordinates) with energy $\nu_\nu^2$, the
The propagator is found by integrating over the macroscopic states

\[ G(l_1, x_1, l_2, x_2) = \int d\tau \sum_j \int_0^\infty \frac{dp}{\pi} \psi_p(l_1)^* \chi_j^m(x_1) e^{-\pi \tau} \psi_p(l_2)^* \chi_j^m(x_2) \]

\[ = \sum_j \int_0^\infty \frac{dp}{\pi} \frac{\psi_p(l_1)^* \chi_j^m(x_1) \psi_p(l_2)^* \chi_j^m(x_2)}{p^2 + \nu_j^2}. \quad (5.9) \]

The matter theories we are interested in have \( \nu_j^2 > 0 \) (no tachyons [18]). Since the integrand is even in \( p \) we can extend the region of integration to \( (-\infty, +\infty) \). Performing the integral by closing the contour in the upper or lower half plane (for every term where it converges) we find for \( l_1 < l_2 \)

\[ G(l_1, x_1, l_2, x_2) = \frac{\chi_j^m(x_1)^* \chi_j^m(x_2)}{4\nu_j} \left( e^{-\nu_j \log \frac{l_1}{l_2}} - \mu^\nu l_2^{\nu} \right) \left( l_2^{\nu} - \mu^\nu l_1^{\nu} \right). \quad (5.10) \]

Returning now to the Liouville problem, the combination \( l_1^{\nu}(l_2^{\nu} - \mu^\nu l_2^{\nu}) \) is replaced by the analogous term

\[ l_1^{\nu}(2\sqrt{\mu} l_1) K_\nu(2\sqrt{\mu} l_2) = \frac{\pi}{2 \sin \nu_j \pi} I_{\nu}(2\sqrt{\mu} l_1) \left( I_{-\nu}(2\sqrt{\mu} l_2) - I_{\nu}(2\sqrt{\mu} l_2) \right). \quad (5.11) \]

This term is a combination of eigenstates of the Hamiltonian with proper boundary conditions at \( l_1 = 0 \) and \( l_2 = \infty \) (in the toy model \( l_2 = \mu^{-1/2} \)). Thus, the sum over the physical microscopic terms in the propagator is equivalent to the integral over the off-shell macroscopic states*.

We conclude this subsection with two remarks.

First, the different terms in eq. (5.10) (and also in eq. (5.11)) correspond to different kinds of trajectories of the system, that is, to different kinds of surfaces. The term \( l_1^{\nu}(l_2^{\nu} - \mu^\nu l_2^{\nu}) \) in eq. (5.10), and similarly the term \( l_1^{\nu}(2\sqrt{\mu} l_1) I_{-\nu}(2\sqrt{\mu} l_2) \) in eq. (5.11), represent the direct motion from \( l_1 \) to \( l_2 \). The corresponding surface with the metric \( e^{\gamma^5} \) looks like fig. 7. Similarly, the second term in eq. (5.10) \( \mu^\nu l_1^{\nu}/l_2^{\nu} \), or, in Liouville theory proper, \( l_1^{\nu}(2\sqrt{\mu} l_1) I_{\nu}(2\sqrt{\mu} l_2) \) in eq. (5.11), represents reflection off the potential. The corresponding surface looks like fig. 8. This term is absent in free-field theory.

* A.B. Zamolodchikov has considered a similar contour deformation from an integral over the macroscopic states to a sum over the microscopic states in a closely related problem.
As $l_1$ and $l_2$ become small, the surface leading to the direct propagation becomes small. According to our general analysis above, this term should be analytic in $\mu$ and indeed, $I_\nu(2\sqrt{\mu} l_1)I_{-\nu}(2\sqrt{\mu} l_2)$ is analytic. The singularity of the propagator in ordinary field theory at coincident points can appear only in this term which is in general not smooth at $l_1 = l_2$. However, as can be seen from (4.8) for the particular state generated by $w(l)$, the singularity is at $l_1 = -l_2$ rather than at $l_1 = l_2$.

The second term $I_\nu(2\sqrt{\mu} l_1)I_{-\nu}(2\sqrt{\mu} l_2)$, corresponds to large surfaces even when $l_1$ and $l_2$ are small and as expected by our general discussion above, is not analytic in $\mu$. This term represents small holes in a big surface and leads to the two-point function of local operators.

Second, comparison of the above propagator with the matrix model result shows that one must include extra signs, namely the sign $(-1)^j$ in (4.35) is not naturally obtained from the norm-squared of the matter states. One can check that the signs in the matrix model are such that $\langle \hat{\sigma}_j \hat{\sigma}_j \rangle$ is always positive. The oscillating signs of the $I_\nu I_{-\nu}$ contributions have the important effect that there is no singularity at $l_1 = l_2$. A related problem is that the matrix-model propagator involves an infinite sum over states, as we have discussed above, these states are probably redundant, but we do not yet understand the appropriate loop-contact terms which might remove them (if they can be removed at all).

The two-macroscopic-loop amplitude was investigated in the critical string in ref. [38]. In ref. [38] peculiar signs in the propagator analogous to those we have just discussed were found, and were attributed to the ghosts.

5.4. A STRING-THEORETIC / SPACE-TIME INTERPRETATION

The minisuperspace theory of the Liouville mode bears a strong resemblance to the proper-time formulation of a particle moving in space-time. In this subsection we reinterpret the above remarks from the space-time point of view.
Recall that in the free-field theory of a particle of mass \( m \), moving in an euclidean space-time with coordinates \( \varphi, x \), the lagrangian is \( \int \Psi (-\partial^2_\varphi - \partial^2_x + m^2) \Psi \) and the analytically continued propagator may be written as

\[
G(q_\varphi, x_1; q_\varphi, x_2) = \int_{-\infty}^{\infty} dE dp \frac{e^{-ipx_1 - iE\varphi_1} e^{ipx_2 + iE\varphi_2}}{E^2 + p^2 + m^2}.
\]

These two very well known equations should be viewed as being analogous to eqs. (5.9) and (5.10) respectively. In the first equation we sum over the off-shell macroscopic states, while in the second we sum over the on-shell microscopic states. A closer approximation to the Liouville problem is therefore obtained by making the euclidean time semi-infinite and imposing the constraint \( \Psi(\varphi = -(\log \mu)/\gamma) = 0^* \). From the space-time field-theoretic point of view, eq. (5.10) can be understood as a solution of \( (-\partial^2_\varphi + \partial^2_x + m^2)G = \delta(\varphi_1 - \varphi_2)\delta(x_1 - x_2) \) subject to the boundary condition that \( G(l_1, l_2) \) vanish for \( l_i = e^{\gamma\varphi_i/2} \to 0 \) and \( l_i = e^{\gamma\varphi_i/2} \to \mu^{-1/2} \).

Returning to Liouville theory, \( K \) diverges in the past and \( I \) diverges in the future representing incoming and outgoing states. The term proportional to \( l_\nu \) in \( K_\nu = \frac{\pi}{2} \sin \pi \nu (l_\nu - I_\nu) \) (for \( \nu \in \mathbb{Z} \)), represents the reflection off the potential – an incoming wave has a subleading outgoing wave in it. The propagator (4.35) we have found is a sum of terms satisfying \( (-\partial^2_\varphi + 4\mu e^{\gamma\varphi} + \nu^2)G = \delta(\varphi_1 - \varphi_2) \) subject to boundary conditions like those discussed above so we expect that there is a string-field-theoretic interpretation of our results with a lagrangian

\[
\int \Psi (-\partial^2_\varphi + 4\mu e^{\gamma\varphi} + H_{\text{matter}}) \Psi.
\]

From this point of view it is quite clear that in this section we have only discussed free string theory. The matrix model results give us an opportunity to investigate in detail the interacting theory.

It would be nice to find a natural origin for (5.13). In particular, we would like to understand how the matter is combined with the Liouville mode into a unified space-time picture. One (speculative) example of the kind of thing we have in mind is the following. It is possible that the Liouville mode should not be identified with euclidean time at all but is instead a radial space-time coordinate. If we redefine our wave functions by \( \tilde{\Psi} \equiv l^{-1/2} \Psi \) (a change of variables in part motivated by

* Semi-infinite time is by no means unphysical. Believers in the big-bang subscribe to it, albeit in Minkowski space.
(4.12)) and rename \( l = r \) then the second-order differential operator in eq. (5.13) is the radial part of the three-dimensional Laplace operator and the energies \( \nu^2 = (j + \frac{1}{2})^2 - \frac{1}{4} = j(j + 1) \) form the spectrum of the square of the three-dimensional angular momentum operator, \( L^2 \). Can it be that the proper space-time picture of the one-matrix model involves three space-time dimensions?

6. Other models

It is important to realize that the ideas of this paper apply to a wide variety of models of two-dimensional gravity. In general we have a "KdV basis" \( \sigma_\nu \) whose wave functions behave as \( \langle \sigma_\nu w(l) \rangle = l^{-\nu} + \text{regular for } l \to 0 \). It is possible to find an analytic change of basis to operators \( \hat{\sigma}_\nu \) with wave functions satisfying the WdW constraint \( \langle \hat{\sigma}_\nu w(l) \rangle = \mu^{\nu/2} K_\nu(2\sqrt{\mu l^2}) \). Comparing the singular parts of the wave functions we find

\[
\hat{\sigma}_\nu = (-1)^\nu 2^\nu \sum_{s \leq [\nu/2]} \frac{\mu^s}{s!\Gamma(s - \nu + 1)} \sigma_{\nu - 2s} \tag{6.1}
\]

From the wave function of the cosmological constant (or of the boundary operator) we obtain \( \langle w(l) \rangle \) as a sum of the wave functions of the cosmological constant and the equation of motion, so that \( \int \sigma_\nu \sigma_\rho \) and \( \int \sigma_\rho e^{\gamma \sigma} \) are the only operators whose one-point functions are nonanalytic in \( \mu \). In terms of the \( \hat{\sigma} \) basis the propagator computed by the two-loop amplitude \( \langle w(l_1)w(l_2) \rangle \) is diagonal, and has an \( IK \) expansion. From this it follows that the correlation functions \( \langle \hat{\sigma}_\nu \hat{\sigma}_\rho \rangle \) are diagonal up to analytic terms in \( \mu \). (This can only happen when \( \nu + \rho \) is an even integer.)

We now briefly sketch how this works for some other models.

6.1. THE ISING MODEL

The Ising model, formulated as a two-matrix model, has a double-scaling limit which, in the formalism of ref. [9] is described in terms of a third-order Lax operator, the continuum limit of the multiplication by \( \lambda \) on orthogonal polynomials, \( Q(\pm) \). We may construct macroscopic loops in terms of this operator as follows*:

\[
\langle \text{Tr } Q(\pm)^{1/\mu} \rangle \to \langle w(\pm l) \rangle = \int_\mu^{\infty} dx \langle x \exp \left[ \pm l(\kappa^3 D^3 + \nu_2 \kappa D + \nu_1) \right] |x\rangle
\]

\[
= \text{const} \times \kappa^{-1} \int_\mu^{\infty} dx (x - \nu_2)^{1/2} e^{\pm \hbar l} K_{1/3}(l(\nu_2)^{1/2}), \tag{6.2}
\]

* Some related formulae have been obtained in ref. [39].
where \( u_2(x,t_n,\alpha) = -\left(2x\right)^{1/3}/4 \) is the specific heat and \( v_1(x,t_n,\alpha) \) is the magnetization. To each Ising operator \( \sigma_n(\mathcal{O}_\alpha) \) associate a fraction \( \nu = n + \alpha/3 \) and an integer \( s = 1 \) for \( n + \alpha \) even and \( s = 2 \) for \( n + \alpha \) odd. Using ref. [32] we differentiate (6.2) to obtain

\[
\langle \sigma_n(\mathcal{O}_\alpha) w(l) \rangle = \kappa^{-1}(\text{sgn } l)^{1/2} \int_{\sqrt{\mu l^2}}^\infty dyy^\nu K_{\nu/3}(y). \tag{6.3}
\]

Using eq. (B.12) and the properties (B.6) of Bessel functions, it easily follows that there is an analytic lower-triangular transformation

\[
\hat{\sigma}_n(\mathcal{O}_\alpha) = \sigma_n(\mathcal{O}_\alpha) + c_{n,\alpha,2}\mu \sigma_{n-2}(\mathcal{O}_\alpha) + \ldots \tag{6.4}
\]

such that

\[
\langle \hat{\sigma}_n(\mathcal{O}_\alpha) w(l) \rangle = (\text{sgn } l)^{1/2} \mu^{\nu/2} K_\nu(\sqrt{\mu l^2}). \tag{6.5}
\]

As in the one-matrix model we can express \( w(l) \) in terms of \( \hat{\sigma} \) and \( I_{n+\alpha/3} \). We can find the coefficients by first working out the two-loop amplitude. The genus-zero two-loop amplitudes are \((l_i > 0)\)

\[
\langle w(l_1)w(l_2) \rangle = (l_1l_2)^{-1/3} \int_0^\infty d\eta \eta \text{Ai}(l_1^{1/3}\mu^{1/3}-l_1^{-1/3}\eta)\text{Ai}(l_2^{1/3}\mu^{1/3}+l_2^{-1/3}\eta),
\]

\[
\langle w(-l_1)w(l_2) \rangle = -(l_1l_2)^{-1/3} \int_0^\infty d\eta \eta \text{Ai}(l_1^{1/3}\mu^{1/3}+l_1^{-1/3}\eta)\text{Ai}(l_2^{1/3}\mu^{1/3}+l_2^{-1/3}\eta).
\]

(6.6)

These integrals are rather difficult, but using eq. (B.11) the integral may be evaluated exactly for vanishing cosmological constant:

\[
\langle w(l_1)w(l_2) \rangle_{\mu=0} = \text{sgn}(l_1)\text{sgn}(l_2)(l_1l_2)^{1/3}l_1^{-1/3}l_2^{-1/3}. \tag{6.7}
\]

Following the reasoning of sect. 4 we can use this to determine the coefficients in the expansion

\[
w(l) = \sum_{n>0} \left\{ (-\text{sgn } l)^n(n + \frac{1}{3})\hat{\sigma}_n(\mathcal{O}_1)\mu^{-n/2-\nu/6}I_{n+1/3}(\sqrt{\mu l^2})
\right.
\]

\[
- (-\text{sgn } l)^{n+1}(n + \frac{2}{3})\hat{\sigma}_n(\mathcal{O}_2)\mu^{-n/2-\nu/6}I_{n+2/3}(\sqrt{\mu l^2}), \tag{6.8}
\]

from which one may obtain the \( IK \) expansion of the propagator, as predicted from
sect. 5. We recall here the discussion at the end of subsect. 5.2 where comparison of the matrix model and Liouville revealed some mysterious signs in the sum over matter states. The same phenomenon occurs here, which is, perhaps, a hint that these signs are not connected to the nonunitary nature of the \((2, 2m - 1)\) conformal field theories.

6.2. UNITARY DISCRETE SERIES

In this section we will calculate the loop and the wave function of the cosmological constant in the general \((n, n + 1)\) model. Using the solution of ref. [32] we can express the Lax operator of the \(n\)th model as

\[
\frac{1}{n} Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n-k-1)!}{k!(n-2k)!} \left( -\frac{u}{2} \right)^k (ip)^{n-2k}
\]

\[
= i^n \left( -\frac{u}{2} \right)^{n/2} T_n \left( p/(-2u)^{1/2} \right), \quad (6.9)
\]

where \(T_n\) is a Tchebyshev polynomial, which satisfies

\[
T_n(i \sinh \theta) = (-1)^{(n-1)/2} i \sinh(n\theta) \quad \text{for } n \text{ odd,}
\]

\[
T_n(i \sinh \theta) = (-1)^{n/2} \cosh(n, \theta) \quad \text{for } n \text{ even.}
\]

Thus we may compute the macroscopic loop amplitude

\[
w(l) = \int_{\mu}^x dx \langle x|e^{\pm i Q_n}|x\rangle
\]

\[
= \text{const} \times \int_{\mu}^x dxx^{1/2n} K_{1/n}(2\sqrt{\mu l^2}), \quad (6.10)
\]

from which we obtain that the loop is given by \(l^{-1} K_{1+1/n}(2\sqrt{\mu l^2})\) and the wave function of the cosmological constant is \(\mu^{1/2n} K_{1/n}(2\sqrt{\mu l^2})\). Using an argument analogous to the case of the Ising model one can show that the operators \(\hat{\sigma}_n(\theta)\) exist. In particular, from the equation for the loop we see that the only operators which have one-point functions which are nonanalytic in \(\mu\) are \(\hat{\sigma}_0(\theta)\) and \(\hat{\sigma}_2(\theta)\). As in subsect. 4.3, one linear combination is the Liouville equation of motion.

The analytic terms in \(\mu\) in (6.10) lead to one-point functions of \(\sigma_{2k+1}(\theta_{n-1})\) (e.g., the energy operator in the Ising model) as found in ref. [32]. We expect that the \(\hat{\sigma}_n(\theta)\) operators exist for all \(r\) in all \(p, q\) models but have not carried out the
detailed proof. It would follow immediately from an $IK$ expansion of the two-loop amplitude that the nondiagonal two-point functions $\langle \sigma_k(\bar{\sigma}_r)\sigma_{\pi}(-\sigma_{\pi-r}) \rangle$ for $k + s$ an odd integer are nonzero and analytic in $\mu$ as found in ref. [32].

6.3. $c = 1$

Recently macroscopic loop amplitudes have been computed for an uncompactified $c = 1$ system [40]. The two-loop formula at genus zero is

$$\langle w_q(l_1) w_{-q}(l_2) \rangle = 4|q| \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\mu l_1 l_2}{\sqrt{\mu(l_1^2 + l_2^2)}} \Gamma(-|q|-k) \left(\frac{\mu l_1 l_2}{\sqrt{\mu(l_1^2 + l_2^2)}}\right)^{|q|+2k} \times K_{|q|+2k}(2\sqrt{\mu(l_1^2 + l_2^2)}), \quad (6.11)$$

where $w_q(l)$ is a macroscopic loop carrying momentum $q$ with loop length $l$ and we have only kept terms with fractional powers of $l$ (we limit ourselves to the case $q \in \mathbb{Z}$). This complicated formula can be substantially simplified by use of the Gegenbauer formula, along the lines of subsect. 4.4. Applying the Gegenbauer formula to each term in (6.11) we obtain an $IK$ expansion. The coefficient of each term $I_{|q|+2k}K_{|q|+2k}$ is a sum of factorials, and one can show that, except for the first term, this sum vanishes. Thus a single term in the $IK$ expansion survives and we obtain the simple result

$$\langle w_q(l_1) w_{-q}(l_2) \rangle^{h=0} = \frac{\pi |q|}{\sin \pi |q|} K_{|q|}(2\sqrt{\mu l_2^2}) I_{|q|}(2\sqrt{\mu l_1^2}), \quad l_1 < l_2. \quad (6.12)$$

As an immediate consequence of this we discover that the wave functions of $\phi_q$ are [40]

$$\langle \phi_q w(l) \rangle = \mu^{\frac{|q|}{2}} K_{|q|}(2\sqrt{\mu l^2}). \quad (6.13)$$

Interestingly, the expansion (6.12) has many good properties lacking in the $c < 1$ expressions we have seen thus far. For one thing, the embarrassing sum over an infinite set of states has disappeared. In ref. [40] it was proposed that the spectrum at $c = 1$ had the characteristic of a topological field theory, with operators $\sigma_{2n}(\bar{\sigma}_q)$. Even if that is correct, eq. (6.12) is a strong hint that the higher redundant operators are automatically decoupled from macroscopic loops. To settle this issue we must check the higher-point functions. One lesson that we may draw from eqs. (6.12) and (6.13) is that one should study macroscopic loop amplitudes in terms of physical wave functions rather than merely expanding them in powers of $l$. Only then does the physics become clear.
6.4. STRINGS IN \(-2\) DIMENSIONS

As a final example we check some of the preceding ideas by investigating a model with a somewhat different flavor from what we have thus far studied, namely, we consider random surface embedded in \(D = -2\) dimensions \([41]\). Note that this theory is different from the \(m = 1\) one-matrix model, although in both cases we have \(c = -2\). The \(m = 1\) theory is actually a theory of loops spanning a world-sheet of zero area (see appendix A). On the other hand, \(D = -2\) has a clear interpretation as tree-like polymers living on the world-sheet \([42]\). The partition function for closed surfaces of spherical topology is

\[
Z(\mu_0) = \sum_{\{G\}} e^{-\mu_0 n} T(G), \quad (6.14)
\]

where we sum over a class of planar graphs having \(n\) vertices. In our case we will use planar \(\varphi^3\) graphs. \(\mu_0\) is the bare cosmological constant and \(T(G)\) is the number of spanning trees of the graph \(G\). The partition sum is evaluated by rewriting (6.14) as

\[
Z(\mu_0) = \sum_{\{T\}} e^{-\mu_0 n} \times \{\text{contractions}\}, \quad (6.15)
\]

i.e. we first sum over three-coordinated trees and then wire up the ends of the trees to form a \(\varphi^3\) planar graph as in fig. 9.

Introducing a loop of length \(L\) into the random graph is simple: As usual we introduce one supervertex with \(L\) legs. The loop partition sum simply becomes

\[
Z(\mu_0, L) = \frac{1}{L} \sum_{\{T^{L}\}} e^{-\mu_0 n} \times \{\text{contractions}\}, \quad (6.16)
\]

Fig. 9. A tree with planar contractions forms a \(\varphi^3\)-random graph.

Fig. 10. A tree with a vertex of coordination number \(L\) makes a random surface with a loop of size \(L\). (The contractions are omitted in this figure.)
where \( \{T^{(L)}\} \) is the class of all three-coordinated trees possessing one supervertex of size \( L \) as illustrated in fig. 10.

Since the number of spanning trees on a planar graph equals the number of spanning trees on the dual graph; we are thus indeed evaluating the partition sum of planar triangulations (with a hole of size \( L \)) embedded in \( D = -2 \) dimensions. \( Z(\mu_0, L) \) is easily calculated using the methods of refs. [41,42]. The generating function for \( \varphi^3 \)-trees with one marked boundary point is

\[
T(z) = \frac{1}{2} \left( 1 - \sqrt{1 - 4z} \right).
\]

Furthermore, a generating function for the number of planar contractions of a \( k \)-legged vertex is

\[
\int_{-2}^{2} d\lambda \rho(\lambda) \lambda^k,
\]

with \( \rho(\lambda) = \sqrt{4 - \lambda^2} \). The reader will recognize \( \rho(\lambda) \) as the Wigner eigenvalue distribution of the \( m = 1 \) one-matrix model. It is now easily seen that eq. (6.16) can be written as \( (g = e^{-\mu_l}) \)

\[
\frac{1}{L} \int_{-2}^{2} d\lambda \sqrt{4 - \lambda^2} \left( 1 - \sqrt{1 - 4g\lambda} \right)^L,
\]

since \( (T(z))^L \) is the generating function for the class \( \{T^{(L)}\} \). It is straightforward to scale (6.19). The critical point is \( g_c = \frac{1}{8} \); so introduce a cutoff \( a \) and put \( g = g_c - a^2 \mu \) as well as \( l = La \). We also note that the singularity in (6.19) comes from the integration region close to the branchpoint at \( \lambda_c = 2 \). Changing variables \( \lambda = 2 - a^2 x \) and neglecting nonuniversal terms we calculate

\[
Z(\mu_0, L) \sim \frac{1}{L} \int_{-2}^{2} d\lambda \sqrt{2 - \lambda} \left( 1 - \sqrt{1 - 4g\lambda} \right)^L
\]

\[
\sim -a^3 1/L \int_{1/a^2}^{0} dx a\sqrt{x} \left[ 1 - \sqrt{1 - 4(1/8 - a^2 \mu)(2 - a^2 x)} \right]^{1/a}
\]

\[
\sim a^4 \frac{1}{L} \int_{0}^{\infty} dx \sqrt{x} e^{-l\sqrt{x} + \mu}.
\]

As explained before we define the loop function \( \langle w(l) \rangle = lZ(\mu, l) \) and the wave function of the puncture operator \( \langle \partial/\partial \mu \rangle \langle w(l) \rangle \) and we find

\[
\frac{\partial}{\partial \mu} \langle w(l) \rangle = l \int_{0}^{\infty} dx \frac{\sqrt{x} e^{-l\sqrt{x} + \mu}}{\sqrt{x} + \mu}.
\]
Fig. 11. The tree skeleton (contractions are omitted) of a tree with two vertices of coordination numbers $L_1$ and $L_2$.

This integral is a Bessel function representation and we obtain

$$\frac{\partial}{\partial \mu} \langle w(l) \rangle = 1/2 \mu^{1/2} K_1(l \mu^{1/2}).$$

(6.22)

as expected. By a standard identity between Bessel functions the wave function of the boundary operator is

$$\langle w(l) \rangle = \mu K_2(l \mu^{1/2}).$$

(6.23)

It is straightforward to extend the above method of calculation to more loops and to added handles (even nonperturbative expressions may be obtained). Here we will be content to present the two-loop function at genus zero. We have two supervertices of size $L_1$ and $L_2$ which are connected by a backbone [42]. The tree skeleton is shown in fig. 11. The blobs indicate a tree. These diagrams are generated by

$$Z(\mu_0, L_1, L_2) = \int_{-2}^{2} d\lambda \rho(\lambda) T(\lambda)^{L_1} \frac{1}{1 - 2 T(\lambda)^{L_2}}.$$  

(6.24)

Note that $T(g, \lambda_c) = 1/2$, so the backbone becomes macroscopic in the scaling limit. Using the above method one readily derives

$$Z(\mu_0, L_1, L_2) \sim a^2 \int_{0}^{\infty} dx \frac{\sqrt{x}}{\sqrt{x + \mu}} e^{-(l_1 + l_2) \sqrt{x + \mu}}.$$  

(6.25)

So, defining $\langle w(l_1)w(l_2) \rangle = l_1 l_2 Z(\mu, l_1, l_2)$, we obtain

$$\langle w(l_1)w(l_2) \rangle = \frac{1}{2} \frac{l_1 l_2}{l_1 + l_2} \mu^{1/2} K_1[(l_1 + l_2) \mu^{1/2}].$$

(6.26)
Using Gegenbauer’s theorem this may be expanded as

\[ \langle w(l_1)w(l_2) \rangle = \sum_{j=0}^{\infty} (-1)^j (j+1)^2 I_{1+j}(l_1 \mu^{1/2}) K_{1+j}(l_2 \mu^{1/2}). \]  

(6.27)

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**Note added in proof**

After completion of this paper we were informed of ref. [49] which also discusses macroscopic loop amplitudes at genus zero. We also realized that the discussion of sect. 3 ignores contact terms when operators hit boundaries. Such contact terms can lead to negative powers of \( l \) which are nonanalytic in \( \mu \). [50].

**Appendix A**

**MACROSCOPIC LOOPS IN TOPOLOGICAL FIELD THEORY**

It is commonly said that analytic terms in \( \mu \) correspond to nonuniversal quantities. This leads to a paradox since in topological field theory all the quantities are analytic in \( t_j \) [43–45]. It would be ludicrous to assert that all topological field theory correlators are therefore meaningless. Nevertheless, it is worth understanding how one distinguishes “meaningful” analytic terms from “meaningless” (i.e. nonuniversal) analytic terms. In subsect. 3.3 we showed how, from the point of view of Liouville theory, one can understand the existence of universal analytic terms in \( \mu \). In this appendix we explain the relation of that insight to the gaussian matrix model. Related remarks have been made in ref. [46].

In the continuum the \( m = 1 \) conformal field theory (the (1,2) minimal model) coupled to gravity does not have the identity operator in the spectrum. Therefore, the area of the surface is not well defined. It follows that we cannot fine-tune parameters of the cutoff theory to make large area surfaces. Instead, we can obtain continuum results by considering macroscopic loops, of length \( l/a \) in lattice units, and then letting \( a \to 0 \). More specifically, in the hermitian matrix model the \( m = 1 \) potential is simply the gaussian potential \( V(\phi) = N \text{tr} \phi^2 \). It is technically easier to study loops by taking the Laplace transform of \( \text{tr} \phi^{l/a} \), thereby computing

\[ \left\langle \prod \frac{1}{N} \text{tr} \left( \frac{1}{\xi_i - \phi} \right) \right\rangle. \]  

(A.1)
The critical behavior is obtained as $\xi_i$ are tuned to their critical values, and not as parameters in the potential are fine-tuned to critical values. For example the one-point function of the resolvent,

$$\langle w(\xi) \rangle = \frac{1}{2} \left( \xi - \sqrt{\xi^2 - 4} \right), \quad (A.2)$$

exhibits a square-root singularity in $\xi$ associated with the Wigner distribution. Thus one can define a continuum limit amplitude unambiguously. The scaled version of $\xi$ is interpreted physically as the boundary cosmological constant $\rho$ [30].

We can incorporate the double-scaling limit with a little more work. Since the discontinuity of the resolvent across the real axis is the eigenvalue density we see that, effectively, the $m = 1$ critical phenomena comes from the edge of the eigenvalue distribution. Now, correlators of the resolvent operator may be computed readily in the fermionic formalism. On the lattice,

$$\frac{1}{N} \text{tr} \left( \frac{1}{\xi_i - \phi} \right) \rightarrow \int d\lambda \Psi^+(\lambda) \frac{1}{\xi - \lambda} \Psi(\lambda), \quad (A.3)$$

where $\Psi(\lambda) = \sum_n a_n \psi_n(\lambda)$ and $\psi_n$ are the orthonormal wave functions made from the orthonormal polynomials. In the case of the gaussian model these are simply Hermite functions. At the edge of the eigenvalue distribution the Hermite functions behave like Airy functions [6,47,48] and the double-scaling limit of $\Psi$ is $\int dz a(z) \text{Ai}(z + \lambda)$, with a Fermi sea defined by $a(z)|\mu\rangle = 0$ for $z < \mu$ and $a(z)^\dagger|\mu\rangle = 0$ for $z > \mu$. Since Airy functions are the Baker functions for the KdV theory with the potential $u(x) = x$ we can obtain, from the gaussian model, the correlation functions predicted from the KdV theory at the $m = 1$ point.

In sum, by introducing loops, or equivalently, resolvents, we provide a cutoff in the matrix model which allows us to distinguish universal from nonuniversal quantities. This is the matrix-model version of the cutoff discussed in Liouville theory in sect. 3.

From this discussion we see that in topological field theory $t_0$, the lowest coupling is in fact that boundary cosmological constant. Indeed at the $m$th multicritical point $t_m$ is the boundary operator [30]. Moreover, drawing the Feynman diagrams and their associated dual graphs in the gaussian model we see that the “surfaces” generated when computing the resolvent correlators in (A.1) are completely degenerate, and more appropriately described as loops. We may connect these observations directly with the discussion of sect. 3 for the Liouville theory by considering (3.6) as $A \rightarrow 0$. In the inside of the disk we obtain $g = 0$ (thus explicitly realizing Witten’s idea that in topological field theory $\langle g_{a\beta} \rangle = 0.$) Note, however, that the metric has nonzero support on the boundary.
Appendix B

SOME USEFUL FACTS ABOUT BESSEL FUNCTIONS

Here we collect some formulae concerning the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$. They are linearly independent solutions of the Bessel equation

$$\left( z \frac{\partial}{\partial z} \right)^2 z^2 - \nu^2 \right] Z_\nu(z) = 0. \quad (B.1)$$

$I_\nu$ may be expanded as

$$I_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k+\nu+1)} \left( \frac{z}{2} \right)^{2k}. \quad (B.2)$$

The expansion of $K_\nu$ is then obtained from

$$K_\nu(z) = \frac{\pi}{2 \sin \nu \pi} \left[ I_{-\nu}(z) - I_\nu(z) \right]. \quad (B.3)$$

Note that $K_{-\nu}(z) = K_\nu(z)$ for all $\nu$, but $I_{-\nu}(z) = I_\nu(z)$ only if $\nu$ is an integer. For large $|z|$ one has asymptotically

$$I_\nu(z) \sim \frac{1}{\sqrt{2 \pi z}} e^z \quad \left( |\arg z| < \frac{\pi}{2} \right),$$

$$K_\nu \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad \left( |\arg z| < \frac{3}{2} \pi \right). \quad (B.4)$$

For half-integer values of the index Bessel functions are actually elementary:

$$I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \left[ e^z \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!(n-k)!} (2z)^{-k} \pm (-1)^{n+1} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} (2z)^{-k} \right],$$

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} (2z)^{-k}. \quad (B.5)$$
Some very useful identities between Bessel functions of differing index are

\[ 2\nu I_{\nu}(z) = zI_{\nu-1}(z) - zI_{\nu+1}(z), \quad -2\nu K_{\nu}(z) = zK_{\nu-1}(z) - zK_{\nu+1}(z), \]

\[ 2z \frac{d}{dz} I_{\nu}(z) = zI_{\nu-1}(z) + zI_{\nu+1}(z), \quad 2z \frac{d}{dz} K_{\nu}(z) = zK_{\nu-1}(z) + zK_{\nu+1}(z), \]

\[ z \frac{d}{dz} I_{\nu}(z) = zI_{\nu-1}(z) \pm \nu I_{\nu}(z), \quad z \frac{d}{dz} K_{\nu}(z) = -zK_{\nu-1}(z) \pm \nu K_{\nu}(z). \]

(B.6)

There exists a large amount of integral representations for the modified Bessel functions. A representation particularly useful for transforming from fixed area to fixed cosmological constant is

\[ K_{\nu}(z) = \frac{1}{2} \left\{ \frac{z}{2} \right\} \int_{0}^{\infty} dt \, t^{\nu-1} e^{-t-z^2/4t}, \quad |\arg z| < \frac{\pi}{2}, \quad \text{Re} \, z^2 > 0. \] (B.7)

For the calculation of the wave functions of the discrete unitary series one needs the representations

\[ K_{\nu}(z) = \frac{1}{\cos \nu \pi/2} \int_{0}^{\infty} dt \cosh(\nu t) \cos(z \sinh t), \]

\[ z = \text{Re} \, z > 0, \quad -1 < \text{Re} \, \nu < 1, \]

\[ K_{\nu}(z) = \int_{0}^{\infty} dt \cosh(\nu t) e^{-z \cosh t}, \quad |\arg z| < \frac{\pi}{2}. \] (B.8)

where the two integrals apply to the odd and even members of the unitary series, respectively. In the special case of the Ising model the wave function is related to an Airy function defined by

\[ \text{Ai}(z) = \frac{1}{\pi} \int_{0}^{\infty} dt \cos \left( \frac{1}{3} t^3 + zt \right) \] (B.9)

which in turn may be expressed as a Bessel function through

\[ \text{Ai}(z) = \frac{1}{\pi} \sqrt{\frac{2}{3}} K_{1/3} \left( \frac{2}{3} z^{3/2} \right). \] (B.10)

For the calculation of the Ising propagator at \( \mu = 0 \) we used the
Weber–Schafheitlin integral

\[ \int_0^\infty dx x^{-\lambda} K_\mu(ax) K_\nu(bx) = \frac{2^{-2\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)} \]

\times \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \]

\times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \]

\times \begin{pmatrix} 1-\lambda+\mu+\nu, & 1-\lambda-\mu+\nu, & 1-\lambda, & 1-b^2 \end{pmatrix},

(B.11)

where \( F \) is a hypergeometric series (in the Ising case it sums to a simple algebraic expression). To prove the existence of the operators \( \hat{\sigma} \) in the Ising case another integral is handy:

\[ \int_1^\infty dx x^{-\nu/2} (x - 1)^{-\mu-1} K_\nu(a\sqrt{x}) = \Gamma(\mu) 2^\mu a^{-\mu} K_{\nu-\mu}(a). \quad (B.12) \]

Bessel functions obey an addition law due to Gegenbauer which, for \( K_\nu \), reads in its most general form

\[ \frac{K_\nu(z)}{z^\nu} = 2^\nu \Gamma(\nu) \sum_{j=0}^\infty \frac{I_{j+\nu}(x)}{x^\nu} \frac{K_{j+\nu}(y)}{y^\nu} C_j^\nu(\cos \phi), \]

\[ z^2 = x^2 + y^2 - 2xy \cos \phi. \quad (B.13) \]

The Gegenbauer polynomials \( C_j^\nu(t) \) are defined as the coefficient of \( \alpha^j \) in the expansion of \((1 - 2at + \alpha^2)^{-\nu}\). Finally, we give an ascending series relevant to the consistency of our definition of the inner product:

\[ I_\nu(z) I_\mu(z) = \sum_{k=0}^\infty \frac{\Gamma(\nu+\mu+2k+1)}{\Gamma(\nu+k+1)\Gamma(\mu+k+1)\Gamma(\nu+\mu+k+1)k!} \left( \frac{z}{2} \right)^{2k+\nu+\mu}. \quad (B.14) \]
References

R. Dijkgraaf, E. Verlinde and H. Verlinde, Notes on topological string theory and 2D quantum gravity, Princeton preprint PUPT-1217, presented at the Cargèse Workshop
J.-L. Gervais, LPTENS 89/14; 90/4
[18] N. Seiberg, Notes on quantum Liouville theory and quantum gravity, Rutgers preprint RU-90-29,
M. Goulian and M. Li, Correlation functions in Liouville theory, Santa Barbara preprint UCSBTH-90-61;
P. Di Francesco and D. Kutasov, Correlation functions in 2D string theory, Princeton preprint PUPT-1237
   P. Di Francesco and D. Kutasov, Integrable models of two-dimensional quantum gravity, Princeton preprint PUPT-1206 (1990), presented at the Cargèse Workshop
[37] D. Gross, I. Klebanov and M. Newman, The two-point correlation function of the one-dimensional matrix model, PUPT-1192
[40] G. Moore, Double-scaled field theory at c = 1, Rutgers preprint RU-91-12
[46] I. Kostov, Strings embedded in Dynkin diagrams, Saclay preprint SPhT/90-133, Lecture given at the Cargèse meeting
[50] G. Moore and N. Seiberg, From loops to fields in 2D quantum gravity, Rutgers/Yale preprint RU-91-29/YCTP-P19-91