Modular Properties of 3D Higher Spin Theory

(Based on 1308.2959)

National Taiwan University 01/10/2014

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I. Modular Properties
Modular Group and Modular Transformation

(Passive Point of View)

Modulus: \( \tau \in \mathbb{C} \)

\[
\begin{align*}
    z \sim z + n + m \tau & \quad n, m \in \mathbb{Z} \\
    z \sim z + n(c\tau + d) + m(a\tau + b) & \quad a, b, c, d \in \mathbb{Z}
\end{align*}
\]

Same lattice, if \( ad - bc = 1 \)

\[
\left( \begin{array}{c}
    \tau \\
    1
\end{array} \right) \rightarrow \gamma \cdot \left( \begin{array}{c}
    \tau \\
    1
\end{array} \right) \quad \gamma \equiv \left( \begin{array}{cc}
    a & b \\
    c & d
\end{array} \right) \in \text{PSL}(2, \mathbb{Z})
\]

- Pick \( \{c\tau + d, a\tau + b\} \) as basis

Modular transformation: \( \tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \)

S-dual: \( \tau \rightarrow \tau' = -\frac{1}{\tau} \)
Solid Torus (A/B cycles)

- A(B)-cycle: (non-)contractible cycle
- Modular Parameter \( \tau \) defined as 
  \[
  \tau = \frac{\int_B dz}{\int_A dz}
  \]

General choice of A/B cycles \( \rightarrow \) Modular Parameter 
\[
\frac{a\tau + b}{c\tau + d}
\]
Focus on the solutions with Euclidean signature.

**Thermal AdS:**

\[
ds^2 = \, d\rho^2 + (e^{2\rho} + \frac{1}{16} e^{-2\rho}) \, dz \, d\bar{z} - \frac{1}{4} (dz^2 + d\bar{z}^2) \quad z \equiv \phi + it_E
\]

\[
I^{(E)}_{\text{AdS}}[\tau] = - \frac{\pi}{4G} \tau_2 \quad \tau_2 \equiv \text{Im}(\tau)
\]

**BTZ black hole:**

\[
ds^2 = \, d\rho^2 + 8\pi G (L \, dz^2 + \bar{L} \, d\bar{z}^2) + \left( e^{2\rho} + (8\pi G)^2 \, L \bar{L} e^{-2\rho} \right) \, dz \, d\bar{z}
\]

\[
I^{(E)}_{\text{BTZ}}[\tau] = - \frac{\pi}{4G} \frac{\tau_2}{|\tau|^2} = I^{(E)}_{\text{AdS}}[-\frac{1}{\tau}]
\]

In general:

- **A-cycle:** \( z \sim z + 2\pi (c\tau + d) \)
- **B-cycle:** \( z \sim z + 2\pi (a\tau + b) \)

\[
I^{(E)}_\gamma[\tau] = I^{(E)}_{\text{AdS}}[\frac{a\tau + b}{c\tau + d}]
\]
Modular Invariant Partition Function

AdS3/CFT2 correspondence:
\[ \left( Z_{AdS} \approx \sum e^{-I} \right) = Z_{CFT} \]
saddle point approximation

- If we include only thermal AdS and BTZ black hole, the result can not be modular invariant. However if we start from AdS and sum over all modular images, the result will be modular invariant.

- This means we sum over the contributions from the “distinct” solutions with general A cycles choices:
  \[ \forall (c, d) \quad \text{with} \quad c, d \in \mathbb{Z}, \quad c \geq 0, \quad \gcd(c, d) = 1. \]

The goal of this work is to extend the above story to higher spin theory.
II. Higher Spin Theory
An extension of ordinary gravity theory including an infinite tower of massless higher spin fields with spin $s \geq 3$ coupled non-linearly.

The theory lives in AdS (or dS) space. The no-go theorems are evaded.

In AdS3, the theory can be realized as a Chern-Simon gauge theory with an infinite-dimensional gauge algebra $h_{s[\lambda]}$.

At $\lambda=N$, the algebra reduce to $sl(N)$. The result theory is a nature generalization of the usual $sl(2)$ Chern-Simon theory. This 3D Chern-Simon theory with $sl(N)$ algebra will be the main topic of this talk.
Basics of 3D Higher Spin Theory

- In D=2+1(or 3), there is a gauge formulation of Einstein gravity in terms of the Chern-Simon Theory:

  The action of the Chern-Simon Theory:

  \[ S = S_{CS}[A] - S_{CS}[^{\bar{A}}] \quad S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr}[A \wedge dA + \frac{2}{3} A \wedge A \wedge A] \]

  \[ A, ^{\bar{A}} \in \text{sl}(2, R) \quad k = \frac{l}{4G} \]

  \text{sl}(2) \text{ algebra: } [L_m, L_n] = (m - n)L_{m+n}, \quad m, n = -1, 0, 1.

  \text{Equation of motion: } dA + A \wedge A = 0

- A convenient gauge choice:

  \[ A = b^{-1} a b + b^{-1} d b \quad b = e^{\rho L_0} \]

  \[ a = a_z \, dz + a_{\bar{z}} \, d\bar{z} \quad \text{is a gauge field lives on the boundary} \]

  \text{E.O.M. for a constant connection: } [a_z, a_{\bar{z}}] = 0 \]
Basics of 3D Higher Spin Theory

- Extend to higher spin theory by $\mathfrak{sl}(2) \rightarrow \mathfrak{sl}(N) \supset \mathfrak{sl}(2)$

- Precise field content will depend on how one embed the gravity sector $\mathfrak{sl}(2)$ into $\mathfrak{sl}(N)$

  Principal embedding:

  $\mathfrak{sl}(N)$ generators $\rightarrow$ Embedded $\mathfrak{sl}(2)$: $L_0$, $L_{\pm 1}$

  Higher spin: $W^{(s)}_m$ for $s = 3, \cdots, N$ and $m = -s + 1, \cdots, s - 1$

  $[L_m, W^{(s)}_n] = [(s - 1)m - n]W^{(s)}_{m+n}$

- “Singularity” and “Horizon” are no longer gauge-invariant concepts.

- The only gauge invariant quantity: holonomy $\text{Hol}_C(A) \equiv \mathcal{P} e^{\frac{c}{\hbar} A}$
Some other features of 3D Higher Spin Theory

- The diffeomorphism and local Lorentz symmetry are contained in the gauge symmetry.

- Asymptotic to AdS3: \((A - A_{AdS})|_{\rho \to \infty} = O(1)\)

- Asymptotic symmetry algebra \(\mathcal{W}_N\) algebra

- For \(hs[\lambda]\), the asymptotic symmetry algebra is \(\mathcal{W}_\infty[\lambda]\)

- Goberdiel and Gopakumar conjecture:

\[
\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}}
\]

\(k, N \to \infty, \quad \lambda = \frac{N}{k+N} \) fixed

Holographic Dual

Higher spin gravity based on \(hs[\lambda]\)
Known smooth solutions in 3D Higher Spin Theory

Black Holes with higher spin charges in SL(3) [M. Gutperle and P. Kraus. `11]

- Identify zero mode of spin 2 field $\mathcal{L}$ and the modulus $\tau$ as thermodynamic conjugate pair (holomorphic formalism).

- Add higher spin chemical potential $\mu_3$:
  \[ a_{\bar{z}} \sim \mu_3 W^{(3)}_2 + \cdots \]

- How to fix the relation between charges/chemical potentials?
  - In normal gravity, using the smooth condition on horizon.
  - Holonomy around $\tau$-cycles match with normal BTZ black hole.

- Use integrability to obtain the partition function and entropy.
  \[ \frac{\partial \mathcal{L}}{\partial \mu_3} = \frac{\partial W}{\partial \tau} \]
  - Entropy depends on higher spin charges.
Conical Surpluses in SL(N) [Castro et al. `11]

- A generalization of AdS3 in higher spin theory
- Characterized by holonomy condition along $\phi$-cycle

$$\text{Hol}_\phi = b^{-1} \exp \left( \int d\phi \left( a_\phi = a_z \right) \right) b \in \text{center of SL}(N)$$

Constraint the vector of eigenvalues: $\Lambda(a_z) = i\vec{n}$, \[\vec{n} = (n_1, \ldots, n_N), \quad n_i \in \begin{cases} \mathbb{Z} & N \text{ odd} \\ \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2} & N \text{ even} \end{cases}, \quad n_i \neq n_j \text{ for } i \neq j, \quad n_i + n_{N+1-i} = 0,\]

- When $\vec{n} = \vec{\rho}$ ($\rho_i = (N + 1)/2 - i$) $\Rightarrow$ Global AdS3
- When $\vec{n} \neq \vec{\rho}$ $\Rightarrow$ Contain conical singularity (conical surplus) 
  - Carry higher spin charges with even spin
General Framework in $\text{sl}(N)$ [de Boer, Jottar `13, Castro et al. `11]

- Highest/Lowest weight gauge convention:

  $Q$ and $M$ are linear in charges and chemical potentials respectively

  $a_z = L_1 + Q$
  $a_{\bar{z}} = M + \text{(terms } \sim W_m^{(s)} \text{)}$
  $Q = \sum_{s=2}^{N} \frac{Q_s}{t^{(s)}} W_s^{(s)}$
  $M = \frac{i}{2\tau_2} \sum_{s=3}^{N} \mu_s W_s^{(s)}$

  Uniquely determined by equation of motion: $[a_z, a_{\bar{z}}] = 0$

- Smooth solutions are characterized by the holonomy condition along A-cycle:

  \[ \text{Hol}_A(A) \equiv \mathcal{P} e^{\oint A} \in \text{center of the SL}(N) \]

  For a constant gauge field: $\text{Hol}_A(A) = b^{-1} e^{2\pi \omega_A} b$

  Holonomy matrix: $\omega_A = (c_1 + d) a_z + (c_{\bar{z}} + b) a_{\bar{z}}$

- Condition constraint the vector of the eigenvalues of holonomy matrix: $\Lambda(\omega_A) = i \bar{n}$

  $\bar{n} = (n_1, \ldots, n_N)$,  \( n_i \in \begin{cases} \mathbb{Z} & \text{N odd} \\ \mathbb{Z} + \frac{1}{2} & \text{N even} \end{cases} \)  \( n_i \neq n_j \text{ for } i \neq j \),  \( n_i + n_{N+1-i} = 0 \)
Modular Images of the Conical Surpluses

For a conical surplus, \( \omega_A = \omega_\phi = a_z + a_{\bar{z}} \)

CS: \[ i \tilde{n} = \Lambda (\omega_\phi [\tau; \mu_s; Q_s]) = \Lambda (a_z [Q_s]) + \Lambda (a_{\bar{z}} [\tau; \mu_s; Q_s]) \]

For a general modular image \( \gamma \), \( \omega_A = (c\tau + d) a_z + (c\bar{\tau} + b) a_{\bar{z}} \)

\( \gamma: \quad i \tilde{n} = \Lambda (\omega_A [\tau; \mu_s; Q_s]) = (c\tau + d)\Lambda (a_z [Q_s]) + (c\bar{\tau} + d)\Lambda (a_{\bar{z}} [\tau; \mu_s; Q_s]) \)

Goal: to figure out some transformations of \( \tau, \mu_s, Q_s \).

Using sl(N) algebra and the lowest/highest weight structure of \( a_z/a_{\bar{z}} \), one can show:

\[ i \tilde{n} = \Lambda (\omega_A [\tau; \mu_s; Q_s]) = \Lambda \left( \omega_\phi \left[ \hat{\gamma}_\tau; \frac{\mu_s}{(c\tau + d)^s}; (c\tau + d)^s Q_s \right] \right). \]

**Modular transformation:**

\[ \tau \mapsto \hat{\gamma}_\tau = \frac{a\tau + b}{c\tau + d}, \quad \mu_s \mapsto \frac{\mu_s}{(c\tau + d)^s}, \quad Q_s \mapsto (c\tau + d)^s Q_s, \]
Passive point of view: coordinate transformation and $Q_s/\mu_s$ redefinition.

In order to sum the partition functions, we need to put them in a particular coordinate and ensemble.

Active point of view: fix coordinate and $\mu_s$ (in grand canonical ensemble)

Different solutions, different solid torus
Coordinate Transformation

• Just like the metrics of AdS3 and BTZ black hole are related by a coordinate transformation, the gauge fields of CS and BTZ are related by the following coordinate transformation up to some constant gauge transformation:

\[ \rho \rightarrow \rho' = \rho + \ln |\tau|, \quad z \rightarrow z' = \frac{z}{\tau}, \quad \bar{z} \rightarrow \bar{z}' = \frac{\bar{z}}{\bar{\tau}}. \]

\[ \hat{h}^{-1} \cdot A^{CS}_{\vec{n}; \ -\frac{1}{\tau}; \ \frac{\mu_s}{\tau^s}}^{(\rho', z', \bar{z}')}(\rho, z, \bar{z}) \cdot \hat{h} = A^{BH}_{\vec{n}; \ \tau; \ \mu_s}^{(\rho, z, \bar{z})}, \]

\[ \hat{h} = \left( \frac{\tau}{\bar{\tau}} \right)^{\frac{L_0}{2}} = e^{-i \arg(\tau) \frac{L_0}{2}} \in SU(N) \]

• This transformation is actually exactly the coordinate transformation that take the metric of thermal AdS3 to BTZ in Fefferman-Graham form.
The coordinate transformation that relate CS to some general modular image $\gamma$:

$$\rho \rightarrow \rho^\gamma = \rho + \ln |c\tau + d|, \quad z \rightarrow z^\gamma = \frac{z}{c\tau + d}, \quad \bar{z} \rightarrow \bar{z}^\gamma = \frac{\bar{z}}{c\bar{\tau} + d}.$$ 

$$\hat{h}_\gamma^{-1} \cdot A^{CS} \left[ \vec{n}; \hat{\gamma}_\tau; \frac{\mu_s}{(c\tau + d)^s} \right]^{(\rho,z,\bar{z})^\gamma} \cdot \hat{h}_\gamma = A^\gamma [\vec{n}; \tau; \mu_s]^{(\rho,z,\bar{z})},$$

$$\hat{h}_\gamma = \left( \frac{c\bar{\tau} + d}{c\tau + d} \right)^{\frac{L_0}{2}} \in SU(N).$$
III. Thermodynamic and Modular Invariant Partition Function
Thermodynamics ("canonical" formalism) [de Boer, Jottar ’13]

\[-\beta F[\tau; \mu_s] = \ln Z[\tau; \mu_s] \approx - I^{(E)}|_{\text{on-shell}}\]

saddle point approximation

Consistent thermodynamic system should have:

\[\delta I^{(E)}|_{\text{on-shell}} = \delta I^{(E)}_{\text{bulk}}|_{\text{on-shell}} + \delta I^{(E)}_{\text{bndy}}|_{\text{on-shell}} = \sum_i (\text{conjugated } q_i)\delta(\mu_i)\]

add boundary action to impose appropriate boundary condition

- Modulus \( \tau \) act as the chemical potential of spin-2 charge
- \( \mu_s \): chemical potential for higher spin charge with \( s > 2 \)

Varying bulk action produce a boundary term:

\[\delta I^{(E)}_{\text{bulk}}[A]|_{\text{on-shell}} = -\frac{ik}{4\pi} \int_{\partial M} \text{Tr}[a \wedge \delta a]\]

When varying the action, one need to vary \( \tau \) (shape of the torus) explicitly. To do that, we can change the coordinate to the rigid torus and shift \( \tau \) dependence to the gauge field, \( a \), and then vary it.

\( \delta a \) involves the variation of charges and chemical potentials including \( \tau \).
Add the following boundary action: \( I_{\text{bndy}}[A] = -\frac{k}{2\pi} \int_{\partial M} d^2 z \text{Tr}[(a_z - 2L_1) a_{\bar{z}}] \)

Varying the whole action yield the desired form (including the part coming from \( A \)):
\[
\delta I^{(E)}_{\text{os}} = \delta I^{(E)}_{\text{bulk}}|_{\text{os}} + \delta I^{(E)}_{\text{bndy}}|_{\text{os}} = -(2\pi ik) \left( T \delta \tau - \bar{T} \delta \bar{\tau} + \sum_{s=3}^{N} (Q_s \delta \mu_s - \bar{Q}_s \delta \bar{\mu}_s) \right)
\]
\[
T = \frac{1}{2} \text{Tr} \left[(a_z)^2\right] + \text{Tr} \left[a_z a_{\bar{z}}\right] - \frac{1}{2} \text{Tr} \left[(\bar{a}_z)^2\right]
\]

- \( T \) is the energy momentum tensor conjugated to the modulus \( \tau \).
- \( T \) is not holomorphic and will depend on the higher spin charges if the chemical potential is not zero.
- In short, the highest/lowest weight gauge choice of the charge/chemical potential separation plus this particular boundary action yield a consistent thermodynamic system.
Evaluation of On-Shell Action (Free Energy) [Banados et al. ‘12]

- Evaluation of the bulk action depends on the choice of A/B cycles.

- Slice the torus along the A-cycle yield the on-shell bulk action:

$$I^{(E)}_{\text{bulk}}[A]|_{\text{os}} = \text{[bulk term]} - \frac{ik}{4\pi} \int_{\partial M} \text{Tr} [\omega_A \omega_B]$$

- For constant gauge fields:  
  $$I^{(E)}_{\text{bulk}}|_{\text{os}} = -(2\pi ik) \frac{1}{2} \text{Tr} [\omega_A \omega_B - \bar{\omega}_A \bar{\omega}_B] .$$

- Using sl(N) algebra and the lowest/highest weight structure of $a_z/a_{\bar{z}}$, one can show that the on-shell boundary action is:

$$I^{(E)}_{\text{bndy}}|_{\text{os}} = (2\pi ik) \frac{1}{2} \sum_{s=3}^{N} (s - 2) (\mu_s Q_s - \bar{\mu}_s \bar{Q}_s) .$$

- The free energy is:  
  $$-\beta F = -(I^{(E)}_{\text{bulk}}|_{\text{os}} + I^{(E)}_{\text{bndy}}|_{\text{os}})$$
A/B cycles of black holes and conical surpluses: \( \omega_A \omega_B = \begin{cases} \omega_\phi \omega_t & \text{CS} \\ \omega_t (-\omega_\phi) & \text{BH} \end{cases} \)

The free energy becomes:

\[
-\beta F = (2\pi i k) \cdot \begin{cases} (T\tau - \overline{T}\overline{\tau}) + \sum_{s=3}^{N} (\mu_s Q_s - \overline{\mu_s} \overline{Q}_s) & \text{CS} \\ -(T\tau - \overline{T}\overline{\tau}) - \sum_{s=3}^{N} (s-1)(\mu_s Q_s - \overline{\mu_s} \overline{Q}_s) & \text{BH} \end{cases}
\]

Solve the holonomy condition in \( \text{sl}(3) \) and expand in \( \mu_3 \)

\[
-\beta F^{\text{CS}} = 4\pi k \cdot n^2 \tau_2 \cdot \left[ 1 - \frac{1}{3}(\alpha^2 + \overline{\alpha}^2) + \frac{10}{27}(\alpha^4 + \overline{\alpha}^4) - \frac{17}{27}(\alpha^6 + \overline{\alpha}^6) + \frac{106}{81}(\alpha^8 + \overline{\alpha}^8) + \cdots \right].
\]

\[
\tau \longleftrightarrow -\frac{1}{\tau} \quad \mu_s \longleftrightarrow \frac{\mu_s}{\tau^s} \quad n \in \mathbb{Z} \quad \alpha \equiv n \frac{i|\tau|^2}{2\tau_2} \frac{\mu_3}{\tau^3} \quad \beta \equiv n \frac{i|\tau|^2}{2\tau_2} \frac{\mu_3}{\tau^3}.
\]

\[
-\beta F^{\text{BH}} = 4\pi k \cdot n^2 \frac{\tau_2}{|\tau|^2} \cdot \left[ 1 - \frac{1}{3}(\beta^2 + \overline{\beta}^2) + \frac{10}{27}(\beta^4 + \overline{\beta}^4) - \frac{17}{27}(\beta^6 + \overline{\beta}^6) + \frac{106}{81}(\beta^8 + \overline{\beta}^8) + \cdots \right].
\]
Free Energy of SL(2,Z) family of solutions

\[-\beta F = - (I_{\text{bulk}}^{(E)}|_{\text{on-shell}} + I_{\text{bdy}}^{(E)}|_{\text{on-shell}})\]

\[I_{\text{bulk}}^{(E)}|_{\text{on-shell}} = -(2\pi i k)^{\frac{1}{2}} \text{Tr} \left[ \omega_A \omega_B - \bar{\omega}_A \bar{\omega}_B \right].\]

\[I_{\text{bdy}}^{(E)}|_{\text{on-shell}} = (2\pi i k)^{\frac{1}{2}} \sum_{s=3}^{N} (s - 2) (\mu_s Q_s - \bar{\mu}_s \bar{Q}_s).\]

Using the relation between the solutions of a conical surplus and a general modular image and the above expression of free energy, one can show that:

\[F^{\gamma} [\bar{n}; \tau; \mu_s] = F^{\text{CS}} \left[ \bar{n}; \hat{\gamma}_\tau; \frac{\mu_s}{(c\tau + d)^s} \right].\]
Modular Invariant Full Partition Function

- Simple result (obtained non-trivially): \( F^\gamma [\vec{n}; \tau; \mu_s] = F^{CS} \left[ \vec{n}; \hat{\gamma} \tau; \frac{\mu_s}{(c\tau + d)^s} \right] \).

- Partition function of CS: \( Z^{CS}_{\vec{n}} = e^{-\beta F^{CS}} \).

- Partition function of a modular image: \( Z^\gamma_{\vec{n}} [\tau; \mu_s] = Z^{CS}_{\vec{n}} \left[ \hat{\gamma} \tau; \frac{\mu_s}{(c\tau + d)^s} \right] \).

- Sum over modular images:

\[
Z_{\vec{n}} [\tau; \mu_s] = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} Z^\gamma_{\vec{n}} [\tau; \mu_s] = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} Z^{CS}_{\vec{n}} \left[ \hat{\gamma} \tau; \frac{\mu_s}{(c\tau + d)^s} \right]
\]

- Sum over \( \vec{n} \): \( Z [\tau; \mu_s] = \sum_{\vec{n}} Z_{\vec{n}} [\tau; \mu_s] \).
IV. Summary and Discussion
Summary

- We found out how a smooth solution in higher spin theory change under modular transformations through the holonomy condition.

- Using canonical formalism, we showed how to construct free energy (or partition function) in higher spin theory and verified the black holes and conical surpluses are S-dual.

- Starting from a conical surplus, one can generate all solutions related by modular transformations. By summing over all modular images, the modular invariant partition function can be formally constructed.

- If the partition function can be explicitly constructed, one can use it to study the phase structure (e.g., Hawking-Page transition) in higher spin theory. However...
  
  - How to solve the holonomy condition in general sl(N)?
  - How to sum over the modular images?
The partition function obtained by canonical formalism is different from the one in holomorphic formalism (1103.4304) which is deduced from the integrability condition:

$$\frac{\partial \mathcal{L}}{\partial \mu_3} = \frac{\partial \mathcal{W}}{\partial \tau}$$

This integrability is incompatible to the modular transformation we found. That is even the integrability is satisfied for a black hole, it is no longer true for a conical surplus.

However, it has been showed that the partition function in holomorphic formalism match with CFT computation (1203.0015). So, what is going on?

Maybe there is a way to build a connection between these two formalisms through field redefinitions (1308.2175).
Thank you!