Parity-violating hydrodynamics from gravity

Debaprasad Maity

LeCosPa, National Taiwan University

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In collaboration with J.W. Chen, N.E. Lee and S.H. Dai
Introduction

- Hydrodynamics ubiquitous phenomena: Nuclear, Astrophysics, Cosmology

- Dynamics of a nearly equilibrated system at high enough temperature may be described by an effective theory called hydrodynamics.

- Hydrodynamics: effective long wavelength description of many classical and quantum many body systems (finite temperature): Conservation equation

\[ \nabla_a T^{ab} = F^{bc} J_c; \quad \nabla_a J^a = CF^* F \delta_{d4} \]

- Guiding principles: Underlying symmetry (Lorenz symmetry) of the system, Thermodynamics laws. Constitutive relations: \( T_{ab} \) and \( J_a \) in terms of fluid dynamic variables: \( T(x), u^a(x), \mu(x) \)

\[ T^{ab} = (\rho + P)u^a u^b + Pg^{ab} + \Pi^{ab}; \quad J^a = nu^a + \gamma^a; \quad \nabla_s J^a_s \geq 0 \]
Introduction

- Hydrodynamics is universal not much depend on the microscopic detail. Transport coefficients?

- When dynamics is well described by kinetic theory, the Boltzmann equation reduces to the equations of fluid dynamics on length scales larger than molecular mean free path.

- If strongly coupled: Hard to make prediction about the transport coefficients. So far no satisfactory answer from field theory.

AdS/CFT?

- Weak classical gravity in ADS space is dual to strongly coupled gauge theory. Any universal prediction would be a useful guide:
  \[
  \frac{\eta}{s} \geq \frac{1}{4\pi}
  \]
  P. Kovtun et al, PRL. 94, 111601 (2005)
Introduction

- **Interesting in its own right**: attempt towards classifying all the solutions of Einstein’s equations would help to learn more on strongly coupled theory dynamics.

- **Systematic perturbation in long wavelength leads to the precise correspondence**: Einstein Equations (1915) $\iff$ Navier Stokes equations (1822).

- **Ability to derive Hydrodynamic equations from holography provides fresh perspectives, complementary to those of the phenomenological approach.**

Introduction

• Fluid/Gravity correspondence: In the long wave length limit, Space of solution of perturbation in asymptotically ADS space $\iff$ space of solutions of the relativistic hydrodynamic equations of a boundary CFT. Important Output: the constitutive relations determined by the bulk gravitational equations. (Thermodynamics is inbuilt in gravity)

• Top down approach (string theory): Widespread application to QCD like theory (AdS/QCD)

• Bottom up approach: General Long-wavelength properties properties of "holographic strongly correlated system"?

Why this could be interesting?
QGP in hydrodynamics regime (at RICH)

2+1d-CFTs arise in condensed matter physics in many different contexts: near the quantum critical points

Strongly coupled system, non-perturbative
Good Analytic control

The ambitious programme: Identify the correct holographic field theory.
The less ambitious programme: By learning about this class of field theories, we may find some universal features like $\frac{n}{s}$.
Some interesting facts

- Parity violating system in 2+1 dimension have been considered in condensed matter literature.

- **Anomalous Hall effect**
  
  N. Nagaosa et al, arXiv:0904.4154

- **Thermal Hall conductivity**
  

- **Hall viscosity**
  
  J. E Avron, Physics/9712050; Nicolis and Son, arXiv: 1103.4851

- In this talk I will talk about all possible first order transport coefficients for a class of charged fluid.
  
Plan

• Review on 2+1 dimensional parity violating Hydrodynamics

• Holographic construction and computational method

• Results so far

• Conclusions
Review on Parity violating Hydrodynamics in 2+1 D

• Identify the fluid dynamical variable: For charge fluid with the external electromagnetic $F_{ab}$ and gravitational source $g_{ab}$. $T(x), u^a(x), \mu(x)$ with normalization $u^a u_a = -1$ (five variables)
Dynamics is determined by conservation equation (Relativistic Navier-Stokes equations);

$$\nabla_a T^{ab} = F^{bc} J_c; \quad ; \quad \nabla_a J^a = 0$$

• Construct constitutive relations based on symmetry

$$T^{ab} = \epsilon_0 u^a u^a + P_0 \Delta^{ab} + \Pi^{ab}; \quad J^a = \rho_0 u^a + \gamma^a$$

• Thermodynamic parameters $P_0(\mu, T), \epsilon_0(\mu, T), \rho_0(\mu, T)$ and $s_0(\mu, T)$ are equilibrium value of pressure, energy density, charge density and entropy density respectively.
Review Contd.

- Thermodynamics relation among them

\[ dP_0 = s_0dT + n_0d\mu \quad ; \quad \epsilon_0 = -P_0 + s_0T + \rho_0\mu \]

- Use second law of thermodynamics: Positive semi definite entropy production

\[ J^a_s = s_0u^a - \frac{\mu}{T}Y^a - \frac{u^a}{T}\Pi^{ab} + J^a_s \quad ; \quad \nabla_a J^a_s \geq 0 \]

- Fluid dynamics is a long wavelength effective description. it is meaningful to present constitutive relations in derivative expansion of the fluid variables.

- It would be useful write down all the quantity in terms of SO(2) symmetry group with respect to the fluid flow \( u^a \).
• Choice of frame: There exists arbitrariness in defining hydrodynamical variable. Some time it is useful to choose particular frame such that

\[ T_{ab} u^a = -\epsilon_0 u^b; \quad J_a u^a = -\rho_0 \]

This is called Landau frame. Above condition essentially transformed into the following equations

\[ \Pi_{ab} u^a = 0; \quad \mathcal{V}_a u^a = 0 \]

• To classify all the quantities we define transverse symmetric and anti symmetric tensor with respect to \( u^a \) as follows

\[ \Delta_{ab} = u_a u_b + g_{ab}; \quad \Sigma_{ab} = \epsilon_{abc} u^b \]
### Review Contd.

<table>
<thead>
<tr>
<th>classification</th>
<th>All data</th>
<th>Equations of motion</th>
<th>Independent data</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scalars</strong></td>
<td>( u^\mu \partial_\mu T ) ( u^\mu \nabla_\mu T \frac{\mu}{T} \partial_\mu u^\mu )</td>
<td>( u^\mu \nabla_\nu T^{\mu\nu} = u^\mu F^{\mu\nu} J_\nu ) ( \nabla_\mu J^\mu = 0 )</td>
<td>( S_1 = \partial_\mu u^\mu )</td>
</tr>
<tr>
<td><strong>Pseudo scalars</strong></td>
<td>( \Sigma^{ab} \nabla_a u_b ) ( \frac{1}{2} \Sigma^{ab} F_{ab} )</td>
<td>( \Omega = -\Sigma^{ab} \nabla_a u_b ) ( B = -\frac{1}{2} \Sigma_{ab} F_{ab} )</td>
<td></td>
</tr>
<tr>
<td><strong>Vectors</strong></td>
<td>( \Delta^\mu_\nu \nabla_\nu T ) ( u^\nu \nabla_\nu u^\mu ) ( \Delta^\mu_\nu \nabla_\nu \frac{\mu}{T} ) ( \Delta^\mu_\nu F^{\nu\rho} u_\rho )</td>
<td>( \Delta^\mu_\nu \nabla_\rho T^{\rho\nu} = \Delta^\mu_\nu F^{\rho\nu} J_\nu )</td>
<td>( V_1^\mu = -\Delta^\mu_\nu \nabla_\nu \frac{\mu}{T} + \frac{F^{\mu\nu} u_\nu}{T} ) ( V_2^\mu = \Delta^\mu_\nu \nabla_\nu T ) ( V_3^\mu = F^{\mu\nu} u_\nu = E^\mu )</td>
</tr>
<tr>
<td><strong>Pseudo Vectors</strong></td>
<td>( \Sigma^\mu_\nu \partial_\nu T ) ( \Sigma^\mu_\alpha u_\nu \partial_\nu u^a ) ( \Sigma^\mu_\nu \partial_\nu \frac{\mu}{T} ) ( \Sigma^\mu_\nu F^{\nu\rho} u_\rho )</td>
<td>( \Sigma^\mu_\nu \nabla_\rho T^{\rho\nu} = \Sigma^\mu_\nu F^{\rho\nu} J_\nu )</td>
<td>( \tilde{V}<em>1^\mu = \Sigma^\mu</em>\nu V_1^\nu ) ( \tilde{V}<em>2^\mu = \Sigma^\mu</em>\nu V_2^\nu ) ( \tilde{V}<em>3^\mu = \Sigma^\mu</em>\nu V_3^\nu = \tilde{E}^\mu )</td>
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<tr>
<td><strong>Tensors</strong></td>
<td>( \sigma_{\mu\nu} )</td>
<td></td>
<td>( T_1 = \sigma_{\mu\nu} )</td>
</tr>
<tr>
<td><strong>Pseudo Tensors</strong></td>
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<td></td>
<td>( \tilde{T}<em>1 = \tilde{\sigma}</em>{\mu\nu} )</td>
</tr>
</tbody>
</table>

\[
\sigma_{\mu\nu} = \frac{1}{2} \Delta^\mu_\alpha \Delta^{\nu\beta} (\nabla_\rho u_\beta + \nabla_\beta u_\rho - g_{\rho\beta} \nabla \cdot u) ; \quad \tilde{\sigma}_{\mu\nu} = -\frac{1}{2} (\Sigma_{\mu\rho} \sigma^{\rho\nu} + \Sigma_{\nu\rho} \sigma^{\rho\mu})
\]
So now we are ready to write down the most general entropy current, particle current and stress tensor in the Landau frame.

\[
J_s^a = s_0 u^a - \frac{\mu}{T} \mathcal{Y}^a + \mathcal{J}_s^a
\]

\[
\mathcal{J}_s^a = (a_0 S_1 + a_2 B + a_3 \Omega) u^a + \sum v_i V_{i\mu} + \sum \tilde{v}_i \tilde{V}_{i\mu}
\]

\[
\mathcal{Y}_a = \sum k_i V_{i\mu} + \sum \tilde{k}_i \tilde{V}_{i\mu}
\]

\[
\Pi_{ab} = (m_0 S_1 + m_2 B + m_3 \Omega) \Delta_{ab} + q \sigma_{ab} + \tilde{q} \tilde{\sigma}_{ab}
\]

Where all the coefficients are functions of \( \mu(x) \) and \( T(x) \).

Now we use \( \nabla_a J_s^a = 0 \)

One simple step: Purely second derivative terms lead to

\[
a_0 = v_1 = v_2 = v_3 = \tilde{v}_2 = 0
\]
Review Contd.: Final result

\[ T^{ab} = \epsilon_0 u^a u^b + (P_0 - \zeta \nabla \cdot u - \tilde{\xi}_B B - \tilde{\xi}_\Omega \Omega) \Delta^{ab} - \eta \sigma^{ab} - \tilde{\eta} \tilde{\sigma}^{ab}, \]
\[ J^a = \rho_0 u^a + \sigma V_1^a + \tilde{\sigma} \tilde{V}_1^a + \tilde{\xi}_E \tilde{E}^a + \tilde{\xi}_T \Sigma^{ab} \nabla_b T \]

- \( \zeta > 0, \eta > 0 \): Bulk and shear viscosity,
- \( \sigma > 0, \kappa = \sigma T > 0 \): Electrical and thermal conductivity:
- Parity violating transport coefficients are dissipationless.
- \( \tilde{\eta}, \tilde{\sigma} \): Hall viscosity and Hall conductivity
- All the other parity violating transport coefficients can be calculated from three independent thermodynamic functions.
- \( \tilde{\xi}_\Omega, \tilde{\xi}_T \): Curl Viscosity and Hall thermal conductivity
- \( \tilde{\xi}_E \) and \( \tilde{\xi}_B \): Anomalous Hall conductivity and Anomalous viscosity

We will holographically compute all these transport coefficients.
Holographic Set Up

- The system we are going to study

\[ \mathcal{L} = R - \frac{6}{L^2} - \frac{1}{4} F^2 + \frac{\lambda}{4} \theta \tilde{F} F - \frac{1}{2} (\partial \theta)^2 - V(\theta) - \frac{\lambda}{4} \theta \tilde{R} R. \]

where

\[ \tilde{R} R = \tilde{R}^M_N P^P R^N_{M P Q}, \]
\[ \tilde{F} F = \tilde{F}^M_N F^N_{M N}, \]

- Scalar potential

\[ V(\theta) = \frac{1}{2} m^2 \theta^2 + \frac{1}{4} c \theta^4. \]
• Find out the stationary charged black brane solution: Dual to an equilibrium field theory with finite temperature $T$ and finite chemical potential $\mu$ in the flat background.

$$ds_0^2 = 2H(r)dt dr - r^2 f(r) dt^2 + r^2 \delta_{ij} dx^i dx^j$$

$$\theta_0 = \theta_0(r), A_0 = -A(r) dt + A^e_\mu dx^\mu,$$

• Do a constant boost by $u^\mu$: Boosted black brane solution Still a solution

$$ds^2 = -2H(r, M, Q) u_\mu dx^\mu dr - r^2 f(r, M, Q) u_\mu u_\nu dx^\mu dx^\nu + r^2 \Delta_{\mu\nu} dx^\mu dx^\nu$$

$$\theta = \theta(r, M, Q) ; \ A = [A(r, M, Q) u_\mu + A^e_\mu] dx^\mu,$$
Strategy Cotd.

• we get 5 parameter family of solutions $u^\mu$, $M$ and $Q$; with $u^\mu u_\mu = -1$. These are identified with the uniform fluid velocity, temperature and chemical potential of the boundary fluid in equilibrium.

• Promotes these parameters to ‘Goldstone fields’ and determines the effective dynamics of these fields, order by order in the boundary derivative expansion.

• Such that locally the solution is approximated as black brane solution.

• Seen from inverse point of view, construction may be regarded as a map from solutions of the relativistic fluid dynamics equations to the space of long wavelength, locally black brane, solutions of gravity in AdS5 Fluid-gravity correspondence.
Strategy Cotd.

• At the origin of the boundary coordinates $x^\mu = 0$ in the co-moving frame, we have $u^\mu = (1, 0, 0)$.

\[
\begin{align*}
    u^\mu(x^\nu) & = (1, x^\nu \partial_\nu \beta^i) = (1, \delta \beta^i), \\
    M(x^\nu) & = M(x = 0) + x^\nu \partial_\nu M, \\
    Q(x^\nu) & = Q(x = 0) + x^\nu \partial_\nu Q, \\
    A_{\mu}^{ext} & = A_{\mu}^{ext}(x = 0) + x^\nu \partial_\nu A_{\mu}^{ext}.
\end{align*}
\]  

• This is no longer a solution of the Equation’s equation. So we have to solve for the correction order by order.

• Correction to the general nth order can be parametrized as

\[
\begin{align*}
    ds^{2(n)} & = r^2 k(r) dv^2 + 2H h(r) dvdr + 2r^2 j_i(r) dv dx^i - r^2 h(r) dx^i dx^i + r^2 \alpha_{ij}(r) dx^i dx^j \\
    A^{(n)} & = a_v(r) dv + a_i(r) dx^i ; \quad \theta^{(n)} = \varphi^1(r)
\end{align*}
\]
General feature of perturbation procedure

- This perturbation theory can be implemented at nth order provided the $u^\mu, T(x)$ and $\mu(x)$ obey

$$\nabla_a T_{n-1}^{ab} = F^{ba} J_{n-1}^a ; \quad \nabla_a J_{n-1}^a = 0$$

Where $T_{n-1}^{ab}$ and $J_{n-1}^a$ are boundary stress tensor and current density of (n - 1) order in derivatives.

- With the above condition a unique solution can be found with the regularity and normalizibility condition.

- Boundary stress tensor and Current density: **Conformal case**
  (Scalar field spontaneously breaks parity in the boundary)
  - solution is asymptotically locally AdS with scalar condensation

$$\langle T_a^a \rangle = 0 ; \quad \langle T_{ab} \rangle = \lim_{r \to \infty} r^3 g_{ab} ; \quad \langle J_a \rangle = \lim_{r \to \infty} \frac{1}{\sqrt{-g}} F^{ra}$$
General feature of the perturbation procedure

- Zero bulk, curl and magnetic ($\zeta, \tilde{\xi}_B, \tilde{\xi}_\Omega$) viscosity.
- Boundary stress tensor and Current density: **Non-conformal case**
  (Scalar field explicitly breaks parity on the boundary)
  - Source of the dual operator of the scalar field breaks conformal invariance

$$\lim_{r \to \infty} \theta(r) = \frac{\theta_0}{r^{(3-\Delta)}} + \frac{\langle O \rangle}{r^\Delta} ; \quad \Delta = \frac{1}{2} \left( 3 + \sqrt{4m^2 + 9} \right)$$

$$\langle T^a_a \rangle = -(3 - \Delta)\theta_0 \left( \langle O^{(1)}_b \rangle \nabla \cdot u + \langle O^{(1)}_\Omega \rangle \Omega + \langle O^{(1)}_B \rangle F_{xy}^{ext} \right)$$

$$\langle T_{ab} \rangle = \lim_{r \to \infty} r^3 g_{ab} ; \quad \langle J_a \rangle = \lim_{r \to \infty} \frac{1}{\sqrt{-g}} F^{ra}$$
Result so far

• Recall

\[ T^{ab} = \epsilon_0 u^a u^b + (P_0 - \zeta \nabla \cdot u - \tilde{\xi}_B B - \zeta_A \Omega) \Delta^{ab} - \eta \sigma^{ab} - \eta_A \tilde{\sigma}^{ab}, \]

\[ J^a = \rho_0 u^a + \sigma V_1^a + \tilde{\sigma} \tilde{V}_1^a + \tilde{\chi} \tilde{E}^a + \tilde{\kappa} \Sigma^{ab} \nabla_b T \]

• Analytic expression: Shere and Hall viscosity

Entropy density is given by

\[ s = \frac{r_H^2}{4G_N} \]

\[ \frac{\eta}{s} = \frac{1}{4\pi} ; \quad \frac{\tilde{\eta}}{s} = -\lambda \frac{r_H^2}{8\pi} \frac{f'\left(r_H\right) \theta'(r_H)}{H(r_H)^2} \]

• At least in two derivative gravity theory \( \frac{\eta}{s} \) seems universal.

• Membrane paradigm
Hall viscosity

Without Condensation

With Condensation

We take $m^2 = -2, c = 0.24$
Parity conserving Conductivity

- Analytic expression: Electrical and Thermal Conductivity

\[ \kappa = T \sigma \; ; \; \sigma = \frac{(M - 4r_H^3)^2}{9M^2} \; ; \; T = \frac{3r_H}{4\pi} (1 - \frac{Q^2}{3r_H^4}) \]

- We have done the perturbative analysis in small value of \( \theta \)
Parity violating conductivities: with scalar source

Hall Conductivity  Anomalous hall conductivity  Hall thermal conductivity
Parity violating conductivities: without scalar source

Hall Conductivity  Anomalous hall conductivity  Hall thermal conductivity
Bulk, Curl and Magnetic viscosity

Bulk Viscosity

Curl Viscosity

Anomalous Viscosity
Conclusions

- Fluid/gravity correspondence reveals a surprising connection between Gravity and Hydrodynamics (Probably expected)
- We provide an explicit model where all the first order transport coefficients can be calculated in strong coupling limit.
- There exist some thermodynamic relations among few transport coefficients. It would be interesting to check those relations from horizon thermodynamics.
- Obviously our result is not most general in the sense of anisotropic hydrodynamics. Recent interest in RICH physics, anisotropic superfluid, anisotropic fermi surface.
Thank You