Introduction to Matrix Models
and Non-Critical String Theory

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These lectures are meant to provide pedagogical introduction to continuum and matrix model formulation of non-critical string theory.

They typically describe strings in 1+0 or 1+1 dimensions with a linear dilaton. They are important because:

- Exactly Solvable
- Context to explore non-perturbative issues (open/closed duality)
- Appears to describe a sector of critical string theory
- Have many connections to topics in mathematics
The subject is reviewed extensively in the literature

- Brezin, Wadia (Collection of Reprints)
- Di Francesco, Ginsparg, Zinn-Justin
- Dijkgraaf
- Ginsparg and Moore
- Klebanov
- Morozov
- Seiberg, Shih

**Warning:** “These theories are exactly solvable. That does not necessarily mean we understand them.”
Plan

1) Review of Liouville Theory and Matrix Model

2) Methods of Orthogonal Polynomials and KdV heirarchy

3) Branes and open/closed string duality
Non-critical string theory

Bosonic string in $d$ dimensions

\[ Z = \int \frac{Dg \, DX}{\text{Vol}(\text{diff})} e^{-\frac{1}{4\pi \alpha'} \int d^2 \xi \sqrt{g} g^{\mu \nu} \partial_\mu X^i \partial_\nu X^i}, \quad i = 0 \ldots d - 1 \]

Measure:

\[ \int Dg e^{-\frac{1}{2} |\delta g^2|} = 1, \quad |\delta g^2| = \int d^2 \xi \sqrt{g} (g^{ac} g^{bd} - 2 g^{ab} g^{cd}) \delta g_{ab} \delta g_{cd} \]

\[ \int DX e^{-|\delta X^2|} = 1, \quad |\delta X^2| = \int d^2 \xi \sqrt{g} \delta X^i \delta X^i \]

Action and measure are diffeomorphism invariant
Gauge fix by choosing conformal gauge

\[ g = e^{\varphi \hat{g}(\tau)} \]

by introducing ghosts

\[ \int DbD\bar{b}DcD\bar{c} e^{-S_{gh}} \quad S_{gh} = \int d^2\xi \sqrt{g} b_{\bar{z}z} \nabla_{\bar{z}} {c^z} - b_{\bar{z}\bar{z}} \nabla_z c^{\bar{z}} \]

Under \( g \rightarrow e^{\sigma}g \), the action is invariant, but the measure transforms

\[ D_gXD_g(gh) \rightarrow D_{e^{\sigma}g}X D_{e^{\sigma}g}(gh) = e^{\frac{d-26}{24\pi\alpha'}} S_{L}(\sigma;g) D_gXD_g(gh) \]
\[ S_L = \int d\xi^2 \sqrt{g} \left( \frac{1}{2} g^{ab} \partial_a \sigma \partial_b \sigma + R \sigma + \mu e^\sigma \right) \]

This is the conformal anomaly which is cancelled by choosing \( d = 26 \)

Alternatively, make \( \phi \) dynamical

\[
\int D\phi e^{-\frac{1}{4\pi \alpha'} \int d^2 \xi \sqrt{\hat{g}} \left( \frac{25-d}{12} \hat{g}^{ab} \partial_a \phi \partial_b \phi + \frac{25-d}{6} \hat{R} \phi \right)}
\]

Then \( \delta \hat{g} = \epsilon(\xi) \hat{g}, \delta \phi = -\epsilon(\xi) \) is a symmetry
Rescale $\sqrt{\frac{25-d}{12}} \varphi \rightarrow \varphi$

$$\int D\varphi e^{-\frac{1}{4\pi\alpha'}} \int d^2\xi \sqrt{\hat{g}} \left( \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + Q \hat{R} \varphi \right)$$

where

$$Q = \sqrt{\frac{25 - d}{3}}$$

The stress tensor

$$T = -\frac{1}{2} \partial \varphi \partial \varphi + \frac{Q}{2} \partial^2 \varphi$$
\[ T(z)T(w) \sim \frac{1}{2(z-w)^4} + ..., \quad c = 1 + 3Q^2 \]

Finally, the \( \mu e^{\gamma \varphi} \) term adjusted so that the operator has dimension

\[ [e^{\gamma \varphi}] = -\frac{1}{2}\gamma(\gamma - Q) = 1 \]

So the bottom line

\[
\int D\varphi e^{-\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\hat{g}} (\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + Q\hat{R}\varphi + \mu e^{\gamma \varphi})} \\
\gamma = \frac{1}{\sqrt{12}}\left(\sqrt{25 - d} \pm \sqrt{1 - d}\right) = \frac{Q}{2} \pm \frac{1}{2}\sqrt{Q^2 - 8}
\]
pick branch

\[ \gamma = \frac{1}{\sqrt{12}}(\sqrt{25 - d} - \sqrt{1 - d}) = \frac{Q}{2} - \frac{1}{2} \sqrt{Q^2 - 8} \]

so that in \( d \to -\infty \) limit, \( \gamma \to 0 \), so that

\[ Q = \frac{2}{\gamma} - \gamma \to \frac{2}{\gamma}, \]

giving rise to “classical” Liouville theory

\[
S = \int d^2 \xi \sqrt{\hat{g}} \left( \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + Q \hat{R} \varphi + \frac{\mu}{\gamma^2} e^{\gamma \varphi} \right)
\]

which is Weyl invariant under \( \hat{g} \to e^{2\rho} \hat{g}, \varphi \to \varphi - \frac{2}{\gamma} \rho \)
These can be coupled to $d$ free bosons (or CFT with central charge $c$) so that

$$c_{\text{matter}} + c_{\text{liouv}} - 26 = 0$$

$c \leq 1$ and $c \geq 25$ picked out because $\gamma$ is real. At $c = 25$ the signature of $\varphi$ coordinate flips. We will focus on $c \leq 1$ for which $\varphi$ coordinate is Euclidean.

- $c < 1$: 1 linear dilaton dimension
- $c = 1$: 1 linear dilaton dimension, one flat direction
linear dilaton: c.f. NS5-brane

\[ \mu e^{\gamma \phi} \] marginal tachyon condensate

no \( X \) for \( c < 1 \), \( c = 1 \) is a barrier.
Other interesting example: \((p, q)\) Minimal CFT

\[
\begin{bmatrix}
\mathcal{O}_{11} & \cdots & \mathcal{O}_{1(p-1)} \\
\mathcal{O}_{21} & \mathcal{O}_{2(p-1)} \\
\vdots & \vdots \\
\mathcal{O}_{(q-1)1} & \cdots & \mathcal{O}_{(q-1)(p-1)}
\end{bmatrix}
\]

\[
c = 1 - \frac{6(p - q)^2}{pq}
\]

- \((p, q) = (2, 3)\): \(c = 0\) pure gravity
- \((p, q) = (3, 4)\): \(c = 1/2\) Ising Model
- \((p, q) = (2, 5)\): \(c = -22/5\) Yang-Lee
- Other \((p, q)\): critical limits of various statistical models
\[
[O_{r,s}] = \frac{(pr - qs)^2 - (p - q)^2}{4pq}
\]

- (3,2): \( (0 \ 0) \)

- (4,3): \( \begin{pmatrix} \frac{1}{2} & \frac{1}{16} & 0 \\ 0 & \frac{1}{16} & \frac{1}{2} \end{pmatrix} \)

- (5,2): \( (0 \ -\frac{1}{5} \ -\frac{1}{5} \ 0) \)
Gravitational dressing

\[ [e^{\alpha \varphi} \mathcal{O}] = 1 \]

\[ [\mathcal{O}] = \Delta_0, \quad [e^{\alpha \varphi}] = -\frac{1}{2} \alpha (\alpha - Q), \quad Q = \sqrt{\frac{25 - d}{3}} \]

so

\[ \alpha = \frac{1}{\sqrt{12}} \left[ \sqrt{25 - d} - \sqrt{1 - d + 24\Delta_0} \right] \]

\[ \alpha < \frac{Q}{2}, \quad \text{Seiberg Bound} \]
Some simple observables: String Susceptibility $\Gamma$

Let

$$Z(A) = \int D\varphi DX \ e^{-S} \delta \left( \int d^2 \xi \sqrt{\hat{g}} e^{\gamma \varphi} - A \right)$$

which for large $A$ scales as

$$Z(A) \sim A^{(\Gamma-2) \chi/2 - 1}$$

String susceptibility is this scaling exponent.

Simple scaling argument: $D\varphi$ invariant under

$$\varphi \rightarrow \varphi + \rho/\gamma$$
Under this transformation

\[
\frac{Q}{8\pi} \int d^2 \xi \sqrt{\hat{g}} \hat{R} \varphi \to \frac{Q}{8\pi} \int d^2 \xi \sqrt{\hat{g}} \hat{R} \varphi + \frac{Q}{8\pi \gamma} d^2 \xi \sqrt{\hat{g}} \hat{R}
\]

so

\[
Z[A] = \int D\varphi DX e^{-S} e^{\frac{Q\rho \chi}{2\gamma}} \delta \left( \int d^2 \xi \sqrt{\hat{g}} e^{\gamma \varphi + \rho} - A \right)
\]

\[
= \int D\varphi DX e^{-S} e^{\frac{Q\rho \chi}{2\gamma} - \rho} \delta \left( \int d^2 \xi \sqrt{\hat{g}} e^{\gamma \varphi} - e^{-\rho} A \right)
\]

\[
= e^{\frac{Q\rho \chi}{2\gamma} - \rho} Z[e^{-\rho} A] = A^{-\frac{Q\chi}{2\gamma - 1}} Z[1], \quad \text{and so}
\]

\[
\Gamma = 2 - \frac{Q}{\gamma} = \frac{1}{12} \left[ (d - 1) - \sqrt{(d - 25)(d - 1)} \right]
\]
Along similar lines, one can define scaling dimension $\Delta_{r,s}$

$$\frac{1}{Z[A]} \int D\varphi DX e^{-S} \delta \left( \int d^2 \xi \sqrt{\hat{g}} e^{\gamma \varphi} - A \right) \int d^2 \xi \sqrt{\hat{g}} e^{\alpha_{r,s} \varphi} \mathcal{O}_{r,s}$$

For $\Delta_0 = \frac{(pr-qs)^2-(p-q)^2}{4pq}$ and $c = 1 - \frac{6(p-q)^2}{pq}$,

$$\alpha = 1 - \Delta_{rs} = \frac{p - q - |pr - qs|}{2q}$$

$$\gamma = 1 - \Delta_{rs} = \frac{p - q - |pr - qs|}{2q}$$
Physical observables are correlation functions of BRST cohomology

For $\mathcal{L} \otimes M(p, q)$,

$$H = \{O_n e^{\alpha_n \phi}\}, \quad O_n : \text{matter, ghosts, Liouville}$$

$$\frac{\alpha_n}{\gamma} = \frac{p + q - n}{2q}, \quad n > 1, \neq 0 \mod q$$

BRST cohomology has infinite element even though number of primary matter fields were finite.
Matrix Model

Theory of random matrices: (applications in disordered systems and quantum chaos)

\[ Z = \int dM e^{-\text{Tr}V(M)} \]

\( M \) is hermitian \( N \times N \) matrix. Measure

\[ DM = \prod_i dM_{ii} \prod_{i<j} dM_{ij} dM_{ij}^* \]

invariant under \( M \rightarrow UMU^\dagger \).
Parametrizing

\[ M = U^\dagger \Lambda U, \quad dM = dU \prod_i d\lambda_i \Delta(\lambda)^2 \]

where

Vandermonde: \[ \Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \]
One way to see this. Let $dU = idTU$. Line element:

$$\text{Tr} \, dM^2 = \sum_i dM_{ii}^2 + \sum_{i<j} dM_{ij}dM_{ij}$$

$$= \text{Tr} \left( U^\dagger (d\Lambda + i[\Lambda, dT])U \right)^2$$

$$= \sum_i d\lambda_i^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2 |dT_{ij}|^2$$

analogue of $dx^2 + dy^2 = dr^2 + r^2 d\Omega^2$. 
Simplest Matrix Model: Gaussian ensemble

\[ Z = \int dM e^{-\text{Tr}M^2} = \int dm_i \prod_{i<j} (m_i - m_j)^2 e^{-\sum_i m_i^2} \]

\[ = \int dm_i e^{-\sum_i m_i^2 + 2 \log(m_i - m_j)} \]

Ensemble of \( N \) particles in harmonic oscillator potential with logarithmic repulsion
Large $N$ limit described in terms of the eigenvalue density $\rho(m)$.

\[ S = - \int dm \, m^2 \rho(m) + \int dm \, dm' \, \log(m - m') \rho(m) \rho(m') \]

Equation of motion

\[ \delta S = -m^2 + 2 \int dm' \, \log(m - m') \rho(m') = 0 \]

and differentiating with respect to $m$,

\[ m = \int dm' \frac{1}{m - m'} \rho(m') \]
Solution:

\[ \rho(m) = \frac{1}{\pi} \sqrt{2N - m^2} \]

Wigner semi-circule distribution
One can also consider more general potential (BIPZ)

e.g. \[ V(M) = \frac{1}{2}M^2 + gM^4 \]

so that equation of motion becomes

\[ \frac{1}{2} \lambda + 2g\lambda^3 = \int d\lambda' \frac{1}{\lambda - \lambda'} \rho(\lambda') \]

solved by

\[ \rho(\lambda) = \frac{1}{\pi} \left( \frac{1}{2} + 4ga^2 + \frac{2g\lambda^2}{N} \right) \sqrt{4a^2N - \lambda^2}, \quad 12ga^4 + a^2 - 1 = 0 \]
Such generalization is interesting because the matrix action

\[ \int dM e^{-\text{Tr}\frac{M^2}{2} + g\frac{M^3}{3!}} \]

can be represented in Feynman expansion

\( \leftrightarrow \) triangulation of 2D surface

\( 1/N \) expansion \( \leftrightarrow \) genus expansion

\( g \rightarrow g_c \leftrightarrow \) continuum limit
Resolvent: useful computational tool

\[ Z = \int dM e^{-NV(M)} \]

Change integration variable \( M \rightarrow M + \frac{1}{M-z} \) Under this transformation

\[ \delta dM = -\text{Tr} \left( \frac{1}{M-z} \right) \text{Tr} \left( \frac{1}{M-z} \right) dM \]

\[ \delta e^{NV(M)} = -\frac{NV'(M)}{M-z} \]
So we arrive at an identity

\[ \langle \text{Tr} \left( \frac{1}{M - z} \right) \text{Tr} \left( \frac{1}{M - z} \right) \rangle + \frac{NV'(M)}{M - z} = 0 \]

In the large \( N \) limit, one can factorize

\[ \langle \text{Tr} \left( \frac{1}{M - z} \right) \rangle \langle \text{Tr} \left( \frac{1}{M - z} \right) \rangle + \frac{NV'(M)}{M - z} = 0 \]

Schwinger-Dyson equation. A little rewriting:

\[ \langle \text{Tr} \left( \frac{1}{M - z} \right) \rangle \langle \text{Tr} \left( \frac{1}{M - z} \right) \rangle + NV'(z)\frac{1}{M - z} = -\frac{NV'(M) - NV'(z)}{M - z} \]

\[ = N^2 f(z), \text{ Polynomial order } p - 2 \]
Let’s denote

\[ R(z) = \frac{1}{N} \text{Tr} \left( \frac{1}{M - z} \right) \]

Then

\[ R(z)^2 + R(z)V'(z) = f(z) \]

For the simple case of \( V(M) = M^2/2 \),

\[ R^2 + zR(z) = c \]

or

\[ R(z) = \frac{-z + \sqrt{4c + z^2}}{2} = \frac{-z + \sqrt{z^2 - 4}}{2} \]
• Constant $c$ fixed by requirement that
$$R(z) = \frac{1}{z} + O(z^{-2})$$

• $R(z)$ imaginary for $-2 < z < 2$.

$$\rho(z) = \frac{1}{2\pi} \sqrt{4 - z^2}$$
For the interacting cubic theory

\[ V = \frac{M^2}{2} + gM^3 \]

one has

\[ R(z) = \frac{-V'(z) + \sqrt{V'(z)^2 + 4f}}{2} \]

\[ f = cz + d, \text{ and the discriminant is quartic in } z. \text{ Arrange } f \text{ so that discriminant has two real roots and one double root.} \]
With this ansatz, one has

\[ R(z) = -\frac{z + 3gz^2}{2} + \frac{(1 + 3g(a + b) + 3gz) \sqrt{(z - 2a)(z - 2b)}}{2} \]

where

\[ 3g(a - b)^2 + 2(a + b)(1 + 3g(a + b)) = 0, \quad (b - a)^2(1 + 6g(a + b)) = 4 \]

so that

\[ R(z) = \frac{1}{z} + O(z^{-2}) \]

This determines the eigenvalue distribution \( \rho(z) \) and it solves the equation of motion.
\[ \mathcal{F}^{(0)} = \int_{2a}^{2b} d\lambda \rho(\lambda) \left( \frac{1}{2} \lambda^2 + g\lambda^3 \right) - \frac{1}{2} \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \ln(\lambda - \lambda') \]

\[ = -\frac{1}{3} \frac{\sigma(3\sigma^2 + 6\sigma + 2)}{\left(1 + \sigma\right)\left(1 + 2\sigma\right)^2} + \frac{1}{2} \ln(1 + 2\sigma) \]

where

\[ \sigma = 3g(a + b) \]

is a solution of

\[ 18g^2 - \sigma(1 + \sigma)(1 + 2\sigma) \]

with a bit more massaging

\[ \mathcal{F}^{(0)} = -\sum_k \frac{1}{2(k + 2)!} \frac{\Gamma(3k/2)}{\Gamma(k/2 + 1)} \approx \sqrt{\frac{2}{3\pi k^7}} \left(108\sqrt{3g^2}\right)^{2k} \]
Now, series

\[ \sum k^{\gamma - 3} \left( \frac{g}{g_c} \right)^{2k} = (g_c - g)^{2 - \gamma} \]

The area scales like

\[ A = \frac{g \frac{\partial}{\partial g} F^{(0)}}{F^{(0)}} \sim \frac{1}{g_c - g} \]

so

\[ Z \sim A^{\gamma - 2} \]

and we can read off

\[ \gamma = -\frac{7}{2} + 3 = -\frac{1}{2} \]
in agreement with

\[
\Gamma = 2 - \frac{Q}{\gamma} = \frac{1}{12} \left[ (d - 1) - \sqrt{(d - 25)(d - 1)} \right]
\]

for \( d = 0 \).
All this was for pure 2D gravity \((c = 0)\). What if we are interested in adding conformal matter?

Consider multi-matrix model

\[
\text{Tr} \left( \frac{1}{2} A^2 + \frac{1}{2} B^2 + gA^3 + gB^3 + cAB \right)
\]

Lasing model \((c = 1/2\ theory)\)
Can also consider

\[ \int dt \frac{1}{2} (\partial_t M)^2 + \frac{1}{2} M^2 + g M^3 \quad (c = 1) \]

Also consider adding higher order terms in the potential

\[ V(M) = \frac{1}{2} M^2 + g_3 M^3 + g_4 M^4 + ... \]

- Different discretization generically lead to same continuum limit (universality)
- Fine tuning coupling gives rise to new critical behavior (multi-criticality)
Double Scaling Limit

If one also compute $1/N$ corrections,

$$\mathcal{F} \sim \sum_{\chi} \sum_{n} N^{\chi} n^{(\Gamma-2)\chi/2-1} (g/g_c)^n \sim (g_c - g)^{(2-\Gamma)\chi/2} N^{\chi}$$

so scale $g \rightarrow g_c$ keeping

$$\kappa = (g_c - g)^{(2-\Gamma)/2} N = \text{fixed}$$

Then

$$\mathcal{F} = \kappa^2 \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \kappa^{-2} \mathcal{F}^{(2)} + \ldots$$
Method of Orthogonal Polynomials

\[ Z = \int dMe^{-\text{Tr}V(M)} = \prod_{i=1}^{N} d\lambda \Delta(\lambda)^2 e^{-\sum_i V(\lambda_i)} \]

Because of anti-symmetry

\[ \Delta(\lambda) = \det(\lambda_i^{j-1}) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{N-1} & \lambda_{N-1}^2 & \cdots & \lambda_{N-1}^{N-1} \end{pmatrix} \]
Now, define a set of polynomials

$$P_n(\lambda) = \lambda^n + \ldots$$

satisfying orthogonality relation

$$\int_{-\infty}^{\infty} e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{mn}$$

Then,

$$\Delta(\lambda) = \det(\lambda_i^{j-1}) = \det(P_{j-1}(\lambda_i))$$
and

\[ Z = N! \prod_{i=0}^{N-1} h_i = N!h_0^N \prod_{i=1}^{N-1} f_k^{N-k}, \quad f_k = \frac{h_k}{h_{k-1}} \]

Finding \( P_n \) (and therefore \( h_n \)) is a finite procedure, so \( Z \) can be computed exactly

If interest is in large \( N \) limit,

\[ \frac{1}{N^2} \mathcal{F} = \frac{1}{N} \sum \left( 1 - \frac{k}{N} \right) \ln f_n \sim \int_0^1 d\xi (1 - \xi) \ln(f(\xi)) \]

where \( \xi = k/N \).
Need $f_n$ for large $n$.

This can be gotten from studying the recursion relation

$$\lambda P_n(\lambda) = \sum_{i=0}^{n+1} c_i P_i(\lambda)$$

but in fact

$$\lambda P_n(\lambda) = P_{n+1} + r_n P_{n-1}$$

because

$$\int \lambda P_n(\lambda) P_i(\lambda) e^{-V(\lambda)} = 0$$

for $i < n - 1$. 
Now,

\[
\int e^{-V(\lambda)} (P_n(\lambda)\lambda) P_{n-1}(\lambda) = r_n h_{n-1}
\]

\[
= \int e^{-V(\lambda)} P_n(\lambda)(\lambda P_{n-1}) = h_n
\]

so

\[
r_n = \frac{h_n}{h_{n-1}} = f_n
\]
Similarly,

\[ \int e^{-V}(P_n'(\lambda)\lambda)P_n(\lambda) = \int e^{-V}(nP_n(\lambda) + \ldots)P_n(\lambda) = nh_n \]

\[ = \int e^{-V}P_n'(\lambda)(\lambda P_n(\lambda)) = \int e^{-V}P_n'(\lambda)r_nP_{n-1}(\lambda) \]

\[ = -\int P_n(\lambda)(r_n e^{-V}P_{n-1})' \]

\[ = \int r_nP_n(\lambda)P_{n-1}(\lambda)e^{-V}V'(\lambda) \]

\[ - \int P_n(\lambda)r_n e^{-V}P'_{n-1} \]

so

\[ nh_n = r_n \int e^{-V}V'(\lambda)P_n(\lambda)P_{n-1}(\lambda) \]
To apply these structures, consider for simplicity a potential with even terms only

\[ V = \frac{1}{2g} \left( \lambda^2 + \frac{\lambda^4}{N} + b\frac{\lambda^6}{N^2} \right) \]

Then

\[ gV' = \lambda + \frac{2\lambda^3}{N} + 3b\frac{\lambda^5}{N^2} \]

Insert this in

\[ nh_n = r_n \int e^{-V} V'(\lambda) P_n(\lambda) P_{n-1}(\lambda) \]
\[ \lambda : P_n(\lambda) \leftrightarrow P_{n+1} \]

\[ r_n P_{n-1} \]

so

\[ gn = r_n + \frac{2}{N}r_n(r_{n-1} + r_n + r_{n+1}) + \frac{3b}{n^2} \text{ (10 terms)} \]

In the large \( n \) limit,

\[ \xi = \frac{n}{N}, \quad r(\xi) = \frac{r_n}{N} \]
and

\[ g\xi = r + 6r^2 + 30br^3 \equiv W(r) \]

In general, if

\[ V(\lambda) = \frac{1}{2g}a_p\lambda^{2p} \]

then

\[ W(r) = a_p \frac{(2p - 1)!}{(p - 1)!^2}r^p \]
For generic $W(r)$,

$$g\xi = W(r) = g_c + \frac{1}{2}W''(r_c)(r - r_c)^2 + ...$$

Then $r - r_c \sim (g_c - \xi g)^{-\Gamma}$ with $\Gamma = -1/2$ so $r = r_c + (g_c - \xi g)^{-\Gamma}$, so that

$$\frac{1}{N^2} F = \int_0^1 d\xi (1 - \xi) f(\xi)$$

$$\sim \int_0^1 d\xi \,(1 - \xi)(g_c - \xi g)^{-\Gamma} \sim (g_c - g)^{-\Gamma+2}$$

Agree with $\Gamma$ for pure gravity computed earlier.
Multi-criticality

In general

\[ W(r) = g_c + c(r - r_c)^2 \rightarrow \Gamma = -\frac{1}{2} \]
\[ W(r) = g_c + c(r - r_c)^3 \rightarrow \Gamma = -\frac{1}{3} \]
\[ \vdots \]
\[ W(r) = g_c + c(r - r_c)^m \rightarrow \Gamma = -\frac{1}{m} \]

\[ \Gamma = 2 - \frac{Q}{\gamma_{min}} = \frac{2}{1 - p - q}, \quad (p, q) = (2l + 1, 2) \]
Now, if one is interested in higher-genus contributions,

\[ g\xi = W(r) + 2r(\xi)(r(\xi + \epsilon) + r(\xi - \epsilon) - 2r(\xi)) \]

where \( \epsilon = 1/N \). Now scale

\[
\begin{align*}
g_c - \xi g &= a^2 z \\
r - r_c &= au(z) \\
N &= a^{-5/2} \\
g - g_c &= \kappa^{-4/5} a^2 \\
\end{align*}
\]

\[ \Rightarrow z = u(z)^2 - \frac{1}{3} u''(z), \quad u(\kappa^{-4/5}) = Z''(\kappa^{-4/5}) \]
This is the KdV equation.

Can be solved perturbatively

\[ u = z^{1/2} \left( 1 - \sum_{k} u_k z^{-5/2k} \right) \]

\[ = z^{1/2} \left( 1 - \frac{1}{24} z^{-5/2} - \frac{49}{1152} z^{-5} - \frac{1225}{6912} z^{-15/2} + \ldots \right) \]

and computes the genus expansion of \( Z \)
Summary of Orthogonal Polynomials

\[ Z = \int dM e^{-\text{Tr}V(M)} = \int \prod_{i=1}^{N} d\lambda \Delta(\lambda) e^{-\sum_i V(\lambda_i) \Delta(\lambda)} \]

\[ |N\rangle = \prod_{i=0}^{N-1} \text{det}(P_{j-1}(\lambda_i)) \]

\[ Z = \langle N | S | N \rangle \]

\[ S = S_{nm} b_n^\dagger b_m, \quad S_{mn} = \delta_{mn} h_n \]
KdV Heirarchy

Normalize $\Pi_n$ so that

$$\int d\lambda e^{-V} \Pi_n \Pi_m = \delta_{nm}$$

and define $Q_{nm}$ by

$$\lambda \Pi_n = \sqrt{\frac{h_{n+1}}{h_n}} \Pi_n + r_n \sqrt{\frac{h_{n-1}}{h_n}} \Pi_{n-1}$$

$$= \sqrt{r_n + 1} \Pi_{n+1} + \sqrt{r_n} \Pi_{n-1} \equiv Q_{nm} \Pi_m$$

$$Q_{nm} = Q_{mn}$$
Along similar lines, define

\[
\frac{\partial}{\partial \lambda} \Pi_n = A_{nm} \Pi_m
\]

which has \([Q, A]\) by definitnion. No particular symmetry

\[
0 = \int d\lambda \frac{\partial}{\partial \lambda} \Pi_n \Pi_m e^{-V} = (A_{nm} + A_{mn} - V') \Pi_n \Pi_m e^{-V}
\]

\[
\Rightarrow A + A^T = V'(Q)
\]

\[
P = A - \frac{1}{2} V'(Q) = \frac{1}{2} (A - A^T) \text{ is antisymmetric}
\]

\[
[P, Q] = 1
\]
In the double scaling limit, $Q_{nm}$ becomes a differential operator.

Anticipate scaling

$$r(\xi) = r_c + a^2 u(z)$$

Then

$$Q = 2r_c^{1/2} + \frac{a^2}{r_c^{1/2}}(u + r_c\kappa^2 \partial_z^2) \sim d^2 + u$$

$$P = d^3 + \frac{3}{4}\{u, d\} \quad \text{Cubic in } d$$
\[ 1 = [P, Q] = \left( \frac{3}{4}u^2 + \frac{1}{4}u'' \right)' \Rightarrow \text{KdV} \]

(2,3) model:

\[ P = (Q^{3/2})_+ \]
Orthogonal Polynomials, Lax Pairs, etc generalizes Multi-matrix model

\[ Z = \int \prod_{a=1}^{n} dM_a \exp \left[ - \text{Tr} V_a(M_a) + c_a M_a M_{a+1} \right] \]

are also solvable.
Key identity: Itzykson-Zuber integral

\[ \int dA e^{\text{Tr} V(A) + cAB} = \int da_i \frac{\Delta(a)}{\Delta(b)} e^{-\sum_i V(a_i) + ca_i b_i} \]

then

\[ Z = \int \prod_{a=1}^{n} dM_a \exp \left[ -\text{Tr} V_a(M_a) + c_a M_a M_{a+1} \right] \]

\[ = \int \prod_{a=1}^{n} d\lambda_a \Delta(\lambda_1) e^{-S(\lambda_a)} \Delta(\lambda_n) \]
Define biorthogonal polynomials

\[
\int \prod_{a=1}^{n} \Pi_i(\lambda_1) \tilde{\Pi}_j(\lambda_n) e^{-V_a(\lambda_a) + c_a \lambda_a \lambda_{a+1}} = \delta_{nm}
\]

from which one derives

\[
Q = d^q + \{v_{q-2}(z), d^{q-2}\} + \{v_{q-4}(z), d^{q-4}\} + \ldots v_0(z)
\]

and adjust \(V\)'s such that

\[
P = (Q)^{p/q}_+
\]
\[ P = (Q^{p/q})_+, \quad [P, Q] = 1 \]
defines differential equation for \( u(z) \)

One can also turn on “coupling” \( t_n \)

\[ P \rightarrow P + \frac{1}{q} \sum_n n t_n (Q^{n/q-1})_+ \]

Generalized KdV flow equation

\[ \frac{\partial}{\partial t_n} Q = [(Q^{n/q})_+, Q] \]
Solve for $u(z, t_i) = \mathcal{F}''(z, t_i)$

$$\tau = \mathcal{Z} = e^{-\mathcal{F}}$$

is called the $\tau$-function: compute correlators

Expectation value of generic single trace operator

$$\text{Tr} f(M)$$

computes insertion of integrated lowest dimension operator. Fine tune for higher dimension operators
\[
\frac{\partial}{\partial t_n} Q = [(Q^{n/q})_+, Q]
\]

\[
\alpha_n \sim \frac{p + q - n}{q}
\]

\[
\gamma \sim \alpha_{p-1,q-1} = 2
\]

\[
\Rightarrow \quad \frac{\alpha_n}{\gamma} = \frac{p + q - n}{2q}
\]

compare with BRST cohomology
Alternative Matrix formulation of KdV flow

\[ e^{\mathcal{F}(\Lambda)} = \frac{\int dM \exp \left[ -\text{Tr} \frac{1}{2} \Lambda M^2 + i \frac{M^3}{6} \right]}{\int dM \exp \left[ -\text{Tr} \frac{1}{2} \Lambda M^2 \right]} \]

define

\[ t_i(\Lambda) = -(2i - 1)!! \text{Tr} \Lambda^{-2i-1} \]

Expand in small \( t_i \)

\[ \ln \tau = \mathcal{F} = \frac{t_0^3}{6} + \frac{t_1}{24} + \frac{t_0^3 t_1}{6} + \frac{1}{24} t_0 t_2 + \frac{t_1^2}{48} + \ldots \]
So we have

Double Scaled Matrix Model (gauge theory)

↕

Non-critical string theory (gravity theory)

↕

Kontsevich Matrix Model (gauge theory)

Can they be thought of as analogues of AdS/CFT in any way?
Think about D-branes

- Matrix point of view

\[ \frac{1}{M} \text{Tr} \Phi^M \]
In double scaling limit

\[
\frac{1}{M} \Phi^M = \frac{1}{M} \left( 2r_c + \frac{a^2}{\sqrt{r_c}}Q \right)^M
\]

scale

\[M = \frac{2r_c \ell}{a^2}\]

Then

\[
\frac{1}{M} \Phi^M = \frac{1}{\ell} e^{\ell Q} \leftarrow \frac{1}{L} \text{Tr} e^{L\Phi}
\]

Laplace transform

\[
\int dL \, e^{-xL} \frac{1}{L} \text{Tr} e^{L\Phi} = \text{Tr} \log(x - \Phi)
\]
Differentiate wrt $x$

$$R(x) = \text{Tr} \frac{1}{x - \Phi}$$

is the resolvent

Interpret as insertion of boundary cosmological constant

$$\int d^2 \xi \sqrt{\hat{g}} \left( \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + Q \hat{R} \varphi + \mu e^{\gamma \varphi} \right) + \int_{\partial \Sigma} K \varphi + \mu_B e^{\gamma \varphi/2}$$

These are branes considered by FZZ/T
What does the resolvent measure?

\[ R(x) = \text{Tr} \frac{1}{\Phi - x} = \sum_i \frac{1}{\lambda_i - x} \]

force due to log interaction with other Eigenvalues

\[ y \equiv V'(x) + 2R(x) = \text{Effective Force} \]
\[ \int y \, dx = \text{Effective Potential} \]

\[ R(x) = \frac{-V'(x) + \sqrt{V'(x)^2 + 4f}}{2} \]
For example for cubic potential theory

\[ y^2 = (1 \text{ double root and pair of single root}) \]

In the continuum limit \( g^2 \to \frac{1}{108\sqrt{3}} \) the double root approach the cut
In this limit, one obtains

\[ T_2(y) = 2y^2 - 1 = x(4x^2 - 3) = T_3(x) \]

for the (2,3) model. Along similar lines, \((2, 2l + 1)\) model gives rise to a cut and \(l\) stationary points

\[ T_2(y) = T_{2l+1}(x) \]
One arrives at a following global picture of 1-point function of FZZT brane
CFT side: one has the Liouville Boundary State

\[ |\mu_B\rangle = \frac{\Gamma(1 + 2iPb)\Gamma(1 + 2iP/b) \cos(2\pi\sigma P)}{2^{1/4}(-2i\pi P)} \mu^{-iPb} |P\rangle \]

\[ \frac{\mu_B}{\sqrt{\mu}} = \cosh \pi b\sigma, \quad b^2 = \frac{q}{p} \]

These branes are semi-localized

\[ \Psi(\varphi) = \langle \varphi | \mu_B \rangle = e^{-\mu_B e^{b\varphi}} \]
CFT and Matrix Model agree e.g. annulus

\[ Z = \int d\tau Z_{\text{ghost}} Z_{\text{Liouville}} Z_{\text{matter}} \]

\[ Z_{\text{Liouville}}(\tau) = \langle l_1 | e^{-\tau(L_0+\bar{L}_0)} | l_2 \rangle \]

\[ Z(l_1, l_2) \sim \sum_{k=1} k \sin(\pi k/q) K_{\frac{k}{q}}(l_1) I_{\frac{k}{q}}(l_2) \]

Small \( l_2 \) limit: loops = \( \sum \) BRST cohomology
FZZT expectation value probe target space (as function of $\mu_B$)
This picture is strictly perturbative

Nice geometrical picture \hspace{1cm} (Seiberg-Shih)

- tachyon backgrounds deforming the Reimann surface preserving the singularity
- adding ZZ-brane opens the root into a cut

Ignores non-perturbative effect such as tunneling of eigenvalues (ZZ-branes)
Non-perturbatively

\[ W(x) = \log(\Phi - x) \]

\[ \langle e^{W(x)} \rangle = \det(\Phi - x) \]

Equivalent to adding fundamental matter

\[ \int d\bar{\chi}d\chi e^{\bar{\chi}(\Phi - x)\chi} \]
To be concrete, pick a simple model: Gaussian potential

$$\langle e^{W(x)} \rangle = \int d\Phi d\bar{\chi} d\chi \ e^{-\frac{1}{2g} \Phi^2 + \bar{\chi}(x-\Phi)\chi}$$
\[ \langle e^{W(x)} \rangle = \int d\Phi d\bar{\chi} d\chi \ e^{\frac{1}{2} \Phi^2 + \bar{\chi}(x - \Phi)\chi} \]
\[ = \int d\bar{\chi} d\chi \ e^{x\bar{\chi}\chi - \frac{g}{2}(\bar{\chi}\chi)^2} \]
\[ = \frac{1}{2\pi g} \int d\bar{\chi} d\chi ds \ e^{x\bar{\chi}\chi - \frac{1}{2g} s^2 + is\bar{\chi}\chi} \]
\[ = \frac{1}{2\pi g} \int ds \ (x + is)^N e^{-\frac{1}{2g} s^2} \]
\[ = \left( \frac{g}{2} \right)^{N/2} H_N(\frac{x}{\sqrt{2g}}) \]
Hermite polynomial is an orthogonal polynomial for Gaussian measure

\[
\det(\Phi - x) = \frac{\det_{ij}(\lambda_j^{i-1})}{\Delta(\lambda)} = \frac{\det_{ij}(P_{i-1}(\lambda_j))}{\Delta(\lambda)}
\]

where \( i = 1..(N + 1) \), \( \lambda_N = x \) So

\[
\langle \det(\Phi - x) \rangle = \int \prod d\lambda \det_{ij}(\lambda_j^{i-1}) \Delta(\lambda) e^{-\frac{1}{2g}\lambda^2} = P_N(x)
\]

FZZT is probing the wavefunction of fermion at the top of fermi-surface
Recursion relation

\[ \lambda P_n(\lambda) = \sqrt{r_{n+1}} P_{n+1}(\lambda) + \sqrt{r_n} P_{n-1} \]

asymptotes to

\[ Q\psi(z, \lambda) = \left( \frac{\partial^2}{\partial z^2} - z \right) \psi(z, \lambda) = \lambda \psi(z, \lambda) \]

in the double scaling limit.

Baker-Akheizer function
Go back to

\[ \frac{1}{2\pi g} \int ds \ (x + is)^N e^{-\frac{1}{2g}s^2} = \frac{1}{2\pi g} \int ds \ e^{-\frac{1}{2g}s^2 + N \log(x+is)} \]

and scale

\[ g = \epsilon^3, \quad N = \epsilon^{-3}, \quad s = i + \epsilon \tilde{s}, \quad x = 2 + \epsilon^2 \tilde{x} \]

\[ \frac{1}{2\pi} \int d\tilde{s} \ e^{-i \left( \frac{\tilde{s}^3}{3} + \tilde{s} \tilde{x} \right)} = Ai(\tilde{x}) \]

• This is the famous Airy function

• This is the famous Kontsevich 1 × 1 matrix model
Multi-FZZT amplitude generalizes this to the matrix Airy integral

\[ \langle \prod_{a=1}^{n} \det(\Phi - x_a) \rangle = \int dS e^{-i \text{Tr} \left( \frac{S^3}{3} + SX \right)} \]

Gaussian matrix model corresponds to \((p, q) = (2, 1)\).

No conformal content: topological gravity. \(c = -2\)
• Gaiotto and Rastelli: **Open SFT of FZZT in topological (2,1) theory** is the Kontsevich matrix integral

Rank $n$ of OSFT is precisely the number of FZZT branes (not $N$)

• Kontsevich: This integral computes **topological closed string amplitudes**

This is AdS/CFT correspondence

OSFT is simple because the theory was topological (much like the duality of Gopakumar-Vafa)
• Airy function: Non-perturbative FZZT amplitude
• The function is entire
• Multi-sheeted structure of FZZT moduli-space is lost at the non-perturbative level

How did this happen?
Stoke’s phenomenon

\[ Ai(\tilde{x}) = \frac{1}{2\pi} \int d\tilde{s} \ e^{-i\left(\frac{\tilde{s}^3}{3}+\tilde{s}\tilde{x}\right)} \]

solution of

\[ \left( \frac{\partial^2}{\partial z^2} - z \right) f(z) = 0 \]

Two solutions: \( Ai(z) \) and \( Bi(z) \). Different \( s \) contour: different linear combination of homogeneous solution.

Pick the solution which gives rise to \( Ai(z) \) (decay for positive real \( z \))
Airy integral has three saddle points

Steepest descent contour hits only one of the saddles

But as $z$ is taken off axis, different saddle points appear and disappear (along the steepest descent contour)

The locus on parameter space where contributing saddles re-arrange themselves is called “stoke’s line”
The branch cut is lost behind the Stoke’s line

Matrix model is powerful enough to address these non-perturbative issues
• Perturbatively, many possible vacua

\[ \times \quad \times \quad \times \quad \times \quad \times \quad \times \]

• non-perturbative FZZT calculation is blind to this, except
• # of stationary point must be even for wave function to decay properly
• \((2, 2l + 1)\) model is well defined non-perturbatively only for \(l\) even.
What are the analogues of all these ideas for $c = 1$ or $\hat{c} = 1$.

What are the analogues of all these ideas for $(p, q)$
Open SFT

\[ S = \int \Psi \ast Q \Psi + \Psi \ast \Psi \ast \Psi \]

Chern-Simions

\[ \int AdA + \frac{2}{3} A^3 \]
back