Theory of critical points between symmetry protected topological states

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Outline

• Introduction.

• Relation between edge theory and bulk critical theory.

• Summary
Symmetry and phases

• Our understandings of phases are mainly based on the “Landau’s paradigm”
  – Phases are described by their symmetry (or spontaneous-symmetry-breaking order)

L. D. Landau

crystal
ferromagnet
superfluid
Symmetry and phases

- Phases are separated by spontaneous-symmetry-breaking phase transitions

- Local order parameters can be measured
  - E.g. Bragg peaks in X-ray diffraction

- Group theory classifies phases
  - E.g. 230 space groups of 3D crystals
Symmetry and phases

• Physics in the “Landau’s paradigm”
  – Low energy physics and phase transition are described by Ginzburg-Landau theory in terms of local order parameters.
  – Example: GL theory of conventional superconductor with pairing order parameter $\psi \sim \langle c_\uparrow c_\downarrow \rangle$,

$$F = F_n + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m} |(-i\hbar \nabla - 2eA)\psi|^2 + \frac{|B|^2}{2\mu_0}$$

naturally leads to Meissner effect, vortex lattice, etc.
Topological phases

- Fractional quantum Hall effect is the first example of “topological phases”:
  - No symmetry breaking
  - No local order parameter
  - Are distinct phases
    \[ \sigma_{xy} = I_x / V_y = v e^2 / h \]
  - Robust against local perturbations
Intrinsic topological order

- FQHE have “intrinsic topological order”
  - FQHE state on closed surface has ground state degeneracy depending on topology only.
  - FQHE excitations have fractional quantum number and fractional statistics.
  - E.g. for filling $\nu=1/m$ Laughlin FQHE state, g.s. degeneracy is $m^g$, $g=$genus
    excitation has charge $e/m$.
  - Can be “measured” by entanglement spectrum/entropy [Li&Haldane PRL’08]

Prof. X.-G. Wen
The “symmetry protected topological (SPT) order”

- Gapped phases with the same symmetry, separated by phase transition if and only if the symmetry is maintained.
- E.g. topological insulator vs. trivial band insulator, need time-reversal symmetry for their distinction
Properties of SPT order

• SPT order has *no local* order parameter
• *No* fractionalization in the bulk: bulk excitations carry ordinary quantum number
• *No* topological ground state degeneracy

• Usually have *gapless edge states*. (exception in 3+1d, Vishwanath & Senthil PRX’13)
• Symmetry acts non-trivially on the edge states, different SPT can be distinguished by this.
Edge states of SPT order

- Example: 1+1D spin-1 Haldane phase (AKLT state)
  - Physical d.o.f. are spin-1 \( [\text{linear representation} \text{ of } \text{SO}(3), \exp(2\pi i S^z)=+1] \), symmetric combination of two spin-1/2s.
  - Spin-1/2s form singlets with neighbors, leaving two spin-1/2s at ends
  - The spin-1/2 end state transforms as \textit{projective representation} of \text{SO}(3): \exp(2\pi i S^z)=-1.
Classification of SPT order

• Free fermions with/without T-reversal and/or “charge conjugation” symmetries have been all classified [Schnyder et al.’08, Kitaev’09]

• Interacting boson SPT with internal symmetry $G$ in $d+1$-dim may be classified by group cohomology $H^{d+1}(G,U(1))$ [Xie Chen et al. Science’12, PRB’13]

• Interacting fermion SPT? (Gu&Wen’12)
Questions

• Is there a Ginzburg-Landau-like theory framework for SPT order and their phase transitions?
  – Recent progress: BF theory, NLσM with topological term, Vishwanath&Senthil PRX’13, Bi&Rasmussen&Xu’13

• How to detect SPT order?
  – Recent progress:
    non-local order parameter, Pollmann&Turner’12;
    “strange correlator”: Cenke Xu et al.’13
Our question

• Relation between critical theory of phase transition between SPT orders and the edge theory.
  – Our answer (conjecture): edge theory is the critical theory confined between two SPTs [NuclPhysB’13]
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Edge theory & bulk critical theory

• For a continuous phase transition between SPTs, it is naturally to expect that a sharp interface and a smooth interface have the same low energy theory.

• A smooth domain wall may be viewed as the bulk critical phase spatially confined in one direction.
Example: 3D topological insulator

• 3D topological insulator and trivial insulator bulk theories can be viewed as the same 3+1d Dirac fermion Hamiltonian with opposite sign of mass term. [see e.g. Qi&Zhang RMP’11]

• On a mass domain wall there will be localized gapless 2+1d chiral fermions: surface states of 3D TI
Cases we consider: SO(3) SPT

- SO(3) SPT in 1+1d: $\mathbb{Z}_2$ classification
- The simplest model exhibiting the two phases: dimerized spin-1/2 chain

- One unit cell contains two spin-1/2, have total spin-0 or -1, is linear representation of SO(3)
Cases we consider: SO(3) SPT

• The critical point is the uniform spin-1/2 chain

• The theory is $SU(2)_W$ Wess-Zumino-Witten theory

$$S_{\text{critical}} = \frac{1}{2\gamma_1} \int dx \, dt \sum_{j=x,t} \text{Tr}(\partial_j g^{-1} \partial_j g) + S_{\text{WZW}}, \quad g \in SU(2)$$

$$S_{\text{WZW}} = \frac{i}{2\pi} \int dx \, dt \int_0^1 du \, \text{Tr}[(g^{-1} \partial_u g)(g^{-1} \partial_t g)(g^{-1} \partial_x g)].$$

– choose $g(x,t,u=0)=$constant, $g(x,t,u=1)=g(x,t)$ is the space-time configuration
Cases we consider: SO(3) SPT

- The “mass term” is $\lambda \int dx \, dt \, \text{Tr}(g)$.
- For $g = \exp(i \theta / 2 \hat{n} \cdot \vec{\sigma})$, $\text{Tr}(g) = \cos(\theta / 2)$.
  - $\lambda \ll 0$, $\theta = 0$, $g = 1$; $\lambda \gg 0$, $\theta = 2\pi$, $g = -1$.
  - Consider a domain wall $-d \ll x \ll d$, with $g(x = -d, t) = 1$, $g(x = +d, t) = -1$.
  - To minimize the “stiffness energy” $\text{Tr}(\partial_x g^{-1} \partial_x g)$

$$g_{dw}(x, t) = \exp\left[i \frac{2\pi (x + d)}{4d} \hat{n}(t) \cdot \vec{\sigma}\right], \quad -d \leq x \leq d$$

the degree of freedom is unit vector $n$.  

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Cases we consider: SO(3) SPT

- The low energy theory of domain wall is
  \[ S_{\text{dw/edge}} = \frac{1}{2\gamma_2} \int dt (\partial_t \hat{n})^2 + \frac{i}{2} \int dt \int_0^1 du \hat{n} \cdot \partial_t \hat{n} \times \partial_u \hat{n} \]
  which is the action for a spin-1/2

- Alternative description: O(4) NLσM with WZW term

\[ g = \Omega_4 I - i \sum_{j=1}^3 \Omega_j \sigma_j \quad S_{\text{critical}} = \frac{1}{4\gamma_1} \int dx \int dt (\partial_j \hat{\Phi})^2 + \frac{i}{\pi} \int dx \int dt \int_0^1 du \epsilon^{abcd} \Omega_a \partial_u \Omega_b \partial_t \Omega_c \partial_x \Omega_d \]

spin-1/2 at a sharp domain wall
Cases we consider: SO(3) SPT

- From the fact that domain wall has spin-1/2, the critical theory can be reconstructed.
  - E.g. consider an array of well-separated domain walls assume only nearest-neighbor coupling, the critical theory should be the gapless uniform chain of these spin-1/2s

- If the chain is non-uniform, it’s gapped and equivalent to one of the SPT phases
Cases we consider: SO(3) SPT

• SO(3) SPT in 2+1d: $\mathbb{Z}$ classification
• Edge theory is gapless $SU(2)_k$ WZW theory protected by $SU(2)_{Right} \ast SU(2)_{Left}$ symmetry
• “Coupled wire” construction of SPT phases: [see e.g. Kane et al. PRL’02 for QHE]  
  $\uparrow/\downarrow$: right/left movers

![Diagram of coupled wire construction]

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Cases we consider: SO(3) SPT

- Critical theory is O(5) NLσM [Senthil&Fisher’06]

\[ \tilde{S}_{\text{critical}} = \frac{1}{2\gamma_4} \int dx \, dt \sum_{j=x,y,t} (\partial_j \hat{M})^2 + \frac{i^3}{4\pi} \int dx \, dy \, dt \int_0^1 du \, \epsilon^{abcde} M_a \partial_u M_b \partial_t M_c \partial_x M_d \partial_y M_e \]

- “Mass” term \[ \lambda \int dt \, dx \, dy \, M_5(x, y, t) \]

- Consider domain wall \[ M_5(y = -d) = +1 \quad \text{and} \quad M_5(y = d) = -1 \]

minimize stiffness energy

\[ \hat{M}_{\text{dw}}(x, y, t) = \left( \sin \frac{\pi(y + d)}{2d} \hat{\Omega}(x, t), \cos \frac{\pi(y + d)}{2d} \right), \quad -d \leq y \leq d \]

edge theory is O(4) NLσM with WZW term

\[ \tilde{S}_{\text{dw/edge}} = \frac{1}{2\gamma_5} \int dx \, dt \sum_{j=x,t} (\partial_j \hat{\Omega})^2 + \frac{i}{\pi} \int dx \, dt \int_0^1 du \, \epsilon^{abcd} \Omega_a \partial_u \Omega_b \partial_t \Omega_c \partial_x \Omega_d \]
Cases we consider: SO(3) SPT

• Bulk theory of non-trivial SO(3) SPT: edge theory with the auxiliary direction $u$ replaced by spatial direction perpendicular to edge WZW term becomes $\theta$-term
  – 1+1d: O(3) NL$\sigma$M with $\theta = 2\pi$, Haldane phase
  – 2+1d: O(4) NL$\sigma$M.
Summary

• We used to know the relation between symmetry and phases, by Landau’s (symmetry-breaking) paradigm.

• Now we need to learn more about symmetry: e.g. about symmetry protected topological order.

• SPT edge theory is related to theory of critical points:
  – Edge theory \(\approx\) critical theory confined in domain wall
  – Critical theory can be constructed by coupled edges.

• Still need a Ginzburg-Landau-like universal theory for these newly identified phases.
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Happy New Year!

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