Time-reversal anomaly in SPT phases

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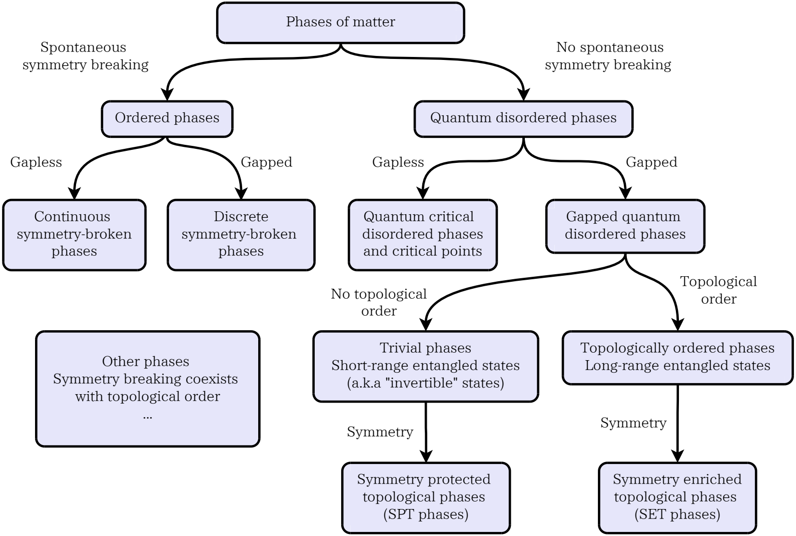
- Symmetry in nature can be broken in at least three different ways: *explicitly*, *spontaneously*, and *anomalously*
- Spontaneous symmetry breaking (SSB) is a key ingredient for the Landau-Ginzburg-Wilson paradigm describing phases of matter and phase transitions.
- Symmetry can be also broken by quantum effects – quantum anomalies. They play a key role in describing topological phases – phases of matter which defy the description by the symmetry breaking paradigm.

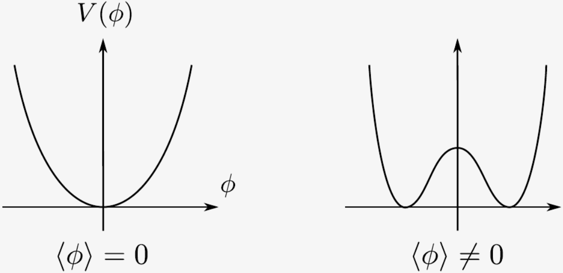
Phases of matter; ordered v.s. disordered and gappped v.s. gapless



Ordered v.s. disordered

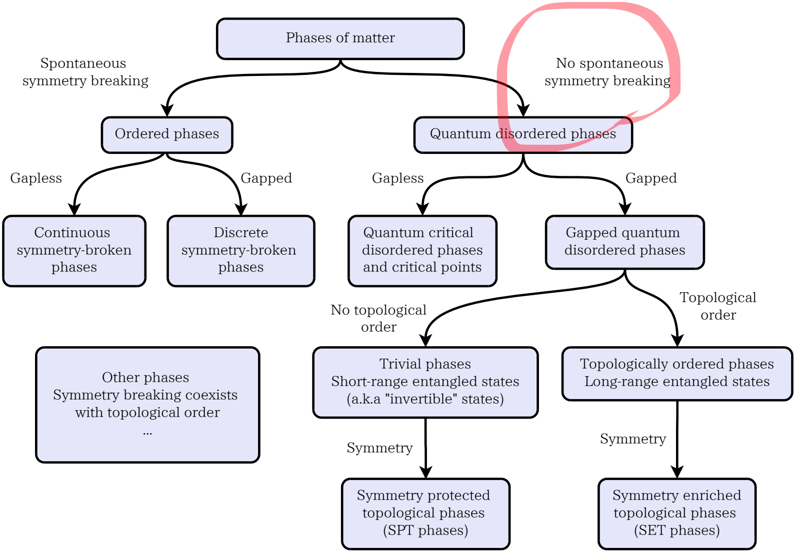
Many phases of matter can be described by local order parameters associated to spontaneous symmetry breaking (SSB)

Example: magnet



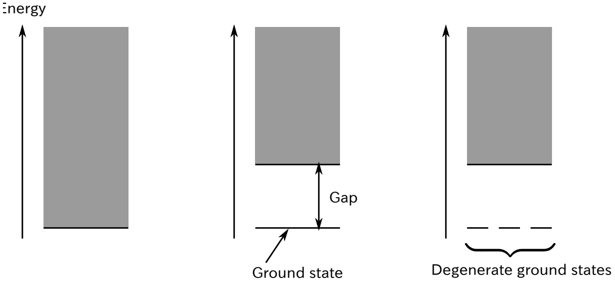
Today, I am interested in disordered phases (phases without any order parameter).

Phases of matter; ordered v.s. disordered and gappped v.s. gapless



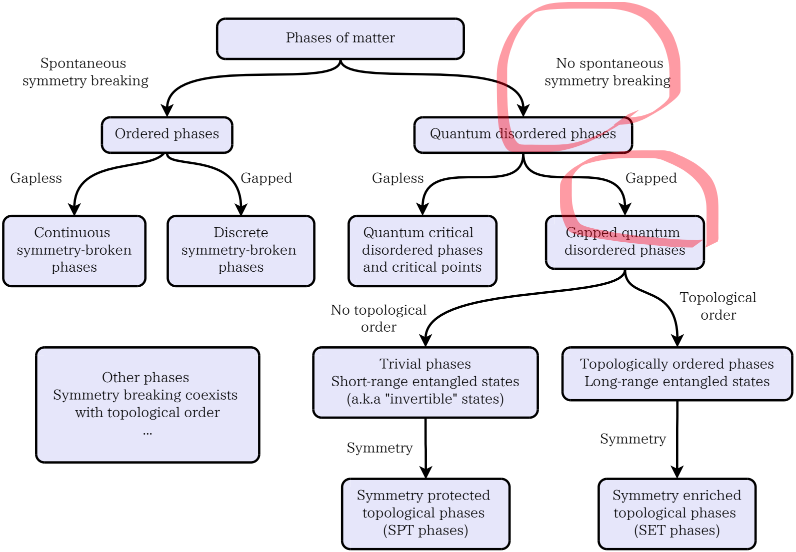
Gapless v.s. gapped, topological order

- For this talk, I will be interested in gapped phases of matter.
- Furthermore, I will not consider topologically ordered phases.

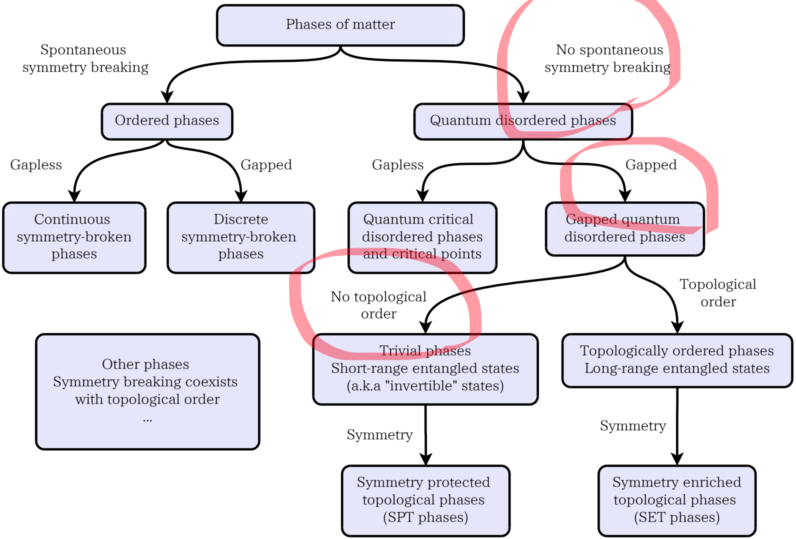


(a) (b) (c)

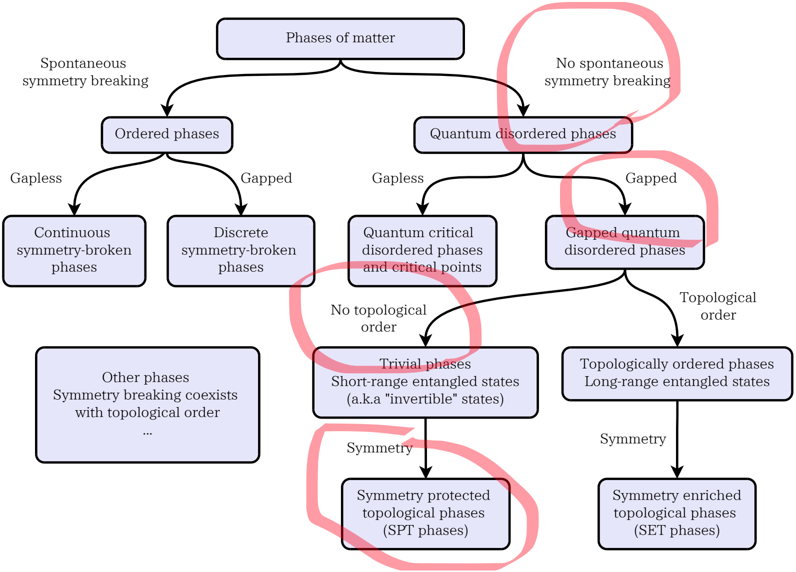
Phases of matter; ordered v.s. disordered and gappped v.s. gapless



Phases of matter; ordered v.s. disordered and gappped v.s. gapless

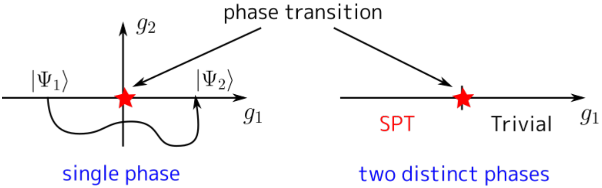


Phases of matter; ordered v.s. disordered and gappped v.s. gapless



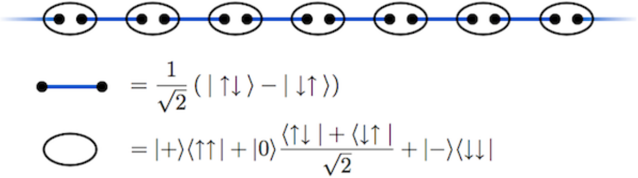
Symmetry-protected topological phases (SPT phases)

- *In the absence of symmetry*, SPT phases are adiabatically connected to a trivial phase
  - Trivial phase = product states $|\Psi\rangle = |\phi\rangle|\phi\rangle\cdots|\phi\rangle$
  - Unique ground state on any manifold
- Nevertheless, *if we impose symmetry*, SPT phases are topologically distinct



Symmetry-protected topological phases (SPT phases)

- Examples: time-reversal symmetric topological insulators; the Haldane phase in 1d spin chains [Haldane, 2016 Nobel Prize]



- No local order parameter: symmetry-breaking paradigm cannot be applied:

Topological phases and anomalies

- Questions:
  - How do we classify SPT phases?
  - How do we characterize/detect/diagnose SPT phase? Effects of interactions?
- Main “tools/concepts”:
  - Response theory (topological quantum field theories)
  - Quantum anomalies
  - Bulk-boundary correspondence

Response theory

- When $\exists$ global symmetry G (unitary, on-site),

$$Z(M,A) = \int \mathcal{D}[\phi]\ \exp[-S(\phi, M, A)]$$

A : background G-gauge field, $\quad \phi$: "matter field"

- Even when no global symmetry,

$$Z(M) = \int \mathcal{D}[\phi]\ \exp[-S(\phi, M)]$$

M : closed spacetime manifold

- (More data maybe needed for other situations.)

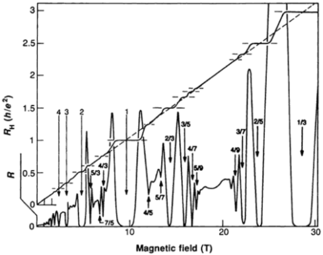
Response theory for SPT phases

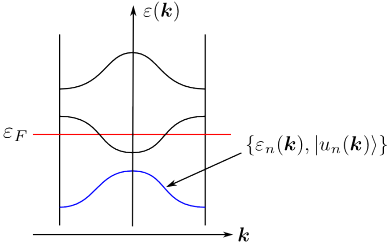
- For topological (SPT) phases,
  - (i) $Z(M, A)$ is expected to have a pure imaginary part:

$$Z(M, A) = \exp[i I_{top}(M, A)]$$

  - (ii) $I_{top}(M, A)$ is expected to be topological (metric independent).
  - (iii) $I_{top}(M, A)$ is not gauge invariant in the presence of boundary.
- I_{top} serves as a "non-local order parameter".
- Generic approach, but very powerful for SPT phases because of unique ground states. [Kapustin et al. (14), Freed (14-16), Witten (15)]

Example: Quantum Hall effect





Example: QHE

- $U(1)$ particle number conservation; can couple the system with an external (probe) gauge field A_μ^{ex}.
- Response of the system is encoded in the effective action:

$$Z(A^{ex}) = \int \mathcal{D}[\psi^\dagger, \psi] e^{-S(A^{ex}, \psi^\dagger, \psi)} = e^{-I_{\mathit{eff}}(A^{ex})}$$

- In the QHE, I_{eff} has a topological contribution; the Chern-Simons term, which is imaginary:

$$I_{\mathit{eff}}(A) = \frac{ik}{4\pi} \int d\tau dx dy \, \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad k = \text{integer}$$

Independent of the metric.

Bulk-boundary correspondence

- In the presence of boundary, the Chern-Simons term is not gauge invariant.
- Necessary to have boundary degrees of freedom which cancel the non-invariance.
- Boundary theory is anomalous.
  - They cannot be gapped trivially while preserving symmetry; Gapless or topologically ordered
- More generally: Bulk $(d+1)$-dim G SPT supports d-dim boundary theory, which has G 't Hooft anomaly.

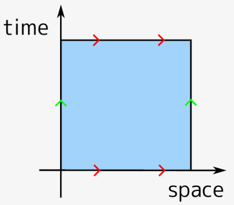
Example: QHE

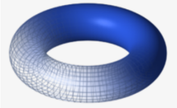
- Chiral edge theory:

$$\mathcal{L} = \frac{1}{2\pi}\psi^\dagger i(\partial_t + \partial_x)\psi$$

Twisted boundary conditions:

$$\psi(t, x+L) = e^{2\pi i a}\psi(t,x), \quad \psi(t+\beta, x) = e^{2\pi i b}\psi(t,x)$$





Example: QHE

- Classical system (Lagrangian + b.c.) is invariant under $a \to a+1$ and $b \to b+1$ (large gauge transformation)
- Quantum mechanics:

$$Z([a,b]) = \int \mathcal{D}[\psi^\dagger, \psi] e^{-S} = \mathrm{Tr}_a \left[e^{-\beta H} e^{2\pi i \left(b+\frac{1}{2}\right) N} \right]$$

  - Tr_a: Spatial b.c. twisted by the phase $e^{2\pi i a}$
  - $e^{2\pi i \left(b+\frac{1}{2}\right) Q}$: Temporal b.c. is twisted by the phase $e^{2\pi i b}$:

- Large gauge anomaly:

$$Z([a,b]) \neq Z([a,b+1]) \quad \text{or} \quad Z([a,b]) \neq Z([a+1,b]).$$

Other symmetries?

- Discrete on-site unitary symmetry ["group cohomology approach": Dijkgraaf-Witten Chen-Liu-Gu-Wen (11)]
- Anti unitary on-site unitary symmetry, e.g., Time-reversal, reflection, etc.
- Crystalline symmetries

Today: Orientation-reversing symmetry:

- Time-reversal, spatial reflection,

Example: Topological insulator

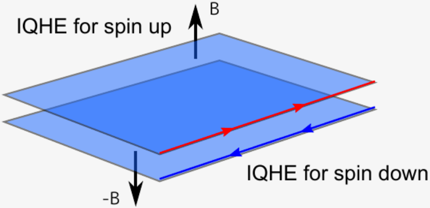
(A.k.a. quantum spin Hall effect)

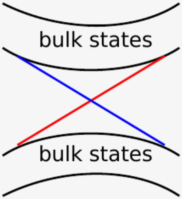
- Time-reversal invariant band insulator with strong spin-orbit interaction

TRS

$$(i\sigma_y)\mathcal{H}^*(-i\sigma_y) = \mathcal{H}$$

- gapless Kramers pair of edge modes





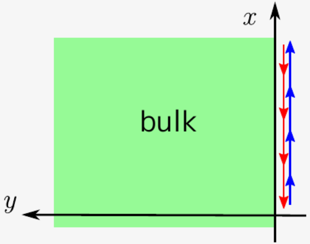
- Characterized by a binary (Z_2) topological quantity

$$W = \prod_{\mathsf{K}} \frac{\mathrm{Pf}\,[w(\mathsf{K})]}{\sqrt{\det\,[w(\mathsf{K})]}}$$

Example: Topological insulator

- Edge Hamiltonian ("helical" edge):

$$H = \int dx \left[\psi_L^\dagger i\partial_x \psi_L - \psi_R^\dagger i\partial_x \psi_R\right]$$

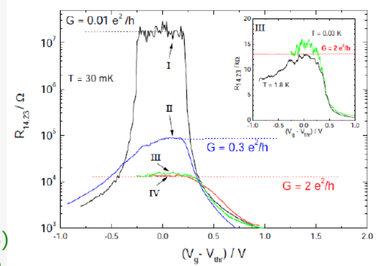


- T symmetry

$$\mathcal{T}\psi_L(x)\mathcal{T}^{-1} = \psi_R(x)$$
$$\mathcal{T}\psi_R(x)\mathcal{T}^{-1} = -\psi_L(x)$$

- Can check no mass terms are allowed

Z_2 classification

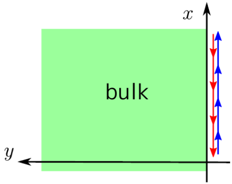


Bernevig-Hughes-Zhang (2006)
M. Koenig et al. Science (2007)

Example:CP symmetry topological insulator

- System charge $U(1)$ and CP symmetry: $P(x, y) \to (-x, y)$.
- "CPT"-dual of (2+1)d topological insulator
- Edge theory (for CP symmetric edge)

$$H = \int dx \left[\psi_L^\dagger i\partial_x \psi_L - \psi_R^\dagger i\partial_x \psi_R \right]$$



- Under CP symmetry

$$\mathcal{U}_{CP} \psi_L(x) \mathcal{U}_{CP}^{-1} = \psi_R^\dagger(-x), \quad \mathcal{U}_{CP} \psi_R(x) \mathcal{U}_{CP}^{-1} = \psi_L^\dagger(-x),$$

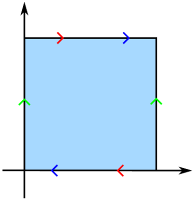
no mass terms are allowed.

- Topological phases with (protected by) time-reversal;
  $\Rightarrow$
  - *Is there any anomaly associate to time-reversal symmetry?*
  - *How can we develop response theory? Or how can we “gauge” time-reversal?*

Anomaly on unoriented surface

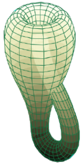
[Hsieh-Sule-Cho-SR-Leigh (14)]

- Twisting by parity symmetry:

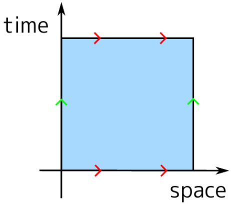


Twisting by parity symmetry

$$Z = \mathrm{Tr}_h \left[P\, e^{-\beta H} \right]$$
$$\Phi(t+T,x) = g \cdot \Phi(t, L-x)$$
$$\Phi(t, x+L) = h \cdot \Phi(t,x)$$

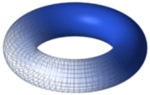


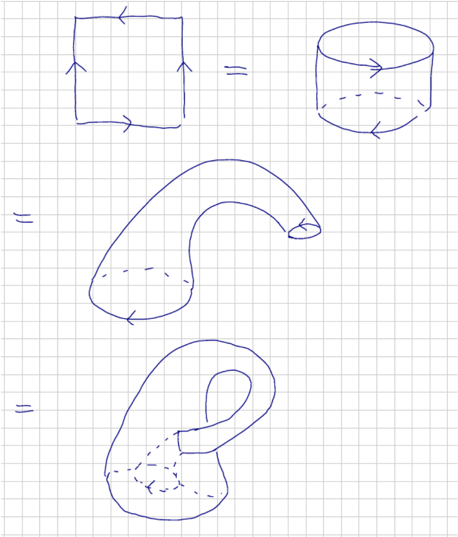
- C.f. Twisting by on-site symmetry:



Twisting by on-site symmetry

$$Z = \mathrm{Tr}_h \left[g e^{-\beta H} \right]$$
$$\Phi(t+T,x) = g \cdot \Phi(t,x)$$
$$\Phi(t, x+L) = h \cdot \Phi(t,x)$$





[Hsieh-Sule-Cho-SR-Leigh (14)]

- Klein bottle partition function: twisting by CP and U(1):

$$\psi_L(t+T,x) = \psi_R^\dagger(L-x,t), \quad \psi_R(t+T,x) = \psi_L^\dagger(L-x,t)$$
$$\psi_L(t,x+L) = e^{2\pi i a}\psi_L(x,t), \quad \psi_R(t,x+L) = e^{2\pi i a}\psi_R(x,t)$$

- Klein bottle (KB) partition func (CP twisted partition func)

$$Z(KB,a) = \mathrm{Tr}_a\left[\mathcal{U}_{CP} e^{-\beta H}\right]$$

- Large gauge anomaly under $a \to a+1$:

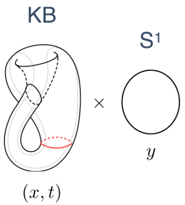
$$Z(KB,a+1) = (-1)Z(KB,a).$$

- C.f. Old work by [Brunner-Hori (03)]

How about the bulk?

- The partition function on Klein bottle $\times$ S^1 with flux:

$$Z(KB \times S^1, A) = (-1)$$

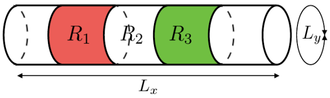


- This $\mathbb{Z}_2$ response is the fundamental response characterizing topological insulators, even in the presence of interactions.

Many-body $\mathbb{Z}_2$ topological invariant for (2+1)d topological insulators

[Shiozaki-Shapourian-SR (17)]

- Setup:



- Formula: (T_1 = fermionic partial transpose)

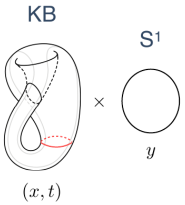
$$Z = \mathrm{Tr}_{R_1 \cup R_3}\left[\rho^{+}_{R_1\cup R_3} C_T^{I_1} [\rho^{-}_{R_1\cup R_3}]^{\mathsf{T}_1} [C_T^{I_1}]^{\dagger}\right],$$

$$\rho^{\pm}_{R_1\cup R_3} = \mathrm{Tr}_{\overline{R_1\cup R_3}}\Big[\underbrace{e^{\pm \sum_{\mathbf{r}\in R_2} \frac{2\pi i y}{L_y} n(\mathbf{r})}}_{\text{partial } U(1) \text{ twist}} |GS\rangle\langle GS| \Big]$$

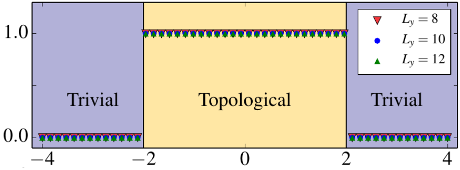
$C_T \sim$ spin flip unitary

Many-body $\mathbb{Z}_2$ topological invariant for (2+1)d topological insulators

- Z is the partition function on Klein bottle $\times S^1$ with flux.

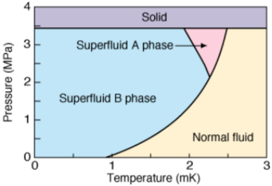


- Phase of Z computed numerically on a lattice:



3D example: ^{3}He B

- B-phase of ^{3}



- BdG hamiltonian:

$$H = \int d^3\mathsf{k}\, \Psi^\dagger(\mathsf{k})\mathcal{H}(\mathsf{k})\Psi(\mathsf{k}), \quad \mathcal{H}(\mathsf{k}) = \begin{bmatrix} \frac{k^2}{2m} - \mu & \Delta\sigma\cdot\mathsf{k} \\ \Delta\sigma\cdot\mathsf{k} & -\frac{k^2}{2m}+\mu \end{bmatrix}$$

$$\Psi(\mathsf{k}) = (\psi_{\uparrow\mathsf{k}}, \psi_{\downarrow\mathsf{k}}, \psi^\dagger_{\downarrow,-\mathsf{k}}, -\psi^\dagger_{\uparrow,-\mathsf{k}})^T$$

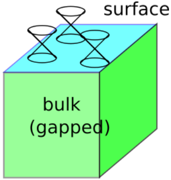
- SPT phase protected by TRS or spatial inversion

$$I\psi^\dagger_\sigma(\mathsf{r})I^{-1} = i\psi^\dagger_\sigma(-\mathsf{r})$$

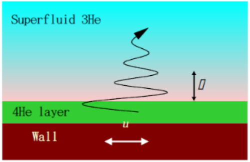
- When non-interaction, characterized by an integer topological invariant ν.

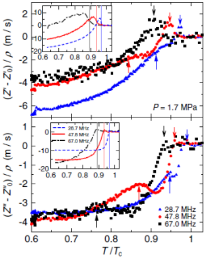
Surface Majorna states

- By bulk-boundary correspondence;surface Majorna cones



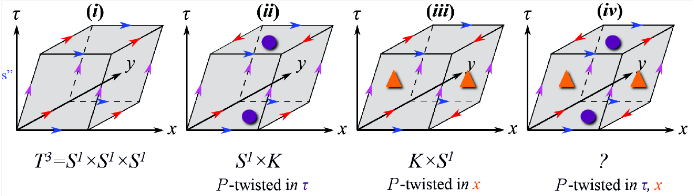
- Detected by surface acoustic impedance measurement [Murakawa et al (09)]:





$\mathbb{Z}_{16}$ classification

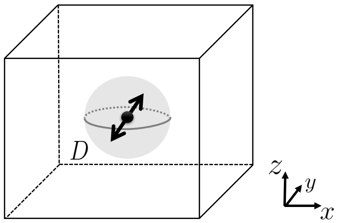
- The non-interacting classification breaks down to $\mathbb{Z}_{16}$ by interaction. Surface topological order, etc. [Fidkowski et al (13), Metlitski et al (14), Wang-Senthil (14), Morimoto-Furusaki-Mudry (15), ..]
- Initial calculation of surface anomaly on T^3 with parity twist reveals $\mathbb{Z}_8$ [Hsieh-Cho-SR (15)]



Many-body topological invariant

- (3+1)d DIII topological superconductors are expected to be detected by $\mathbb{R}P^4$. [Kapustin et al (14-15), Freed-Hopkins (14-15), Witten (15), ...]
- We consider partial inversion I_{part} on a ball D:

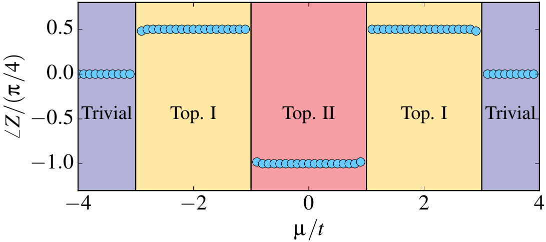
$$Z = \langle \Psi | I_{part} | \Psi \rangle = \mathrm{Tr}_D (I_{part} \rho_D)$$



- The spacetime is effectively four-dimensional projective plane, $\mathbb{R}P^4$.

Bulk calculations

- Numerics on a lattice:



- $Z(\mathbb{R}P^4) = \exp[2\pi i\nu/16]$ with $\nu = 0, \ldots, 15$ is the fundamental response of (3+1) topological superconductors protected by orientation-reversing symmetry.

Boundary calculations

- The topological invariant can be computed from the Majorana surface theory [Shiozaki-Shapourian-SR(16)]

$$Z = \mathrm{Tr}_D \left(I_{part} \rho_D \right) = \frac{\mathrm{Tr}_{\partial D} (I_{part} e^{-H_{surf}})}{\mathrm{Tr}_{\partial D} (e^{-H_{surf}})}$$

  where H_{surf} is the entanglement Hamiltonian $\simeq$ physical surface Hamiltonian

- Result when $\nu = 1$:

$$Z = \exp \left[-\frac{i\pi}{8} + \frac{1}{12} \ln(2) - \frac{21}{16} \zeta(3) \left(\frac{R}{\xi} \right)^2 + \cdots \right]$$

Boundary calculations

- With interactions, TSC surface can be gapped topologically ordered. [Senthil-Vishwanath, Fidkowski-Chen-Vishwanath (13), Wang-Senthil (15), Metlitski-Fidkowski-Chen-Vishwanath (14), ...]
- When surface is topologically ordered: [Wang-Levin, Tachikawa-Yonekura, Barkeshli et al (16)]

$$
\begin{aligned}
Z &= \frac{\mathrm{Tr}_{\partial D}(I_{part} e^{-H_{surf}})}{\mathrm{Tr}_{\partial D}(e^{-H_{surf}})} \\
&= \sum_p e^{-2\pi i h_p} \eta_p d_p = \exp\left[\frac{2\pi i \nu}{16}\right]
\end{aligned}
$$

where sum is over symmetric anyons with topological spin h_p, quantum dimension d_p, and eigenvalues of T^2.

Summary

- Topological insulators and topological superconductors protected by orientation reversing symmetry can be detected/defined by their coupling to unoriented spacetime.
- Constructed explicit many-body topological invariants.
- Essentially the same construction of many-body topological invariants for other cases, e.g., the Kitaev Majorana chain, etc.
- Numerically useful?