Wigner functions for chiral fermions

Qun Wang

University of Science and Technology of China

Recent Developments in Chiral Matter and Topology
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Outline

- Introduction
- Wigner functions of chiral fermions in external electromagnetic field
  [J.H. Gao, et al., 1203.0725; J.H. Gao, QW, 1504.07334]
- Disentangling covariant Wigner functions for chiral fermions
  [J.H. Gao, Z.T. Liang, QW, X.N. Wang, 1802.06216]
- The chiral vortical effect in Wigner function approach
  [J.H. Gao, J.Y. Pang, QW, 1810.02028]

Introduction

- High energy HIC with $\sqrt{s}$:

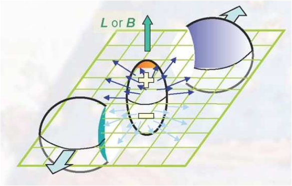
$$v \sim 1 - \frac{m_p^2}{2s}, \quad \gamma \sim \frac{\sqrt{s}}{m_p}$$

- Electric field in the rest frame of one nucleus

$$\mathbf{E} = \frac{Ze}{R^2}\hat{\mathbf{r}}$$

Boost to Lab frame ($v_z \approx 0.999989$c for 200 GeV, MeV$^2 \approx 1.44 \times 10^{13}$ Gs)

$$\begin{aligned} \mathbf{B} &= -\gamma \mathbf{v}_z \times \mathbf{E} = B\mathbf{e}_\phi \\ eB &\sim 2\gamma v_z \frac{Ze^2}{R^2} \sim 10 m_\pi^2 \sim 3 \times 10^{18}\mathrm{Gs} \end{aligned}$$



Fukushima, Kharzeev, Warringa (2008)

Skokov (2009), Deng, Huang (2012)

many others ...

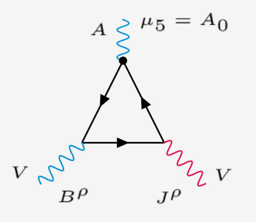
Talks: Jinfeng Liao, Igor Shovkovy

Introduction

- Anomalies: the classical symmetry of the Lagrangian broken by quantum effects:
  (a) chiral symmetry by axial anomaly
  (b) scale symmetry by scale anomaly
- Chiral anomaly implies VVA coupling: if A^μ ($\mu_5 = A_0$) and V^μ (Magnetic field B^μ) are background field, then V^μ (vector current J^μ) is induced.
- Chiral Magnetic effect:

$$J^\rho = \hbar \frac{Q^2}{2\pi^2} \mu_5 B^\rho$$

It's a quantum and topological effect (protected by symmetry).



Kharzeev, McLerran, Warringa (2008)
Fukushima, Kharzeev, Warringa (2008)

Recent reviews, e.g.:

Kharzeev, Landsteiner, Schmitt, Yee (2013)

Kharzeev, Liao, Voloshin, Wang (2015)

Hattori, Huang (2017)

Classical transport equation

- Distribution function in phase space $f(t, \mathbf{x}, \mathbf{p})$
- Classical Boltzmann equation in background EM field

$$p^{\mu}(\partial_{\mu} - QF_{\mu\nu}\partial_{\mathbf{p}}^{\nu})f(t, \mathbf{x}, \mathbf{p}) = C[f] \tag{1}$$

- Conserved current and energy-momentum tensor

$$\begin{aligned} j^{\mu}(x) &= \int \frac{d^3p}{(2\pi)^3 E_p} p^{\mu} f(t, \mathbf{x}, \mathbf{p}) \\ T^{\mu\nu}(x) &= \int \frac{d^3p}{(2\pi)^3 E_p} p^{\mu} p^{\nu} f(t, \mathbf{x}, \mathbf{p}) \end{aligned} \tag{2}$$

- Classical transport theory: one $f(t, \mathbf{x}, \mathbf{p})$ and one equation!

Quantum transport in Wigner function

- The Wigner function for fermions in EM field

$$W_{\alpha\beta}(X,p)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot y/\hbar}\left\langle\bar{\psi}_\beta\left(x_+\right)U\left(x_+,x_-\right)\psi_\alpha\left(x_-\right)\right\rangle$$

$[\hat{X},\hat{p}]=0$

Gauge link:

$U(x_1,x_2)=\exp\left[-iQ\int_{x_2}^{x_1}dz_\mu A^\mu(z)\right]$

where $x_\pm = X\pm y/2$.

- $W(X,p)$ (4×4 complex matrix)$\Rightarrow$ 32 variables $\Rightarrow W^\dagger=\gamma_0 W\gamma_0$
  $\Rightarrow$ 16 variables (Scalar, Pseudoscalar, Vector, Axial vector, Tensor)
  $\Rightarrow$ 8 variables for chiral fermions (Vector, Axial vector)
  $\Rightarrow$ 4 for RH and 4 for LH.

Some early works on Wigner functions of QED/QCD:

Heinz (1983); Elze, Gyulassy, Vasak (1986); Vasak, Gyulassy, Elze (1987);

Zhuang, Heinz (1996); Blaizot, Iancu (2002); QW, Redlich, Stoecker, Greiner (2002)

Wigner function

- The WF can be decomposed in 16 independent generators of Clifford algebra,

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu\mathscr{V}_\mu + \gamma^5\gamma^\mu\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

whose coefficients $\mathscr{F}(1)$, $\mathscr{P}(1)$, $\mathscr{V}_\mu(4)$, $\mathscr{A}_\mu(4)$ and $\mathscr{S}_{\mu\nu}(6)$ are the scalar, pseudo-scalar, vector, axial-vector and tensor components.

- For chiral fermions, $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ are decoupled from $\mathscr{F}$, $\mathscr{P}$ and $\mathscr{S}_{\mu\nu}$.

Some early works on Wigner functions of QED/QCD:

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WF-EOM for chiral fermions in constant field

- The vector component $\mathscr{J}^{s}_{\mu}(x,p)$ for chiral fermions (with chirality $s=\pm$) is defined as

$$\mathscr{J}^{s}_{\mu}(x,p) = \frac{1}{2}[\mathscr{V}_{\mu}(x,p)+s\mathscr{A}_{\mu}(x,p)] \quad (3)$$

- The EOM for $\mathscr{J}^{\mu}_{s}(x,p)$ in constant field read

$\nabla_{\mu}=\partial^{X}_{\mu}-QF_{\mu\nu}\partial^{\nu}_{p}$

3 evo eqs for $\mathscr{J}^{i}$ ($\mu\nu=0i$)

3 constr eqs for $\mathscr{J}^{i}$ and $\mathscr{J}_{0}$ ($\mu\nu=ij$)

$$\begin{aligned} p^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0 \quad \longrightarrow \text{Mass-shell condition} \\ \nabla^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0 \quad \longrightarrow \text{Evolution equation for } \mathscr{J}_{0} \\ 2s(p^{\lambda}\mathscr{J}^{\rho}_{s}-p^{\rho}\mathscr{J}^{\lambda}_{s}) &= -\hbar\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}^{s}_{\nu} \end{aligned} \quad (4)$$

- There are 8 equations for 4 variables in $\mathscr{J}^{\mu}_{s}(x,p)$ for a given s.

Vasak, Gyulassy, Elze (1987); Gao, Liang, Pu, QW, Wang (2012); Chen, Pu, QW, Wang (2013);

Gao, QW (2015); Gao, Pu, QW (2017)

WF-EOM in non-constant field

- The set of (eight) equations for the vector component $\mathscr{J}_s^\mu(x,p) = (\mathscr{J}_0, \mathscr{J}^i)$ for chiral fermions (with chirality s)

Equation 3-8:
3 evolution eqs for $\mathscr{J}^i$ $(\mu\nu = 0i)$
3 constraint eqs for $\mathscr{J}^i$ and $\mathscr{J}_0$ $(\mu\nu = ij)$

$$\Pi^\mu \mathscr{J}_\mu^s(x,p) = 0 \longrightarrow \text{Equation 1: mass-shell condition}$$

$$G^\mu \mathscr{J}_\mu^s(x,p) = 0 \longrightarrow \text{Equation 2: evolution equation for } \mathscr{J}_0$$

$$2s(\Pi^\lambda \mathscr{J}_s^\rho - \Pi^\rho \mathscr{J}_s^\lambda) = -\hbar\epsilon^{\mu\nu\lambda\rho} G_\mu \mathscr{J}_\nu^s \quad (5)$$

- where

$j_1(z) = (\sin z - z\cos z)/z^2$, $j_1(0) = 0$

$$\Pi^\mu = p^\mu - \hbar\frac{1}{2} j_1\left(\frac{\hbar}{2}\partial_x \cdot \partial_p\right) Q F^{\mu\nu}\partial_\nu^p$$

∂_x only acts on $F^{\mu\nu}$

$$G^\mu = \partial_x^\mu - j_0\left(\frac{\hbar}{2}\partial_x \cdot \partial_p\right) Q F^{\mu\nu}\partial_\nu^p \quad (6)$$

$j_0(z) = \sin z/z$, $j_0(0) = 1$

Vasak, Gyulassy, Elze (1987)

Semiclassical expansion in $\hbar$

- Semiclassical expansion in powers of $\hbar$

$$
\begin{aligned}
\mathscr{J}_\mu &= \sum_{n=0}^{\infty} \hbar^n \mathscr{J}_\mu^{(n)} \\
\Pi^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} \Pi^\mu_{(2n)} \\
G^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} G^\mu_{(2n)}
\end{aligned} \tag{7}
$$

- where

∂_x acts only on $F^{\mu\nu}$

$$
\begin{aligned}
G^\mu_{(2n)} &= \frac{(-1)^{n+1}}{2^{2n}(2n+1)!} (\partial_x \cdot \partial_p)^{2n} F^{\mu\nu} \partial^p_\nu \\
\Pi^\mu_{(2n)} &= \frac{(-1)^n n}{2^{2n-1}(2n+1)!} (\partial_x \cdot \partial_p)^{2n-1} F^{\mu\nu} \partial^p_\nu
\end{aligned} \tag{8}
$$

∂_x acts only on $F^{\mu\nu}$

Question to be answered

- The question has long been asked: to what extent a chiral fermion quantum system in electromagnetic fields can be described by the quasi-classical distribution function.
- The answer is yes. The rigorous proof is based on the semi-classical expansion in $\hbar$ for the covariant Wigner functions.
- This remarkable property of chiral fermions will significantly simplify the kinetic simulation of chiral effects in heavy ion collisions and Dirac/Weyl semimetals.

The 0-th and 2nd order operators

- 0-th order operators (in comoving frame)

$$
\begin{aligned}
G_{(0)}^{\mu} &= \partial_x^{\mu} - QF^{\mu\nu}\partial_{\nu}^{p} = (G_0^{(0)}, -\mathbf{G}^{(0)}) \\
G_0^{(0)} &= \partial_t + Q\mathbf{E}\cdot\nabla_p \\
\mathbf{G}^{(0)} &= \nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B}\times\nabla_p \\
\Pi_{(0)}^{\mu} &= (p_0, \mathbf{p})
\end{aligned} \tag{9}
$$

Only $G_0^{(0)}$ contains time derivative ∂_t on the Wigner function

- 2nd order operators

$$
\begin{aligned}
G_{(2)}^{\mu} &= \frac{Q^2}{24}(\partial_x\cdot\partial_p)^2 F^{\mu\nu}\partial_{\nu}^{p} \\
\Pi_{(2)}^{\mu} &= -\frac{Q^2}{12}(\partial_x\cdot\partial_p)F^{\mu\nu}\partial_{\nu}^{p}
\end{aligned} \tag{10}
$$

EOM for Wigner function

- Let us write EOM explicitly in time and spatial components

G_0 contains 0-th order: $G_0^{(0)} = \partial_t + Q\mathbf{E} \cdot \nabla_p$

$$
\begin{aligned}
G_0 \mathscr{J}_0 + \mathbf{G} \cdot \boldsymbol{\mathscr{J}} &= 0 \quad (11) \\
\hbar [G_0 \boldsymbol{\mathscr{J}} + \mathbf{G} \mathscr{J}_0] &= 2s(\mathbf{\Pi} \times \boldsymbol{\mathscr{J}}) \quad (12) \\
\Pi_0 \mathscr{J}_0 - \mathbf{\Pi} \cdot \boldsymbol{\mathscr{J}} &= 0 \quad (13) \\
-\hbar \mathbf{G} \times \boldsymbol{\mathscr{J}} &= 2s(\mathbf{\Pi} \mathscr{J}_0 - \Pi_0 \boldsymbol{\mathscr{J}}) \quad (14)
\end{aligned}
$$

The 0-th order equations

- The evolution equations at $O(\hbar^0)$

$G_0^{(0)}=\partial_t+Q\mathbf{E}\cdot\nabla_p$ $\quad$ $\mathbf{G}^{(0)}=\nabla_x+Q\mathbf{E}\partial_{p_0}+Q\mathbf{B}\times\nabla_p$

$$G_0^{(0)}\mathscr{J}_0^{(0)}+\mathbf{G}^{(0)}\cdot\mathscr{J}^{(0)} = 0 \quad (15)$$

$\hbar G_0^{(0)}\mathscr{J}^{(0)}$ is absent at $O(\hbar^0)$ ⟶

$$2s(\mathbf{p}\times\mathscr{J}^{(0)}) = 0 \quad (16)$$

- Constraint equations at $O(\hbar^0)$

$$p_0\mathscr{J}_0^{(0)}-\mathbf{p}\cdot\mathscr{J}^{(0)} = 0 \quad (17)$$

$$2s(p_0\mathscr{J}^{(0)}-\mathbf{p}\mathscr{J}_0^{(0)}) = 0 \quad (18)$$

The 0-th solution

- From Eqs. (18,17) we obtain

$$
\begin{aligned}
\mathscr{J}^{(0)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)} \\
(p_0^2-\mathbf{p}^2)\mathscr{J}_0^{(0)} &= 0 \\
\Rightarrow \mathscr{J}_0^{(0)} &= p_0 f^{(0)}(x,p)\delta(p^2)
\end{aligned}
\tag{19}
$$

- The (evolution) equation (16) is satisfied automatically. The evolution for $f^{(0)}$ is described by Eq. (15),

$$
\begin{aligned}
&(\partial_t + Q\mathbf{E}\cdot\nabla_p)\mathscr{J}_0^{(0)} \\
&+(\nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B}\times\nabla_p)\cdot\left(\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)}\right) = 0
\end{aligned}
\tag{20}
$$

The 1-st order equations

- The evolution equations at $O(\hbar)$

Contains $\partial_t \mathscr{J}_0^{(1)}$ $$G_0^{(0)} \mathscr{J}_0^{(1)} + \mathbf{G}^{(0)} \cdot \boldsymbol{\mathscr{J}}^{(1)} = 0 \quad (21)$$

Contains $\partial_t \boldsymbol{\mathscr{J}}^{(0)}$ $$G_0^{(0)} \boldsymbol{\mathscr{J}}^{(0)} + \mathbf{G}^{(0)} \mathscr{J}_0^{(0)} = 2s(\mathbf{p} \times \boldsymbol{\mathscr{J}}^{(1)}) \quad (22)$$

- The constraint equations at $O(\hbar)$

Mass-shell condition $$p_0 \mathscr{J}_0^{(1)} - \mathbf{p} \cdot \boldsymbol{\mathscr{J}}^{(1)} = 0 \quad (23)$$

Constraint equation to relate $\boldsymbol{\mathscr{J}}$ in terms of $\mathscr{J}_0$ $$\mathbf{G}^{(0)} \times \boldsymbol{\mathscr{J}}^{(0)} = 2s(p_0 \boldsymbol{\mathscr{J}}^{(1)} - \mathbf{p} \mathscr{J}_0^{(1)}) \quad (24)$$

The 1-st order solution

- From Eq. (24) we obtain

$$\mathscr{J}^{(1)} = \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(1)} + \frac{s}{2p_0}\mathbf{G}^{(0)} \times \mathscr{J}^{(0)} \quad (25)$$

and insert it into Eq. (23) we have

$\mathbf{G}^{(0)} = \nabla_x + Q\mathbf{E}\partial_{p_0} + Q\mathbf{B} \times \nabla_p$

$$(p_0^2 - \mathbf{p}^2)\mathscr{J}_0^{(1)} = \frac{s}{2}\mathbf{p} \cdot \left(\mathbf{G}^{(0)} \times \mathscr{J}^{(0)}\right) \quad (26)$$

$\mathscr{J}^{(0)} = \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)}$

So we obtain the solution

for proof, see backup slide

$$\mathscr{J}_0^{(1)} = p_0 f^{(1)}\delta(p^2) + sQ(\mathbf{p} \cdot \mathbf{B})f^{(0)}\delta'(p^2) \quad (27)$$

$\delta'(y) \equiv \frac{d\delta(y)}{dy} = -\frac{1}{y}\delta(y)$

- The evolution $\mathscr{J}_0^{(1)}$ or $f^{(1)}$ is described by Eq. (21).

Two evolution equations for $\mathscr{J}_0^{(0)}$

- Insert Eq. (25) into Eq. (22), we have $\mathbf{p}\times\boldsymbol{\mathscr{J}}^{(1)}\sim\mathbf{p}\times\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(1)}=0$, so we obtain another evolution equation for $\mathscr{J}_0^{(0)}$ through $\boldsymbol{\mathscr{J}}^{(0)}$

$G_0^{(0)}=\partial_t+Q\mathbf{E}\cdot\nabla_p$ $\quad$ $\mathbf{G}^{(0)}=\nabla_x+Q\mathbf{E}\partial_{p_0}+Q\mathbf{B}\times\nabla_p$ $\quad$ $\boldsymbol{\mathscr{J}}^{(0)}=\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)}$

$$G_0^{(0)}\boldsymbol{\mathscr{J}}^{(0)}+\mathbf{G}^{(0)}\mathscr{J}_0^{(0)}=\frac{1}{p_0}\mathbf{p}\times\left(\mathbf{G}^{(0)}\times\boldsymbol{\mathscr{J}}^{(0)}\right)\tag{28}$$

$\boldsymbol{\mathscr{J}}^{(0)}=\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)}$

Remember that the original evolution equation (15) for $\mathscr{J}_0^{(0)}$

$$G_0^{(0)}\mathscr{J}_0^{(0)}+\mathbf{G}^{(0)}\cdot\boldsymbol{\mathscr{J}}^{(0)}=0\tag{29}$$

- These two evolution equations can be shown to be consistent with each other!

The n-th order

- 8 equations can be reduced to 2 equations at the n-th order:
- 1 evolution equation and 1 mass-shell constraint equation for $\mathscr{J}_0^{(n)}$ once lower order components $\mathscr{J}_0^{(n-1)}, \mathscr{J}_0^{(n-2)}, \cdots, \mathscr{J}_0^{(0)}$ are known.
- Only the time-component is independent while spatial components depend on time-component.
- The detailed proof: [Gao, Liang, QW, Wang, 1802.06216]

Second order results

- Solving mass shell equations we obtain $\mathscr{J}_0^{(0,1,2)}$ up to $O(\hbar^2)$

$$\delta'(y) \equiv \frac{d\delta(y)}{dy} = -\frac{1}{y}\delta(y)$$

$$\delta''(y) = \frac{2}{y^2}\delta(y)$$

$$\begin{aligned}
\mathscr{J}_0^{(0)} &= p_0 f^{(0)}\delta(p^2) \\
\mathscr{J}_0^{(1)} &= p_0 f^{(1)}\delta(p^2) + sQ(\mathbf{p}\cdot\mathbf{B})f^{(0)}\delta'(p^2) \\
\mathscr{J}_0^{(2)} &= p_0 f^{(2)}\delta(p^2) + sQ(\mathbf{p}\cdot\mathbf{B})f^{(1)}\delta'(p^2) + Q^2\frac{(\mathbf{p}\cdot\mathbf{B})^2}{2p_0}f^{(0)}\delta''(p^2) \\
&\quad +\frac{1}{4p^2}\mathbf{p}\cdot\left\{\mathbf{G}^{(0)}\times\left[\frac{1}{p_0}\mathbf{G}^{(0)}\times\left(\mathbf{p}f^{(0)}\delta(p^2)\right)\right]\right\} \\
&\quad -\frac{p_0}{p^2}\Pi_\mu^{(2)}p^\mu f^{(0)}\delta(p^2) \\
&\quad +\frac{1}{p^2}\mathbf{p}\cdot\left(\mathbf{\Pi}^{(2)}p_0 - \Pi_0^{(2)}\mathbf{p}\right)f^{(0)}\delta(p^2) \qquad (30)
\end{aligned}$$

$f^{(0,1,2)}$ are described by evolution equation at each order

Second order results

- Spatial components $\boldsymbol{\mathscr{J}}^{(n)}$ are given by $\frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(n)}$ and a function of lower order time-components $\mathscr{J}_0^{(n-1)}, \mathscr{J}_0^{(n-2)}, \cdots \mathscr{J}_0^{(0)}$:

$$\begin{aligned}
\boldsymbol{\mathscr{J}}^{(0)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(0)} \\
\boldsymbol{\mathscr{J}}^{(1)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(1)} + \frac{s}{2p_0}\mathbf{G}^{(0)} \times \boldsymbol{\mathscr{J}}^{(0)} \\
\boldsymbol{\mathscr{J}}^{(2)} &= \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(2)} + \frac{s}{2p_0}\mathbf{G}^{(0)} \times \boldsymbol{\mathscr{J}}^{(1)} - \frac{1}{p_0}\Pi_0^{(2)}\boldsymbol{\mathscr{J}}^{(0)} \\
&\quad + \frac{1}{p_0}\boldsymbol{\Pi}^{(2)}\mathscr{J}_0^{(0)}
\end{aligned} \tag{31}$$

Mass-shell condition up to $O(\hbar)$

- We collect first three lines of Eq. (30), $\mathscr{J}_0^{(0)} + \hbar \mathscr{J}_0^{(1)} + \hbar^2 \mathscr{J}_0^{(2)}$, to obtain

$$\mathscr{J}_0 \approx p_0 f(x,p)\delta(\tilde{p}^2) \tag{32}$$

where

Quantum effect

$$\begin{aligned} \tilde{p}^2 &\equiv p^2 + \hbar s Q \frac{\mathbf{p}\cdot\mathbf{B}}{p_0} \\ f(x,p) &\equiv f^{(0)} + \hbar f^{(1)} + \hbar^2 f^{(2)} \end{aligned} \tag{33}$$

- The mass-shell condition $\delta(\tilde{p}^2)$ gives

$\boldsymbol{\Omega}_p \equiv \frac{\mathbf{p}}{2|\mathbf{p}|^3}$

$$E_p^{(\pm)} = \pm E_p (1 \mp \hbar s Q \mathbf{B}\cdot\boldsymbol{\Omega}_p) \tag{34}$$

Son, Yamamoto (2013); Manuel, Torres-Rincon (2013); Gao, QW (2015); Hidaka, Yang, Pu (2017); Huang, Shi, Jiang, Liao, Zhuang (2018); Gao, Liang, QW, Wang (2018)

CKE in three-momentum

- For $|\mathbf{p}|\neq 0$, we obtain CKE for particle by $\int_{0^+}^{\infty} dp_0$,

$$\mathbf{v}\equiv\nabla_p E_p^{(+)} \qquad \mathbf{\Omega}_p\equiv\frac{\mathbf{p}}{2|\mathbf{p}|^3} \qquad \tilde{\mathbf{E}}\equiv\mathbf{E}-Q^{-1}\nabla_x E_p^{(+)}$$

$$\begin{aligned}
&(1+\hbar sQ\mathbf{\Omega}_p\cdot\mathbf{B})\,\partial_t f(x,E_p,\mathbf{p})\\
&+\left[\mathbf{v}+\hbar sQ(\mathbf{E}\times\mathbf{\Omega}_p)+\hbar sQ\frac{1}{2|\mathbf{p}|^2}\mathbf{B}\right]\cdot\nabla_x f(x,E_p,\mathbf{p})\\
&+\left[Q\tilde{\mathbf{E}}+Q\mathbf{v}\times\mathbf{B}+\hbar sQ^2(\mathbf{E}\cdot\mathbf{B})\mathbf{\Omega}_p\right]\cdot\nabla_p f(x,E_p,\mathbf{p}) = 0 \quad (35)
\end{aligned}$$

- At $|\mathbf{p}|=0$, there are two additional terms in the above CKE which are singular but were previously neglected,

$$\begin{aligned}
&\hbar s\,(\mathbf{E}\cdot\mathbf{B})\,(\nabla_p\cdot\mathbf{\Omega}_p)\,f(x,E_p,\mathbf{p})\\
&-\lim_{\Lambda\to 0}\frac{2\hbar s}{\Lambda}(\mathbf{E}\cdot\mathbf{p})\,(\mathbf{B}\cdot\mathbf{p})\,\delta'(\Lambda^2-\mathbf{p}^2)f(x,\Lambda,\mathbf{p}) \quad (36)
\end{aligned}$$

Derivation of Eq. (35):

Son, Yamamoto (2013); Manuel, Torres-Rincon (2013); Chen, Pu, QW, Wang (2013); Hidaka, Yang, Pu (2017);

Gao, Liang, QW, Wang (2018); Huang, Shi, Jiang, Liao, Zhuang (2018)

New source of chiral anomaly

- The second term comes from the total derivative in p_0 and are relevant to the anomalous conservation equation

Previous terms (I_1)

New terms ($I_2 = -I_1$)

New terms ($I_3 = I_1$)

$$\begin{aligned}\partial_t j_0 + \nabla_x \cdot \mathbf{j} &= -\frac{\hbar s Q^2}{2}\int d^3\mathbf{p}\Big[(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}_p\cdot\nabla_p f \\ &\quad +(\mathbf{E}\cdot\mathbf{B})\left(\nabla_p\cdot\boldsymbol{\Omega}_p\right) f \\ &\quad -\lim_{\Lambda\to 0}\frac{2}{\Lambda}(\mathbf{E}\cdot\mathbf{p})(\mathbf{B}\cdot\mathbf{p})\delta'(\Lambda^2-\mathbf{p}^2)f\Big],\end{aligned} \quad (37)$$

- Since

$$\begin{aligned} I_1 + I_2 &\sim \int d^3\mathbf{p}\nabla_p(\boldsymbol{\Omega}_p f) = 0 \\ I_1 &= I_3 = \hbar\frac{sQ^2}{4\pi^2}(\mathbf{E}\cdot\mathbf{B}) \end{aligned} \quad (38)$$

so I_3 is left. Also $I_2 + I_3 = 0$ to leave I_1 to contribute (previous result).

Related works: Landsteiner, Rebhan (2011); Mueller, Venugopalan (2017, 2018); Fujikawa (2018)

Decomposition of WF in a general reference frame

- With an auxiliary time-like vector n^μ ($n^2 = 1$), we decompose X^μ as

$$X^\mu = (X \cdot n)n^\mu + \bar{X}^\mu \tag{39}$$

Eq. (31) becomes ($\nabla^\mu \equiv G^\mu_{(0)} = \partial^\mu_x - QF^{\mu\nu}\partial^p_\nu$)

$$\begin{aligned} \bar{\mathscr{J}}^{(0)}_\mu &= \bar{p}_\mu \frac{n \cdot \mathscr{J}^{(0)}}{n \cdot p} \\ \bar{\mathscr{J}}^{(1)}_\mu &= \bar{p}_\mu \frac{n \cdot \mathscr{J}^{(1)}}{n \cdot p} - \frac{s}{2(n \cdot p)} \epsilon^{\mu\nu\rho\sigma} n_\nu \nabla_\sigma \mathscr{J}^{(0)}_\rho \end{aligned} \tag{40}$$

Eq. (30) becomes

$$\begin{aligned} \mathscr{J}^{(0)}_{\parallel\mu} &= n_\mu (n \cdot p) f^{(0)} \delta\left(p^2\right) \\ \mathscr{J}^{(1)}_{\parallel\mu} &= n_\mu (n \cdot p) f^{(1)} \delta\left(p^2\right) - sQn_\mu (B \cdot p) f^{(0)} \delta'\left(p^2\right) \end{aligned} \tag{41}$$

WF is independent of n^μ

- We can combine Eq. (40) and (41) as

$$
\begin{aligned}
\mathscr{J}_\mu^{(0)} &= p_\mu \frac{n\cdot\mathscr{J}^{(0)}}{n\cdot p} \\
\mathscr{J}_\mu^{(1)} &= p_\mu \frac{n\cdot\mathscr{J}^{(1)}}{n\cdot p} - \frac{s}{2(n\cdot p)}\epsilon^{\mu\nu\rho\sigma} n_\nu \nabla_\sigma \mathscr{J}_\rho^{(0)}
\end{aligned}
\tag{42}
$$

- We can also choose any other time-like vector n'_μ to make the decomposition

$$
\begin{aligned}
\mathscr{J}'^{\mu}_{(0)} &= p^\mu \frac{n'\cdot\mathscr{J}^{(0)}}{n'\cdot p} \\
\mathscr{J}'^{\mu}_{(1)} &= p^\mu \frac{n'\cdot\mathscr{J}^{(1)}}{n'\cdot p} - \frac{s}{2n'\cdot p}\epsilon^{\mu\nu\rho\sigma} n'_\nu \nabla_\sigma \mathscr{J}_\rho^{(0)}
\end{aligned}
\tag{43}
$$

WF is independent of n^μ

- We can easily check $\mathscr{J}'^{\mu}_{(0)} = \mathscr{J}^{\mu}_{(0)}$ as

$$\begin{aligned}\delta \mathscr{J}^{\mu}_{(0)} &= \mathscr{J}'^{\mu}_{(0)} - \mathscr{J}^{\mu}_{(0)} \\ &= p^\mu \frac{(n\cdot p)\left(n'\cdot \mathscr{J}_{(0)}\right) - (n'\cdot p)\left(n\cdot \mathscr{J}_{(0)}\right)}{(n'\cdot p)(n\cdot p)} \\ &= 0 \end{aligned} \tag{44}$$

  where we have used $\mathscr{J}^{\mu}_{(0)} \propto p^\mu$. Then we see $\mathscr{J}^{(0)}_{\mu}$ is independent of the choice of n^μ.
- We can also verify $\mathscr{J}'^{\mu}_{(1)} = \mathscr{J}^{\mu}_{(1)}$ (more complicated but straightforward).
- So up to $O(\hbar)$, we see $\mathscr{J}^{\mu}$ is independent of n^μ.

Lorentz covariance and side-jump

- We look at the difference of distribution function $\delta f_{(1)}$ from the change of reference frame

$$\begin{aligned} \delta f_{(1)} &= \frac{n\cdot \mathscr{J}_{(1)}}{n\cdot p} - \frac{n'\cdot \mathscr{J}_{(1)}}{n'\cdot p} \\ &= p^{\mu}\frac{n^{\rho}n'^{\sigma}\left(p_{\rho}\mathscr{J}_{\sigma}^{(1)} - p_{\sigma}\mathscr{J}_{\rho}^{(1)}\right)}{(n'\cdot p)(n\cdot p)} \\ &= -\hbar s\epsilon_{\mu\nu\rho\sigma}\frac{n^{\rho}n'^{\sigma}\nabla^{\mu}\mathscr{J}_{(0)}^{\nu}}{2(n'\cdot p)(n\cdot p)} \end{aligned} \tag{45}$$

where we have used $2s(p_{\rho}\mathscr{J}_{\sigma}^{(1)} - p_{\sigma}\mathscr{J}_{\rho}^{(1)}) = -\hbar\epsilon_{\mu\nu\rho\sigma}\nabla^{\mu}\mathscr{J}_{(0)}^{\nu}$.

Chen, Son, Stephanov, Yee, Yin (2014); Hidaka, Pu, Yang (2017);

Gao, Liang, QW, Wang (2018); Huang, Shi, Jiang, Liao, Zhuang (2018)

CVE and Lorentz covariance

- If we neglect EM field we have

$$\delta f_{(1)} = -s\frac{n'_\alpha p_\gamma \tilde{\Omega}^{\alpha\gamma}}{2\,(n'\cdot p)}\frac{df_{(0)}}{d(\beta\cdot p)} + s\frac{n_\alpha p_\gamma \tilde{\Omega}^{\alpha\gamma}}{2\,(n\cdot p)}\frac{df_{(0)}}{d(\beta\cdot p)} \tag{46}$$

Then we have

$$f_{(1)} = -s\frac{n_\alpha p_\gamma \tilde{\Omega}^{\alpha\gamma}}{2\,(n\cdot p)}\frac{df_{(0)}}{d(\beta\cdot p)}$$

Then insert it to (42), we obtain

$$\begin{aligned}\mathscr{J}_\mu^{(1)} &= -p^\mu \frac{s}{2\,(n\cdot p)} n_\alpha p_\gamma \tilde{\Omega}^{\alpha\gamma}\frac{df_{(0)}}{d(\beta\cdot p)}\delta(p^2) \quad \rightarrow j^\mu_{CVE}(1)\\ &\quad -\frac{s}{2n\cdot p}\epsilon^{\mu\nu\rho\sigma} n_\nu p_\rho (\partial^x_\sigma f_{(0)})\delta(p^2) \quad \rightarrow j^\mu_{CVE}(2)\\ &= -\frac{s}{2}\tilde{\Omega}^{\mu\nu} p_\nu \frac{df_{(0)}}{d(\beta\cdot p)}\delta(p^2) \quad \rightarrow \text{independent of } n^\mu \end{aligned} \tag{47}$$

Gao, Liang, Pu, QW, Wang (2012); Gao, Liang, QW, Wang (2018); Gao, Pang, QW (2018)

Chiral Vortical Effect

- Assuming Fermi-Dirac distribution for $f_{(0)}$, the CVE current

$$\begin{aligned} j^{\mu}_{CVE}(1) &= \frac{1}{3} T \xi_s \omega^{\mu} \\ j^{\mu}_{CVE}(2) &= \frac{2}{3} T \xi_s \omega^{\mu} \end{aligned} \tag{48}$$

where $(s = \pm)$

$s = \pm 1$

$$\begin{aligned} \xi_s &= \frac{1}{2}(\xi + s\xi_5) \\ \xi &= \frac{\mu\mu_5}{\pi^2} \\ \xi_5 &= \frac{1}{6}T^2 + \frac{1}{2\pi^2}(\mu^2 + \mu_5^2) \end{aligned} \tag{49}$$

CVE conductivity

Vlienkin (1978); Son, Surowka (2011); Ermenger et al. (2009); Landsteiner et al. (2011); Gao, Liang, Pu, QW, Wang (2012); Huang, Sadofyev (2018); ......

CVE: normal and magnetization current

- Choosing $n^\mu = u^\mu = (1,0,0,0)$, we have 3D form

$$
\begin{aligned}
\mathbf{j}_{CVE}(1) &\approx \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{|\mathbf{p}|}\left[f_{\mathrm{FD}}\left(\beta|\mathbf{p}| - \beta\mu_s - s\hbar\frac{\mathbf{p}\cdot\boldsymbol{\omega}}{2|\mathbf{p}|}\right)\right. \\
&\quad \left. + f_{\mathrm{FD}}\left(\beta|\mathbf{p}| + \beta\mu_s - s\hbar\frac{\mathbf{p}\cdot\boldsymbol{\omega}}{2|\mathbf{p}|}\right)\right] \\
\mathbf{j}_{CVE}(2) &= \nabla\times\int\frac{d^3p}{(2\pi)^3}s\hbar\frac{\mathbf{p}}{2|\mathbf{p}|^2} \\
&\quad \times\left[f_{\mathrm{FD}}(\beta|\mathbf{p}| - \beta\mu_s) + f_{\mathrm{FD}}(\beta|\mathbf{p}| + \beta\mu_s)\right] \qquad (50)
\end{aligned}
$$

$\rightarrow$ Normal current

Spin-vorticity coupling

Magnetic moment of chiral fermion

$\rightarrow$ Magnetization current $\nabla\times\mathbf{M}$

Chen, Son, Stephanov, Yee, Yin (2014); Chen, Son, Stephanov (2015); Gao, Pang, QW (2018)

Summary of main results

- Covariant Wigner function method is a very useful tool for chiral fermions (many properties).
- Proof of a theorem: (1) to any order of $\hbar$ that only the time-component of the WF is independent while other components are explicit derivatives.
- Proof of a theorem: (2) to any order of $\hbar$ that a system of kinetic equations for multiple-components of WF can be reduced to one CKE involving only the single-component distribution function.
- In this formalism we can also obtain the CVE current as a sum of two parts, which can be identified as the normal current and magnetization current. Each part depends on n^μ, but the sum is frame independent provided the distribution function is modified corresponding to the change of reference frames ("side-jump").

Backup slides

Proof of Eq. (28)

$$
\begin{aligned}
\frac{1}{p_0}\mathbf{p}\cdot[(\mathbf{B}\times\nabla_p)\times\mathbf{p}\mathscr{J}_0^{(0)}] &= \frac{1}{p_0}p_i\epsilon_{ijk}(\mathbf{B}\times\nabla_p)_j p_k\mathscr{J}_0^{(0)} \\
&= \frac{1}{p_0}p_i\epsilon_{ijk}\epsilon_{jlm}B_l\times\partial_m^p p_k\mathscr{J}_0^{(0)} \\
&= -(\delta_{il}\delta_{km}-\delta_{im}\delta_{kl})p_iB_l\partial_m^p[p_kf_0\delta(p^2)] \\
&= -(\mathbf{p}\cdot\mathbf{B})\partial_k^p[p_kf_0\delta(p^2)]+p_iB_k\partial_i^p[p_kf_0\delta(p^2)] \\
&= -2(\mathbf{p}\cdot\mathbf{B})f_0\delta(p^2)
\end{aligned}
$$

The n-th order evolution equations

- The evolution equations at $O(\hbar^n)$ read

Contains $\partial_t \mathscr{J}_0^{(n)}$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \mathscr{J}_0^{(n-2i)} + \mathbf{G}^{(2i)} \cdot \boldsymbol{\mathscr{J}}^{(n-2i)} \right] = 0 \tag{51}$$

Contains $\partial_t \boldsymbol{\mathscr{J}}^{(n)}$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \boldsymbol{\mathscr{J}}^{(n-2i)} + \mathbf{G}^{(2i)} \mathscr{J}_0^{(n-2i)} \right] = 2s \sum_{i=0}^{[(n+1)/2]} \mathbf{\Pi}^{(2i)} \times \boldsymbol{\mathscr{J}}^{(n-2i+1)} \tag{52}$$

The n-th order constraint equations

- The constraint equations at $O(\hbar^n)$ read

Mass-shell condition
$$\sum_{i=0}^{[n/2]}\left[\Pi_0^{(2i)}\mathscr{J}_0^{(n-2i)}-\mathbf{\Pi}^{(2i)}\cdot\boldsymbol{\mathscr{J}}^{(n-2i)}\right]=0 \tag{53}$$

$$2s\sum_{i=0}^{[(n+1)/2]}\left[\mathbf{\Pi}^{(2i)}\mathscr{J}_0^{(n-2i+1)}-\Pi_0^{(2i)}\boldsymbol{\mathscr{J}}^{(n-2i+1)}\right]$$

Can solve $\boldsymbol{\mathscr{J}}^{(n+1)}$

Constraint equation to relate $\boldsymbol{\mathscr{J}}$ in terms of $\mathscr{J}_0$

$$=-\sum_{i=0}^{[n/2]}\mathbf{G}^{(2i)}\times\boldsymbol{\mathscr{J}}^{(n-2i)} \tag{54}$$

Solve $\mathscr{J}^{(n+1)}$ as function of $\mathscr{J}_0^{(0)}, \cdots \mathscr{J}_0^{(n+1)}$

- From Eq. (54) we can solve

$$\mathscr{J}^{(n+1)} = \frac{\mathbf{p}}{p_0}\mathscr{J}_0^{(n+1)} + \frac{s}{2p_0}\sum_{i=0}^{[n/2]}\mathbf{G}^{(2i)} \times \underbrace{\mathscr{J}^{(n-2i)}}_{\mathscr{J}^{(n)},\, \mathscr{J}^{(n-2)}\ldots} + \frac{1}{p_0}\sum_{i=1}^{[(n+1)/2]}\left[\mathbf{\Pi}^{(2i)}\underbrace{\mathscr{J}_0^{(n-2i+1)}}_{\mathscr{J}_0^{(n-1)},\, \mathscr{J}_0^{(n-3)}\ldots} - \Pi_0^{(2i)}\underbrace{\mathscr{J}^{(n-2i+1)}}_{\mathscr{J}^{(n-1)},\, \mathscr{J}^{(n-3)}\ldots}\right] \quad (55)$$

- By recursively using the above for $\mathbf{G}^{(2i)} \times \mathscr{J}^{(n-2i)}$ and $\Pi_0^{(2i)}\mathscr{J}^{(n-2i+1)}$ one can finally express $\mathscr{J}^{(n+1)}\left[\mathscr{J}_0^{(0)}, \cdots, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n+1)}\right]$.

Evolution equation for $\mathscr{J}^{(n)}$

- Evolution equation for $\partial_t \mathscr{J}^{(n)}$ in (52) becomes

$\mathbf{F}[\partial_t \mathscr{J}^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \ldots]$

$= 2s\,(\mathbf{p} \times \mathscr{J}^{(n+1)})$

$$\sum_{i=0}^{[n/2]} \left[G_0^{(2i)} \underbrace{\mathscr{J}^{(n-2i)}}_{\mathscr{J}^{(n)},\, \mathscr{J}^{(n-2)} \ldots} + \mathbf{G}^{(2i)} \underbrace{\mathscr{J}_0^{(n-2i)}}_{\mathscr{J}_0^{(n)},\, \mathscr{J}_0^{(n-2)} \ldots} \right]$$

$$= 2s\,\mathbf{p} \times \mathscr{J}^{(n+1)} + 2s \sum_{i=1}^{[(n+1)/2]} \mathbf{\Pi}^{(2i)} \times \underbrace{\mathscr{J}^{(n-2i+1)}}_{\mathscr{J}^{(n-1)},\, \mathscr{J}^{(n-3)} \ldots} \tag{56}$$

- We use Eq. (55)

$$\mathscr{J}^{(n+1)} = \frac{\mathbf{p}}{p_0} \mathscr{J}_0^{(n+1)} + \mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \ldots , \right] \tag{57}$$

$\mathbf{F}[\partial_t \mathscr{J}^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \ldots]$

$= 2s\,\mathbf{p} \times \mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)} \ldots \right]$

Evolution equation for $\mathscr{J}^{(n)}$

- The evolution equation for $\partial_t \mathscr{J}^{(n)}$ in (52) is now converted to another evolution for $\partial_t \mathscr{J}_0^{(n)}$

$$\begin{aligned} & \mathbf{F}\left[\frac{\mathbf{p}}{p_0}\partial_t \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \cdots\right] \\ = & \, 2s\, \mathbf{p} \times \mathbf{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \mathscr{J}_0^{(n-2)}, \cdots\right] \end{aligned} \qquad (58)$$

- The original evolution equation (51)

$$\begin{aligned} & (\partial_t + Q\mathbf{E}\cdot\nabla_p)\, \mathscr{J}_0^{(n)} + \mathbf{G}^{(0)} \cdot \mathscr{J}^{(n)} \\ & \sum_{i=1}^{[n/2]} \left[G_0^{(2i)} \mathscr{J}_0^{(n-2i)} + \mathbf{G}^{(2i)} \cdot \mathscr{J}^{(n-2i)}\right] \quad = \quad 0 \end{aligned} \qquad (59)$$