Interplay of symmetry & topology in noninteracting (and interacting) systems

Haruki Watanabe
University of Tokyo

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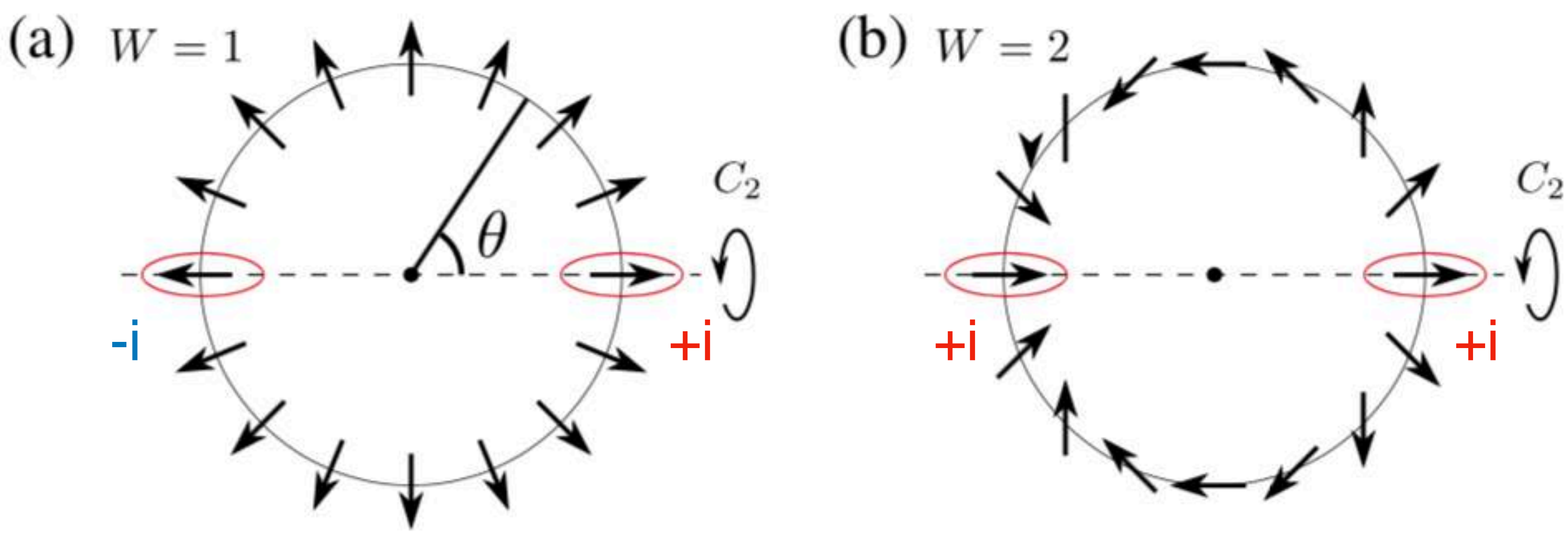

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- Symmetry detects topology

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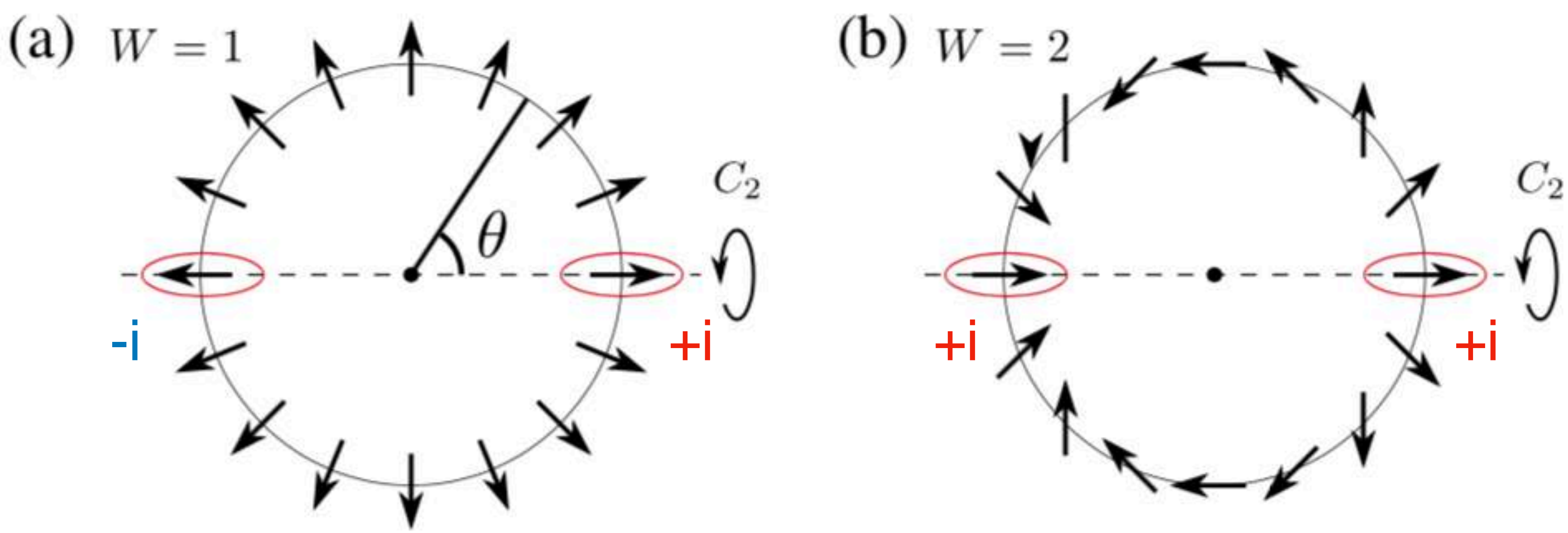

W mod 2 can be seen from the product of two rotation eigenvalues!

cf. Professor Tay-Rong Chang's talk

Fu-Kane formula

- Z_2 index for Quantum Spin Hall insulators (2D, TR)

Requires a careful gauge fixing and integration of Pfaffian in k space

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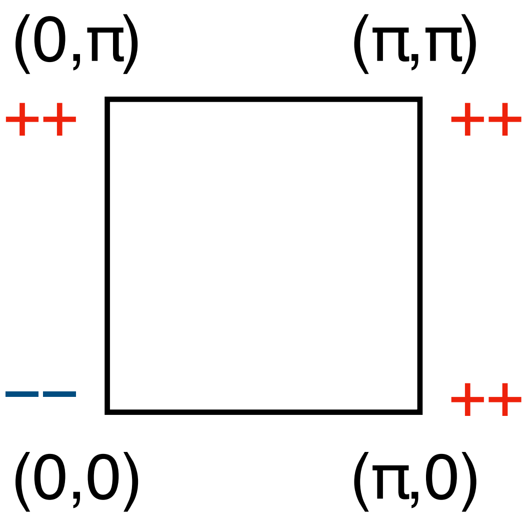
Fu-Kane formula: $\nu = \Pi_{k=\mathrm{TRIMs}}\, \xi_k = \pm 1$

Easy & helpful for material search!



Combination of inversion eigenvalues indicates Z2 QSH

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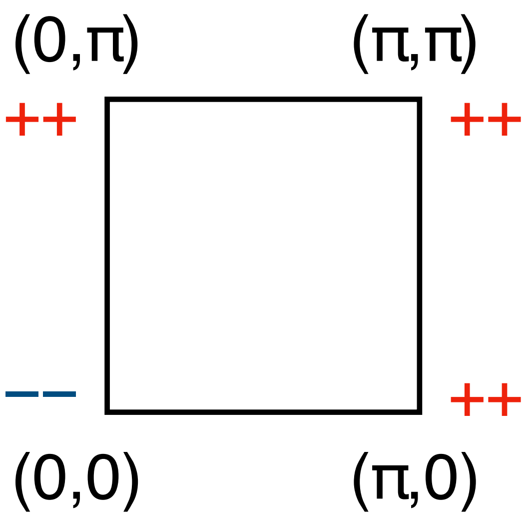

Irreducible representations of a more general space group

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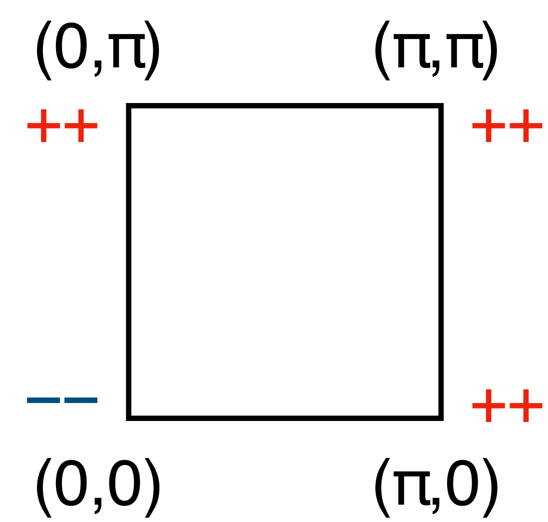


Irreducible representations of a more general space group

Combination of ~~inversion eigenvalues~~ indicates ~~Z2 QSH~~

more general topology including HOTI

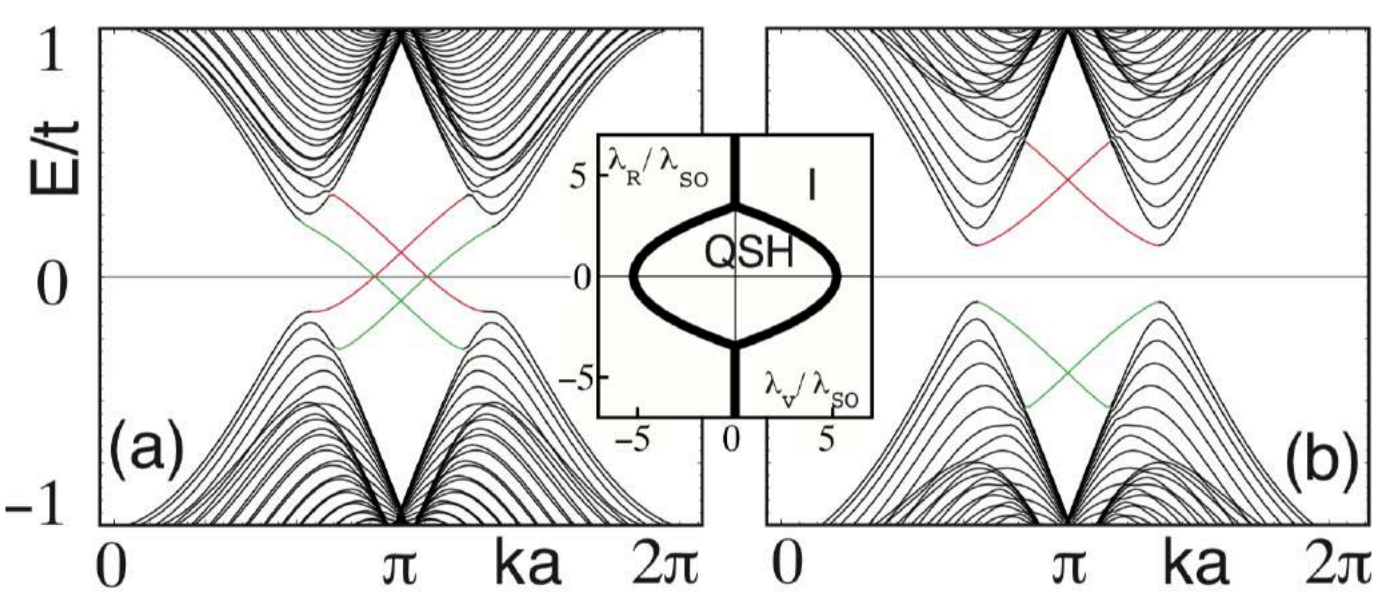
“topological” insulators

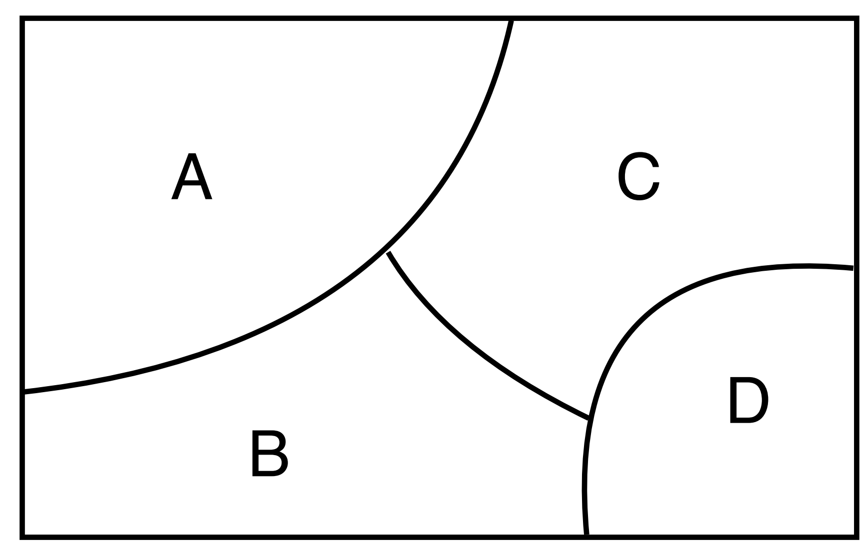
1. Presense of protected gapless edge/surface states

2. Winding number (e.g. Chern number, Z2 QSH index)

3. Obstruction in adiabatically connectinng to trivial states

   Most general definition / applicable to interacting systems





Plan of my talk

- Basics of symmetry indicators

- What can we "see" from it?

1. Conventional topological insulators (Chern, Z2 TI, etc)
2. Higher-order topological insulators
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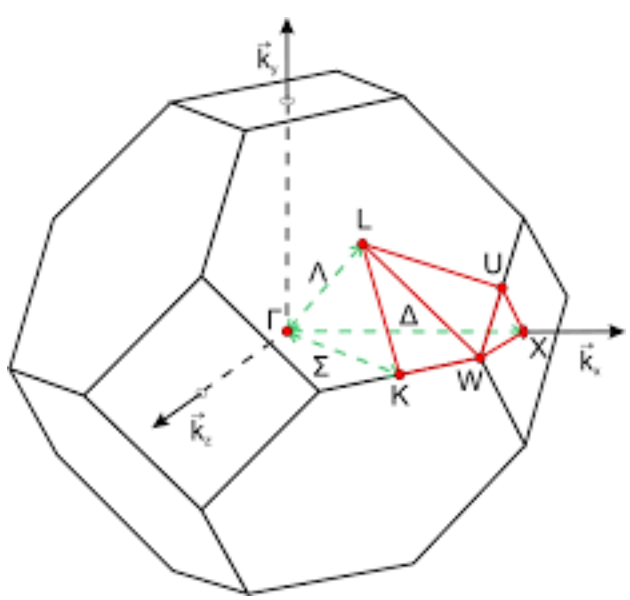
Characterizing band structure by irreducible representations

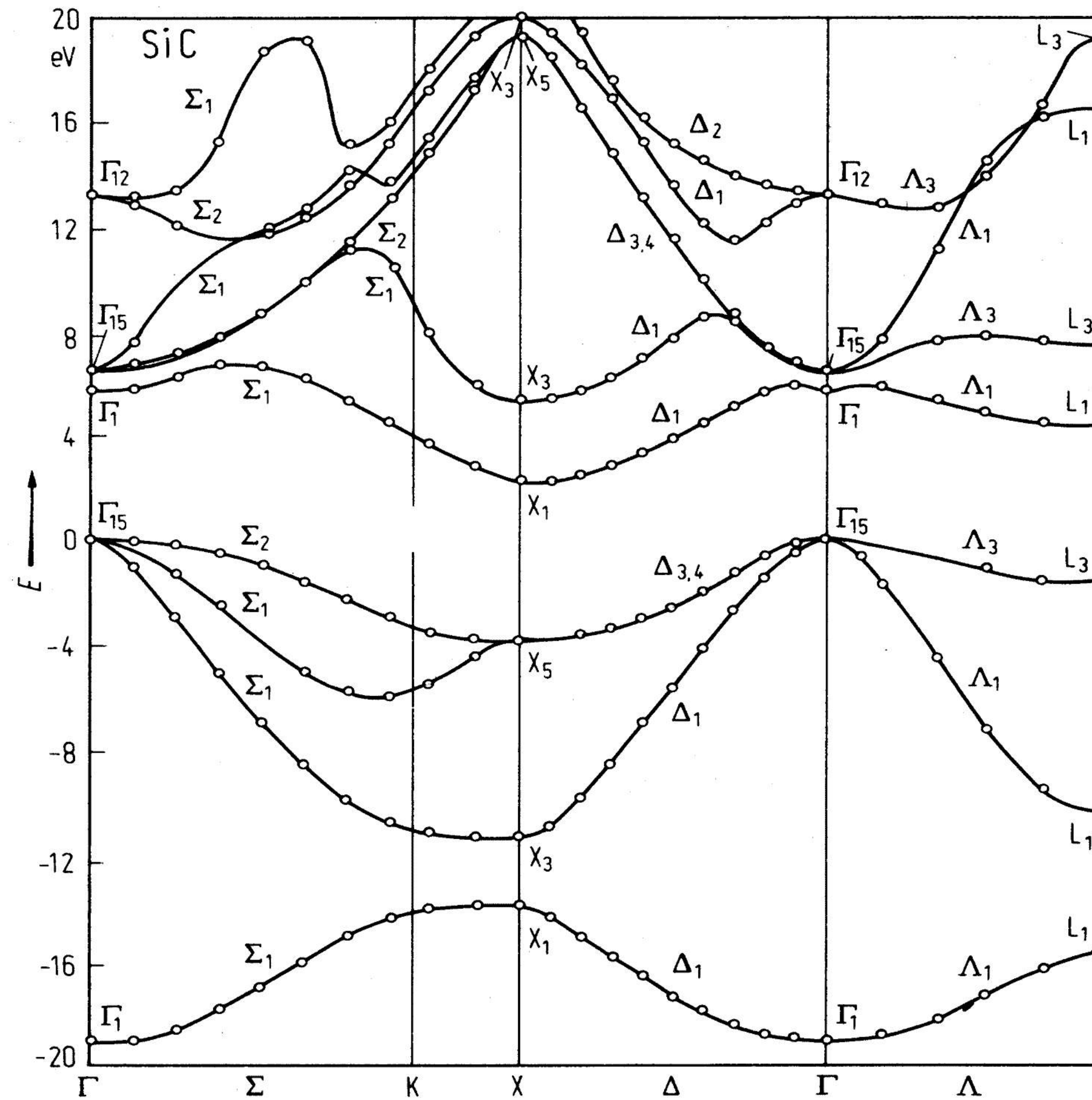
Po-Vishwanath-Watanabe, Nat. Commun. (2017)

Related works:
Bradlyn-…-Bernevig (2017)
Shiozaki-Sato-Gomi (2018)
Song-…-Fang (2018)

Representations in band structures



Hemstreet & Fong (1974)



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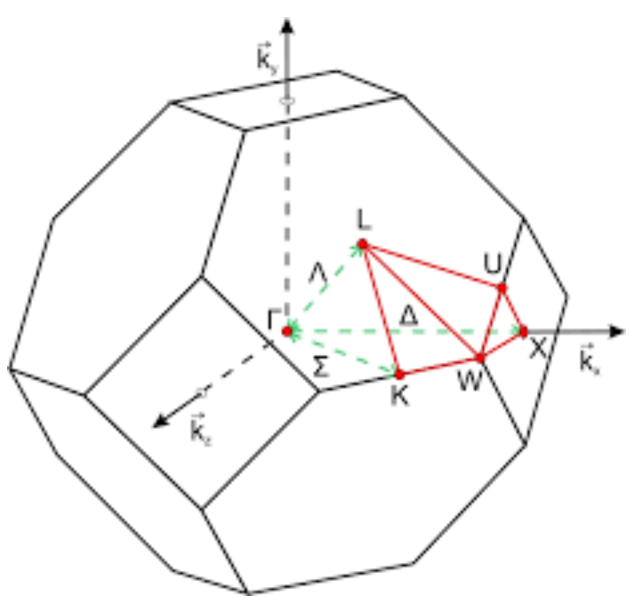

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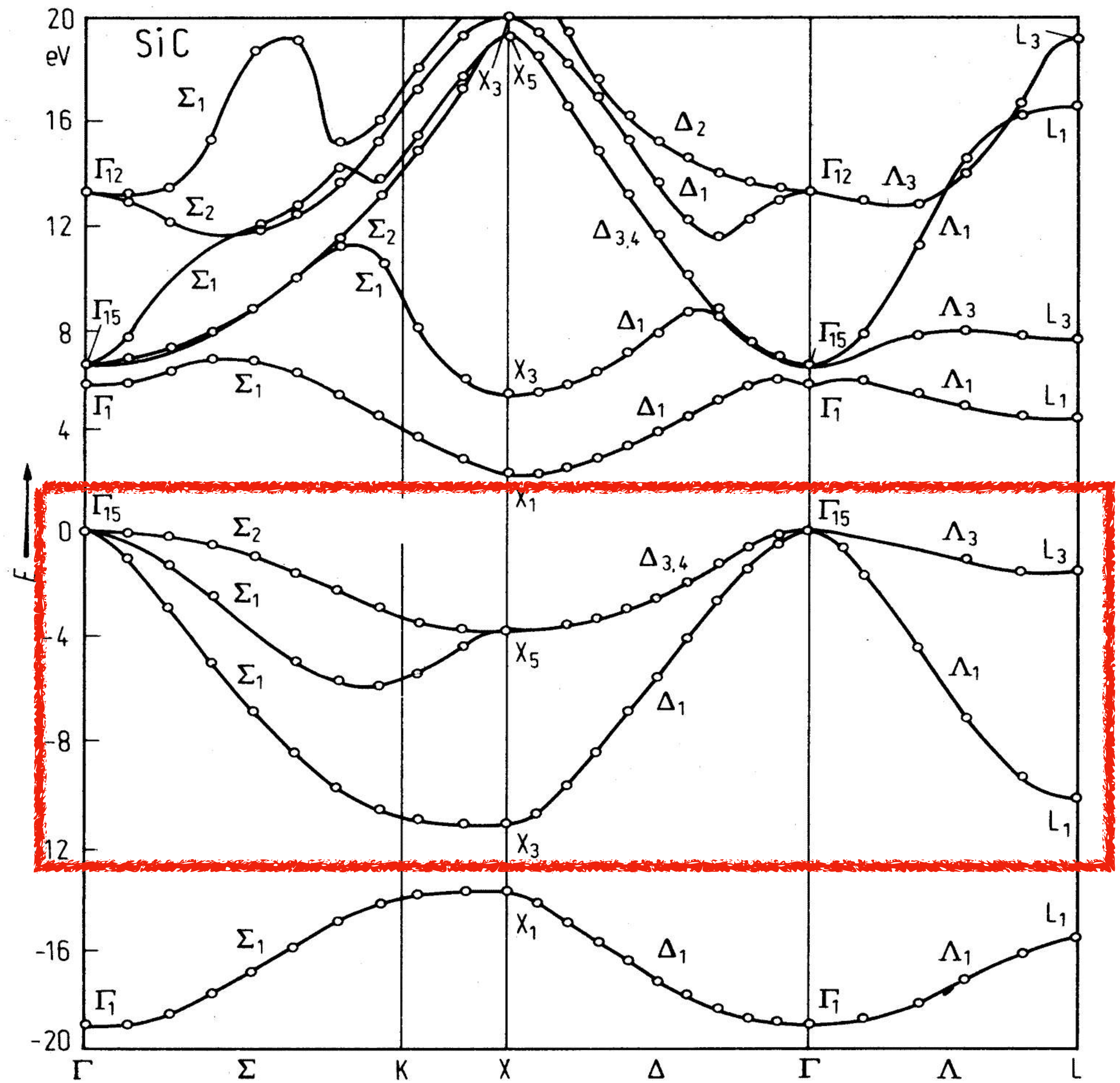

Step-by-step process

Given symmetry setting (e.g., space group G, TR, spin-orbit)

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4. Count the number of times $u_{\boldsymbol{k}}^{\alpha}$ appears in band structure $\{n_{\boldsymbol{k}}^{\alpha}\}$

※ Note compatibility relations among $\{n_{\boldsymbol{k}}^{\alpha}\}$

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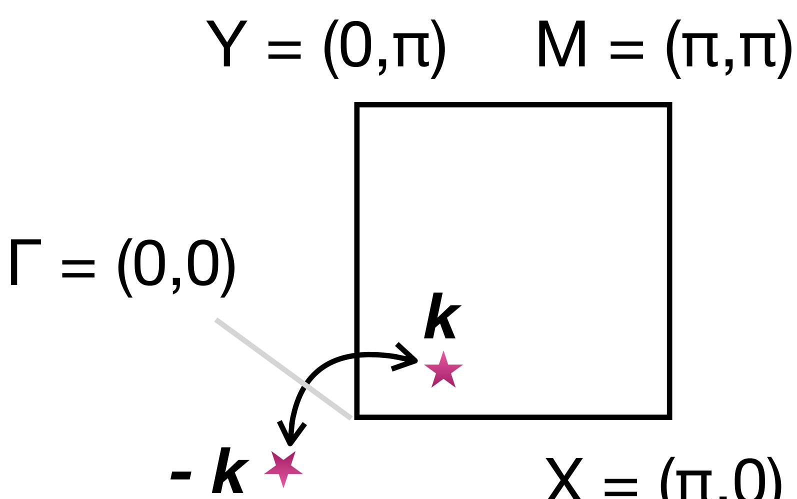
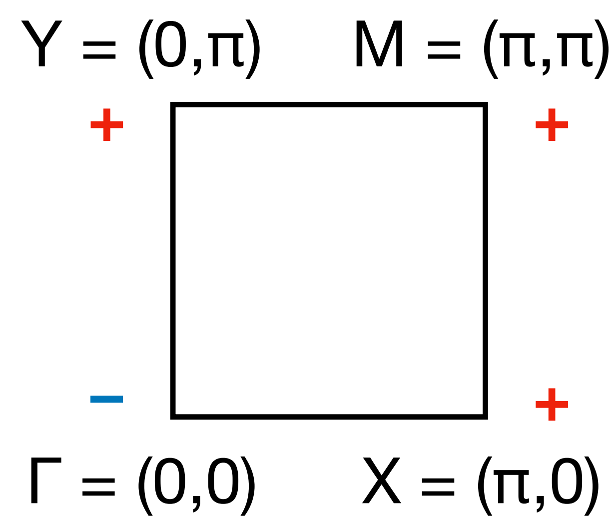
5. Form a vector $\boldsymbol{b} = (n_{\boldsymbol{k}1}^{1}, n_{\boldsymbol{k}1}^{2}, \ldots\, n_{\boldsymbol{k}2}^{1}, n_{\boldsymbol{k}2}^{2}, \ldots)$

Example: 2D lattice with inversion symmetry



Inversion $I^2 = +1$

→ eigenvalues **+1** or **-1**



(Γ,X,Y,M) : (−,+,+,+)

b = (0,1,1,0,1,0,1,0)

Band structure space {BS}

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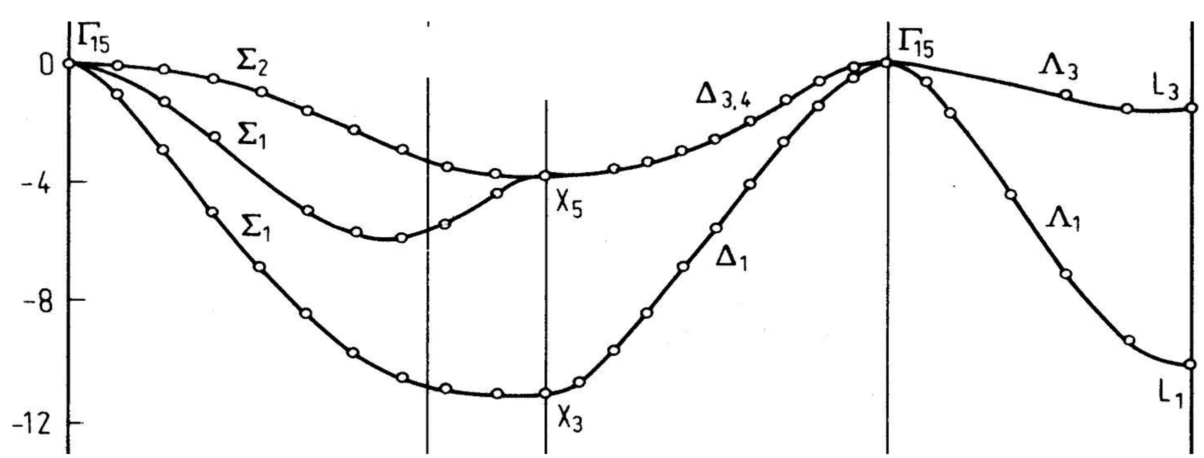
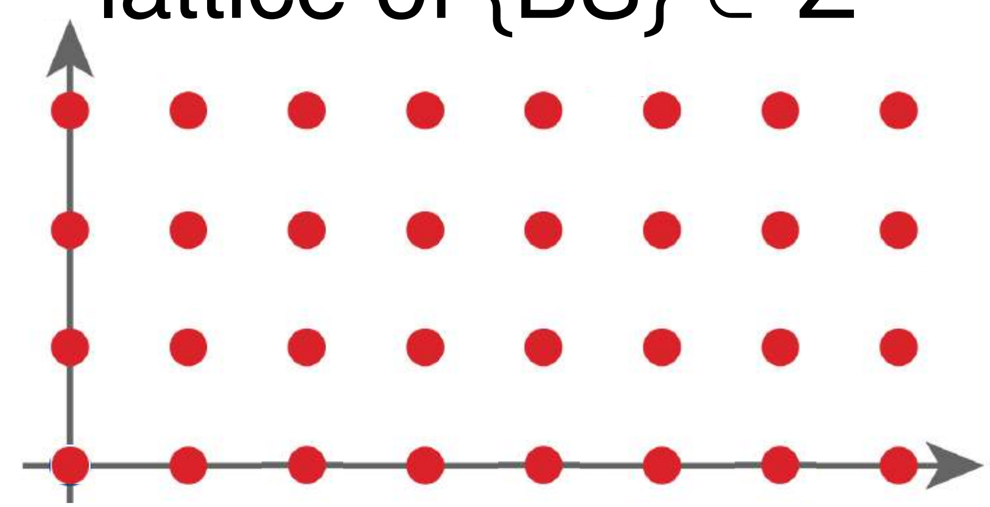
- Form a set $\boldsymbol{b}$'s (band structure space) :

$$\{BS\} = \{\ \boldsymbol{b} = \{n_{\boldsymbol{k}}^{\alpha}\}\ |\ \text{satisfying compatibility relations}\ \} \subset Z^{d_{BS}}$$



lattice of $\{BS\} \subset Z^{d_{BS}}$

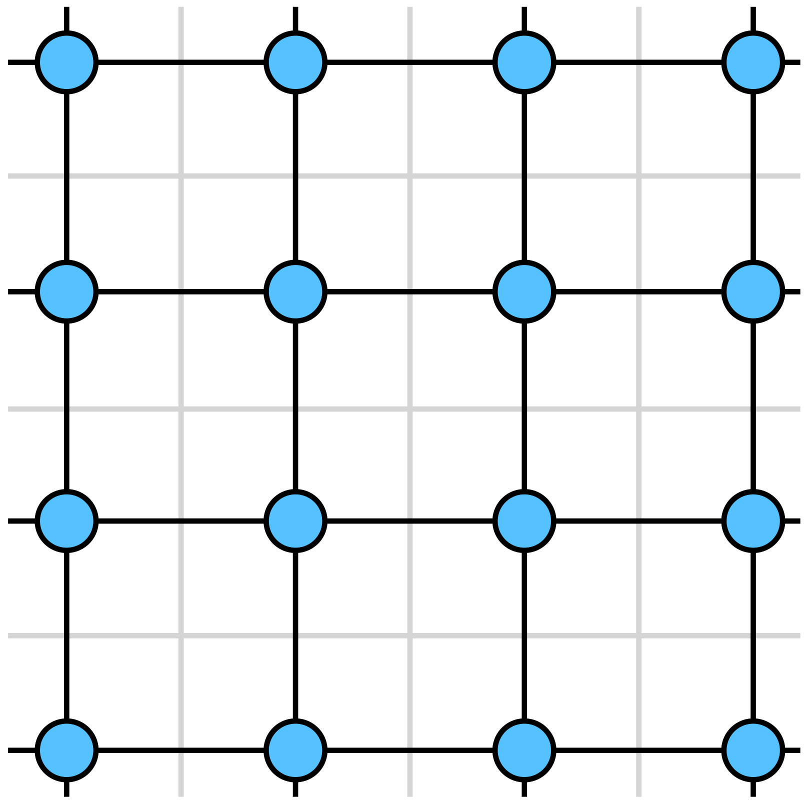


Trivial subset of {BS}

Atomic Insulators

TB model but no hopping (trivial flat bands)

Product state in real space (trivial) ⇔ Wannier orbitals



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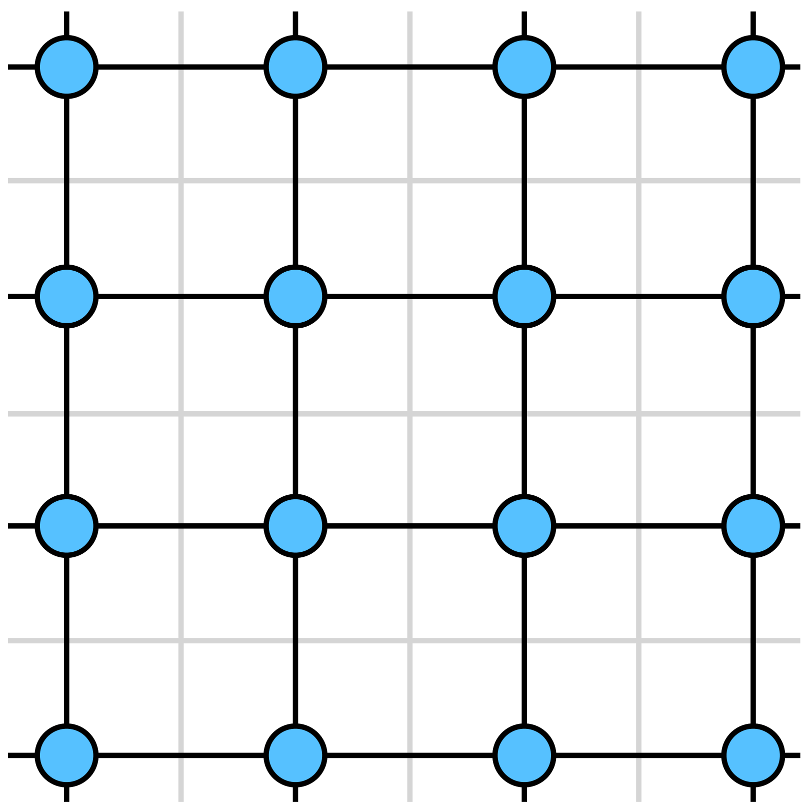
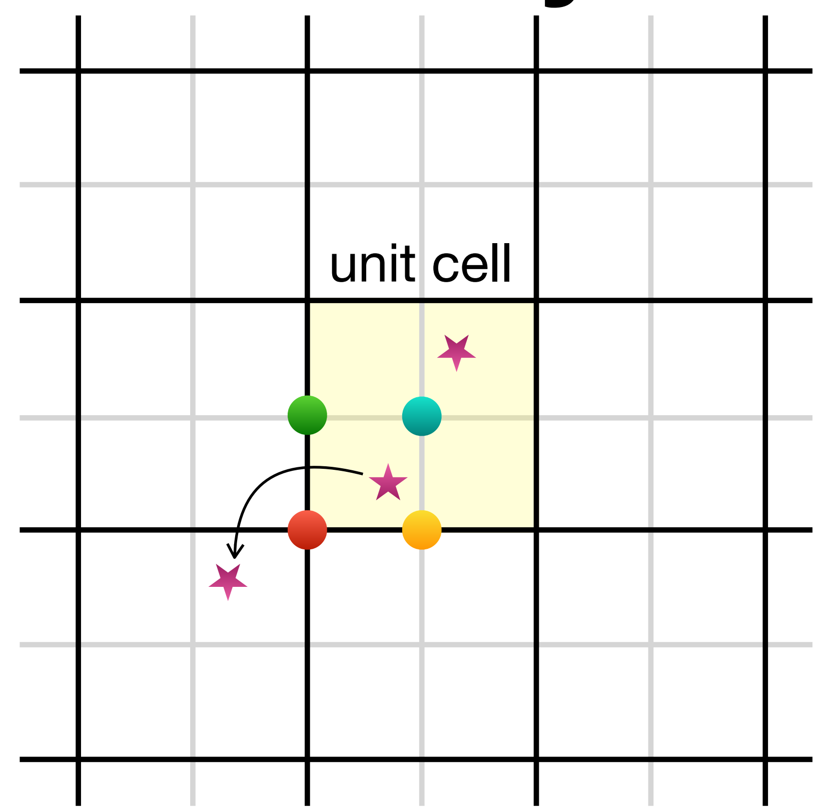
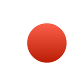
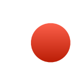
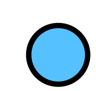
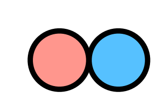
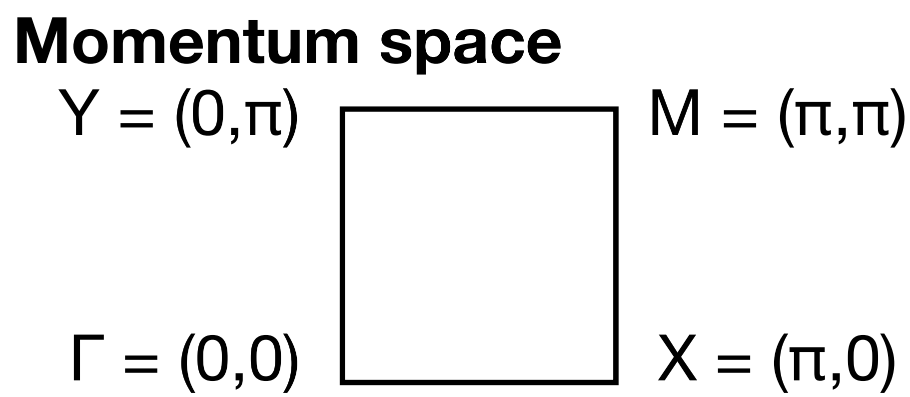


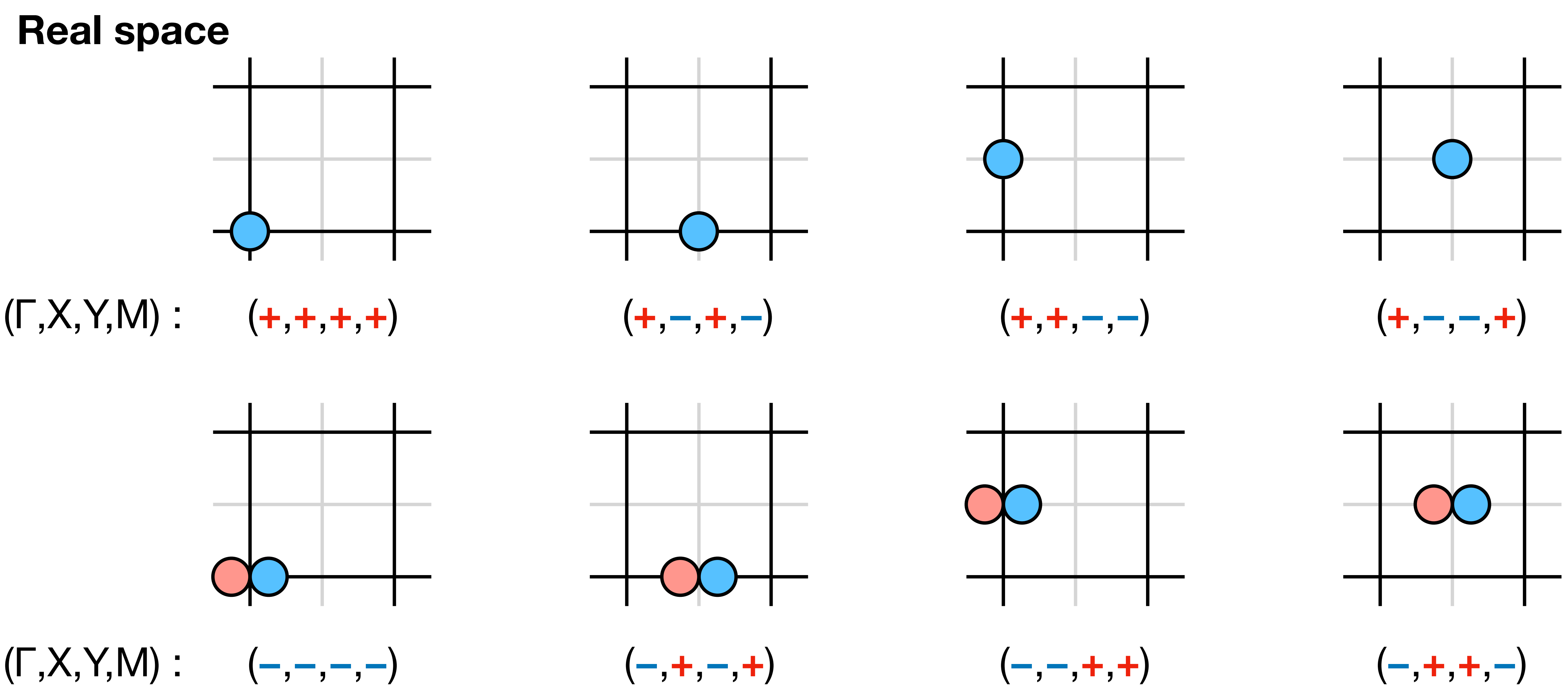


We have to specify the position x and the orbital type

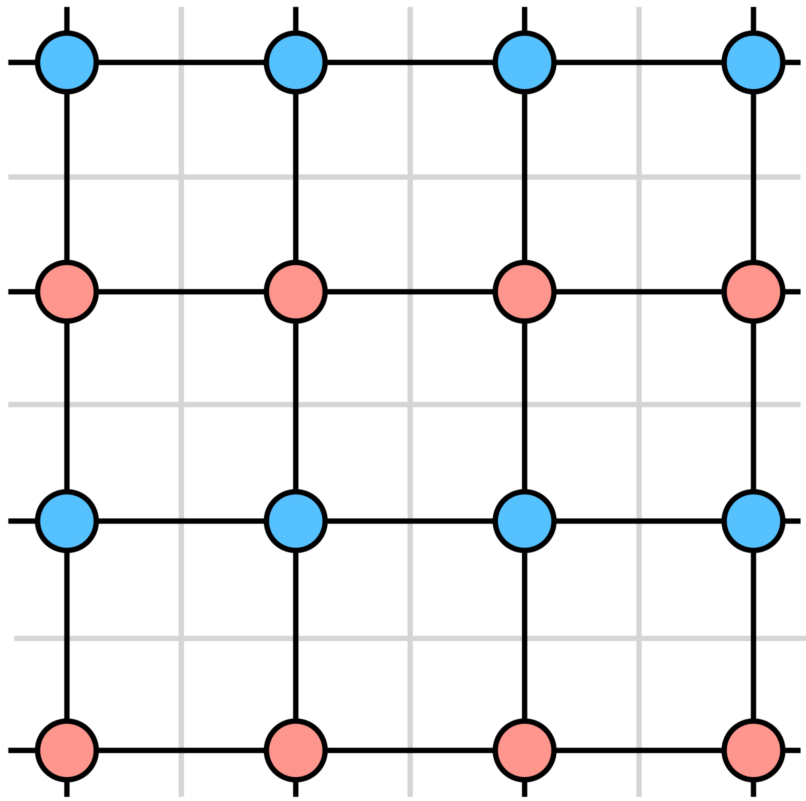
1. Choose $\boldsymbol{x}$ in unit cell. e.g. $\boldsymbol{x}$ = 
2. Find little group (site-symmetry) $G_{\boldsymbol{x}}$. $G_{\boldsymbol{x}} = \{e, I\}$ at $\boldsymbol{x}$ = 
3. Choose an orbit (an irrep of $G_{\boldsymbol{x}}$).  ($I = +1$)  ($I = -1$)

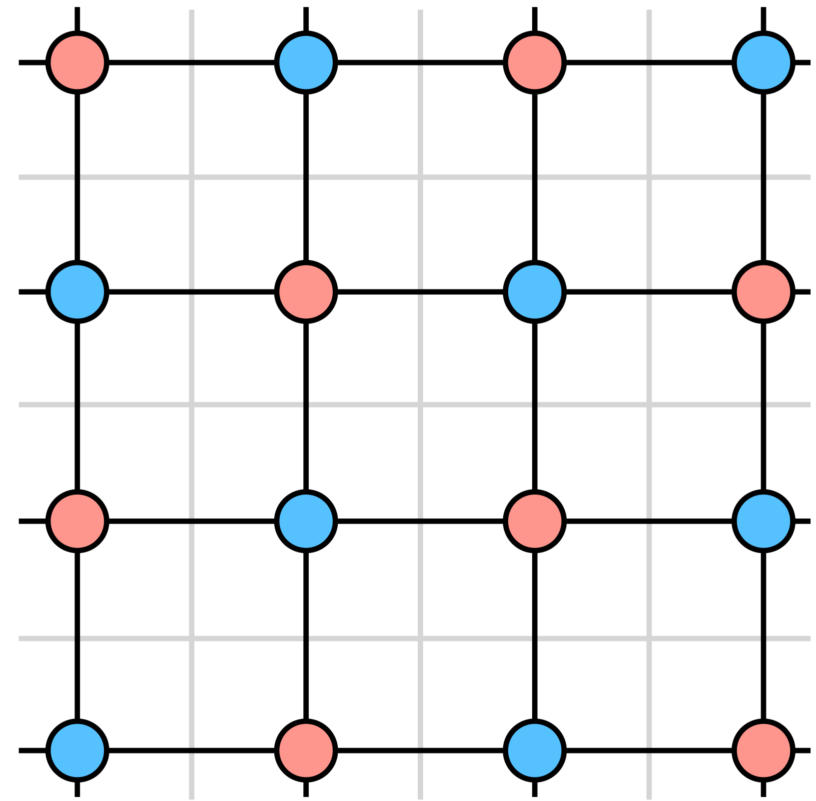
Irrep contents of AI





k = (0, π)
l = **+1**





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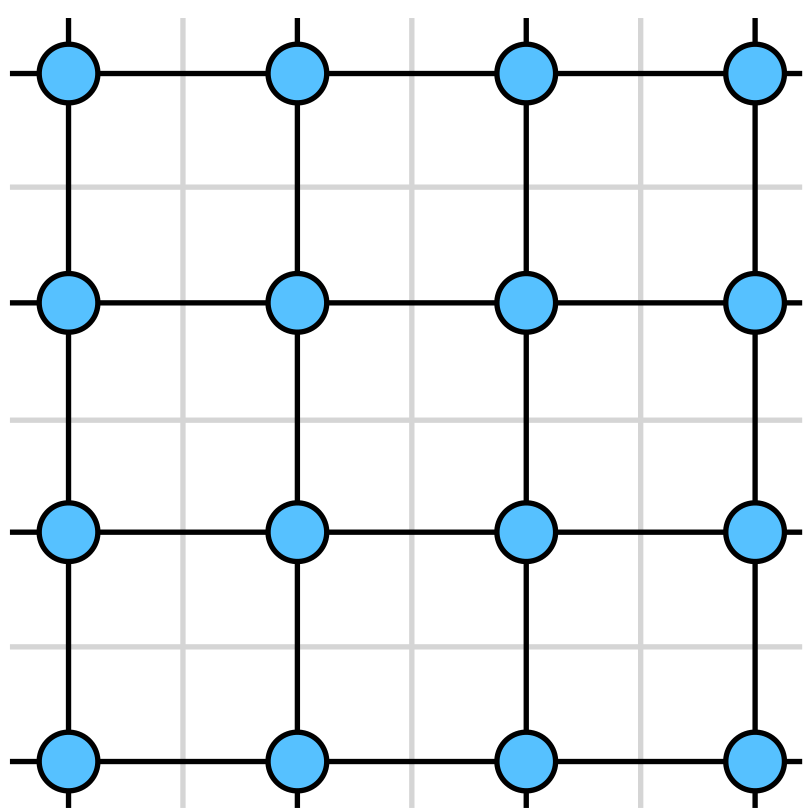

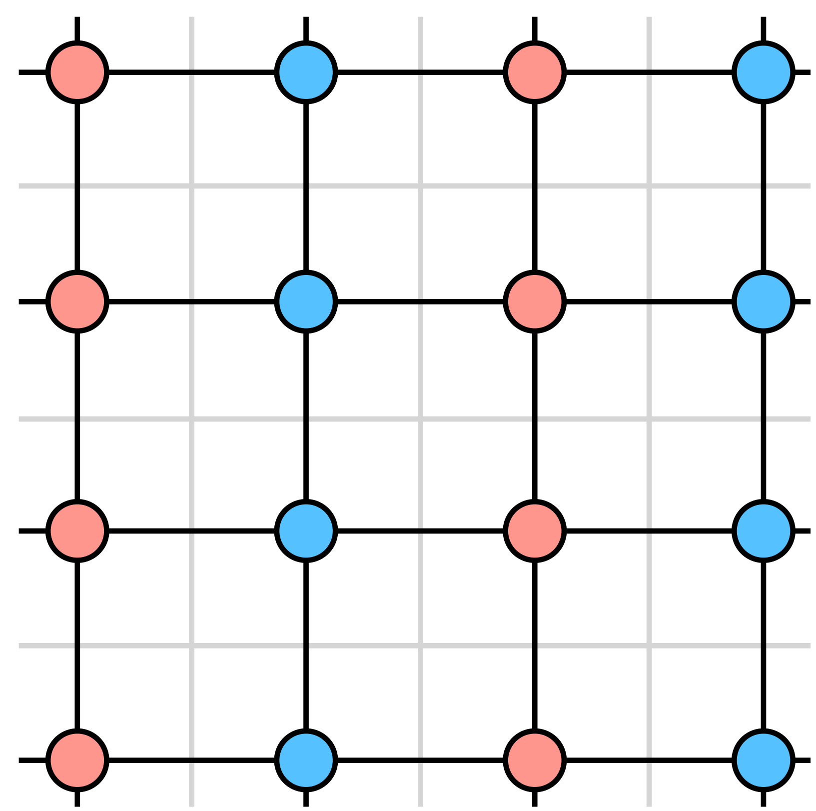

k = (π, 0)
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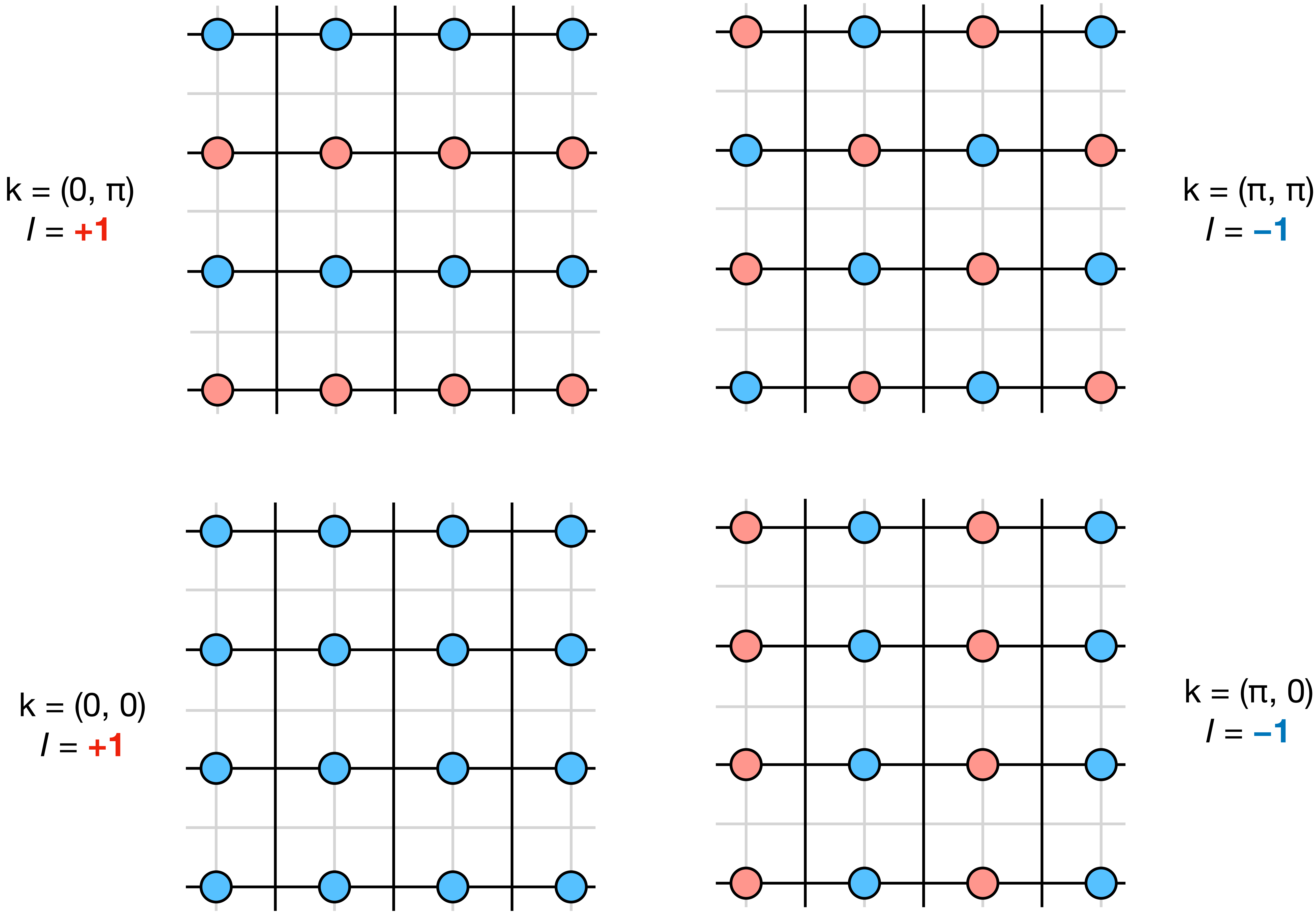
k = (0, π)
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Atomic insulator space {AI}

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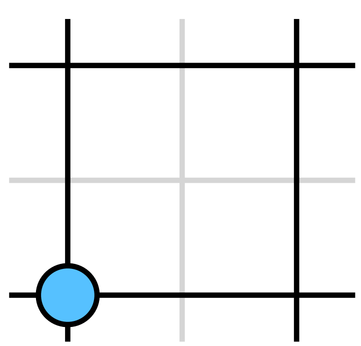
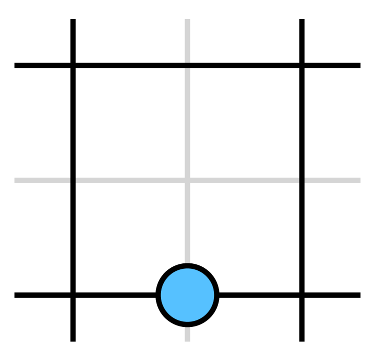
- Form the set $\boldsymbol{a}$'s (atomic insulator space) :

$$\{AI\} = \{\ \boldsymbol{a} = \{n_{\boldsymbol{k}}^{\alpha}\}\ |\ \text{corresoinding to AI}\} \subset Z^{d_{AI}}$$



(+,+,+,+)

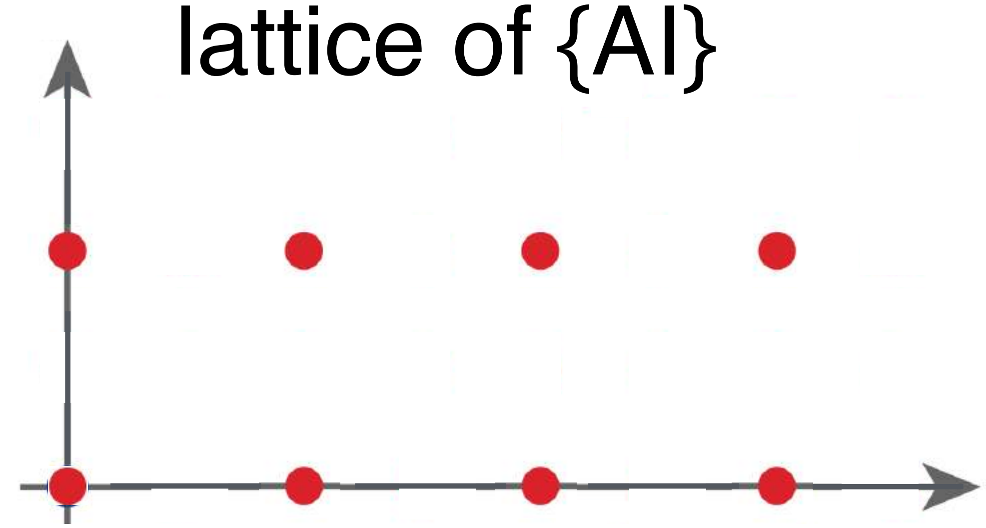
$\boldsymbol{a}_1 = (1,0,1,0,1,0,1,0)$



(+,−,+,−)

$\boldsymbol{a}_2 = (1,0,0,1,0,1,1,0)$

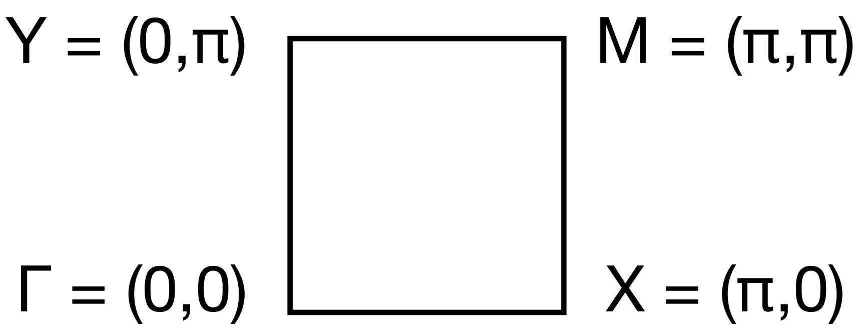
lattice of {AI}



Diagnosing the topology

Band Structures {BS}: set of ***b***'s

(Γ,X,Y,M) : $\boldsymbol{b}_1 = (+,-,-,+)$

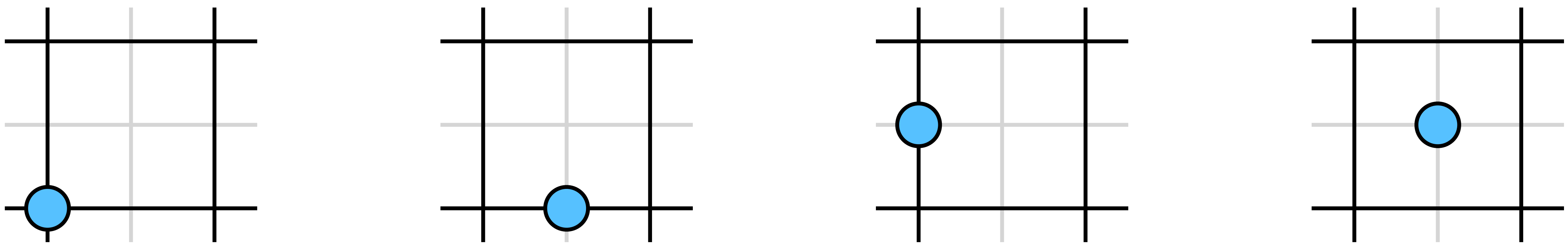


$\boldsymbol{b}_2 = (++,+-,+-,++)$

$\boldsymbol{b}_3 = (+,+,+,-)$

$\boldsymbol{b}_4 = (++,++,++,--)$

Atomic Insulators {AI}: set of ***a***'s



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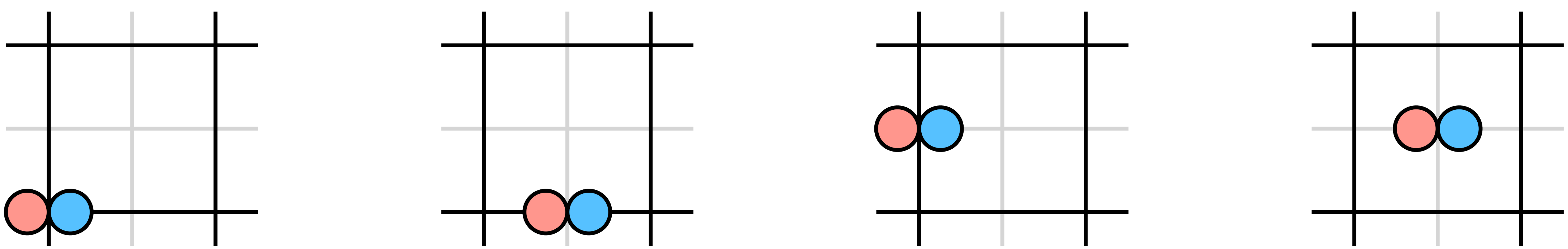

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Po-Vishwanath-Watanabe
Nat. Commun. (2017)

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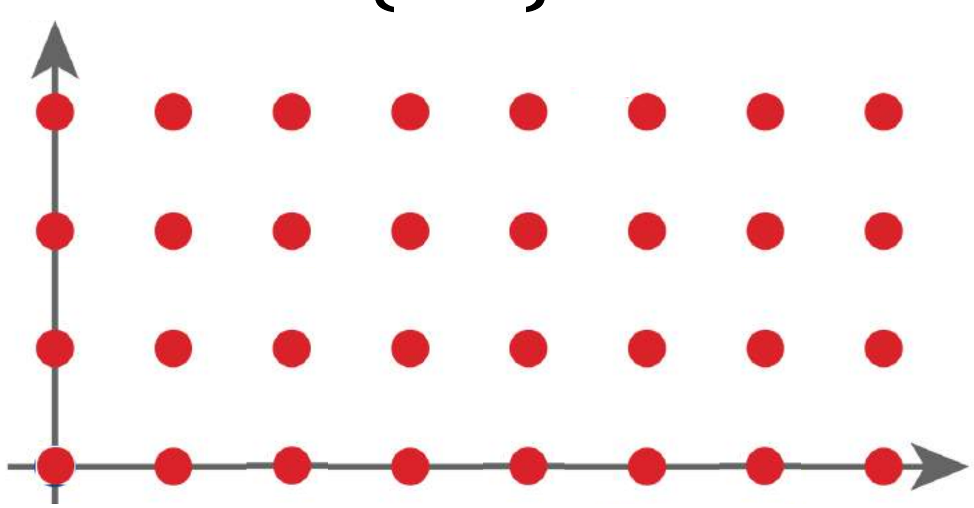

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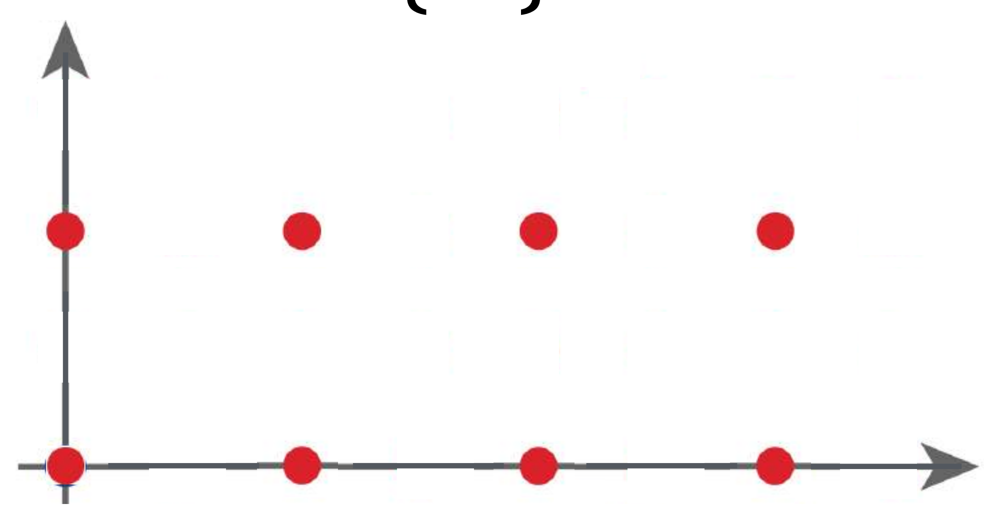

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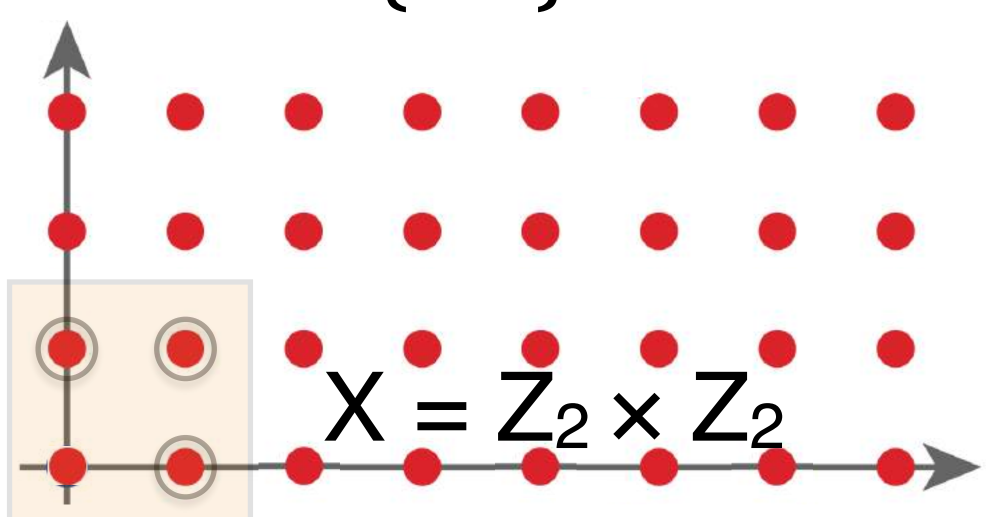

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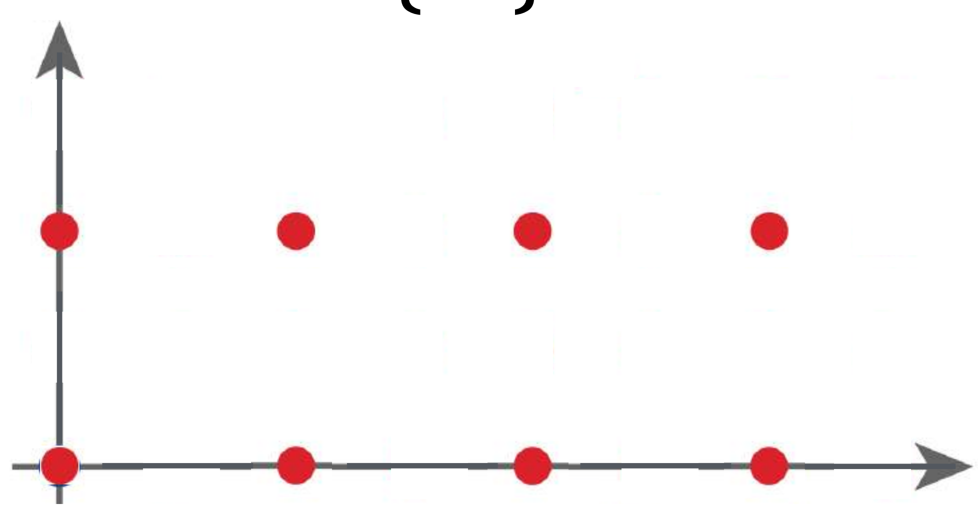

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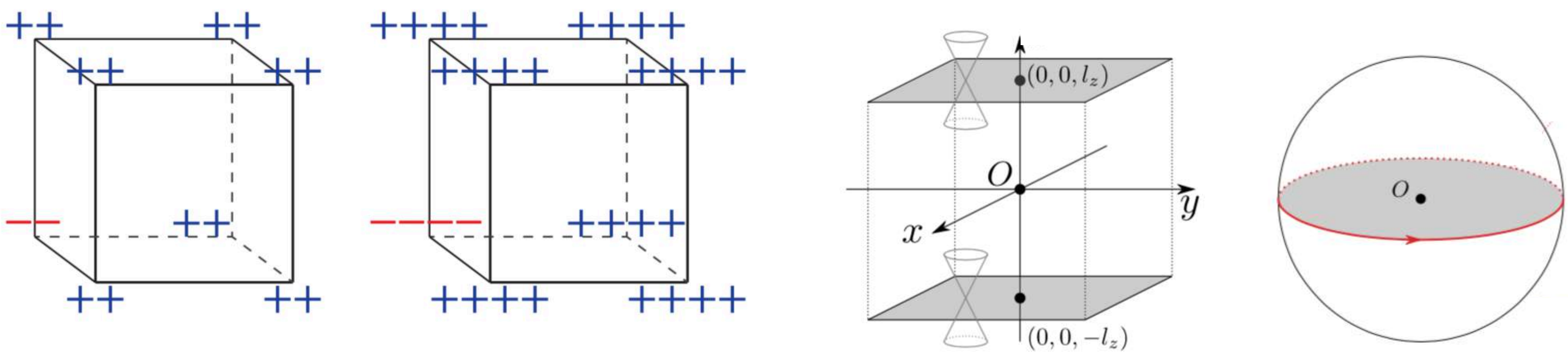
Symmetry indicator for inversion &TRS with SOC in 3D

$X = Z_2 \times Z_2 \times Z_2 \times Z_4$

Sum of inversion parities

$$\kappa_1 \equiv \frac{1}{4} \sum_{K \in \mathrm{TRIMs}} (n_K^+ - n_K^-) \in \mathbb{Z}.$$



Po-Vishwanath-Watanabe, Nat. Commun. (2017)
Chen Fang & Liang Fu, arXiv:1709.01929
Khalaf-Po-Vsiwanath-Watanabe, PRX (2018)

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weak TI

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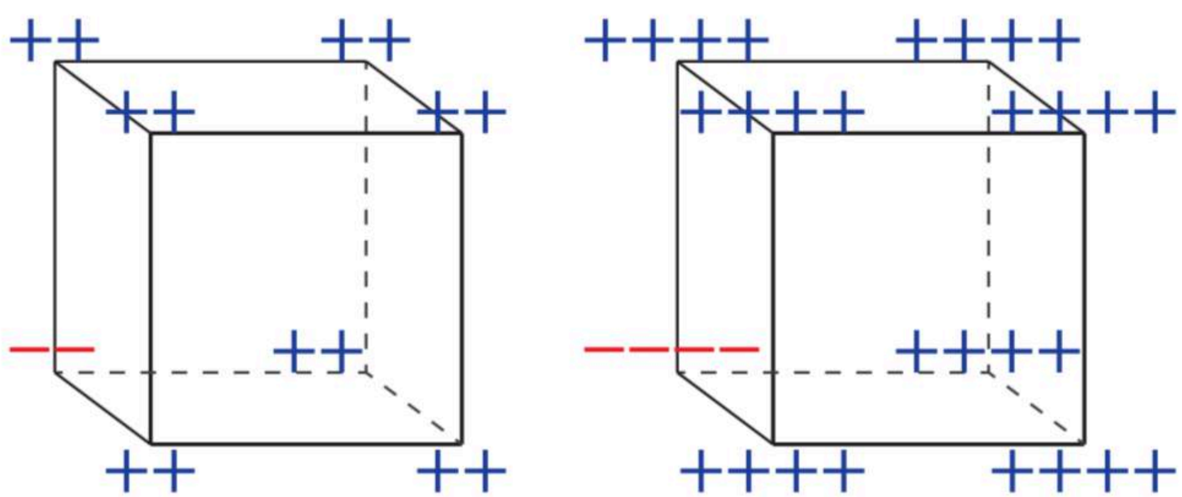

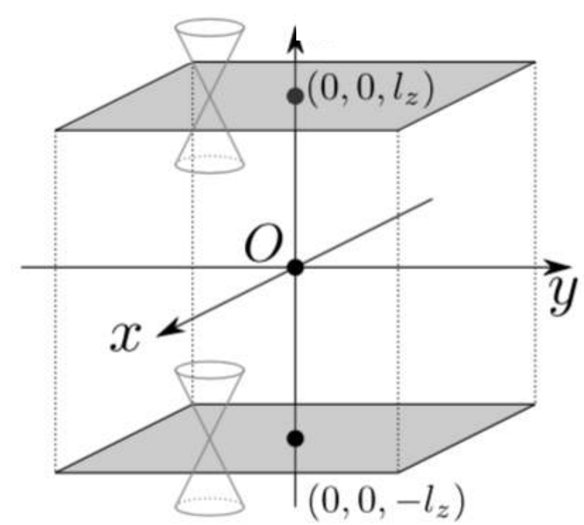

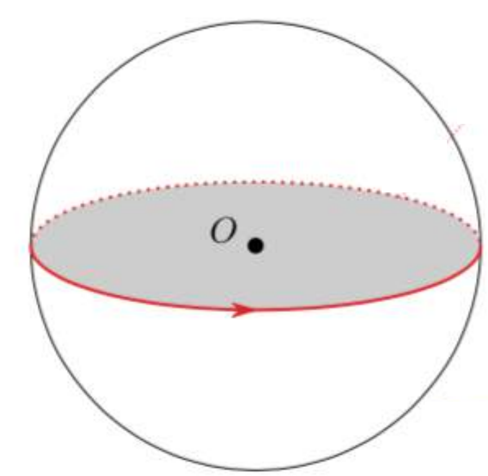

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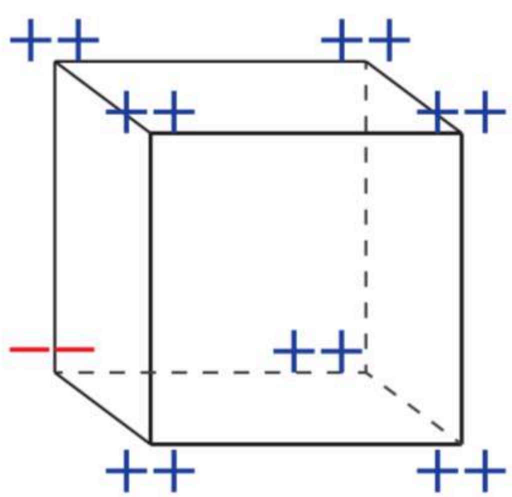
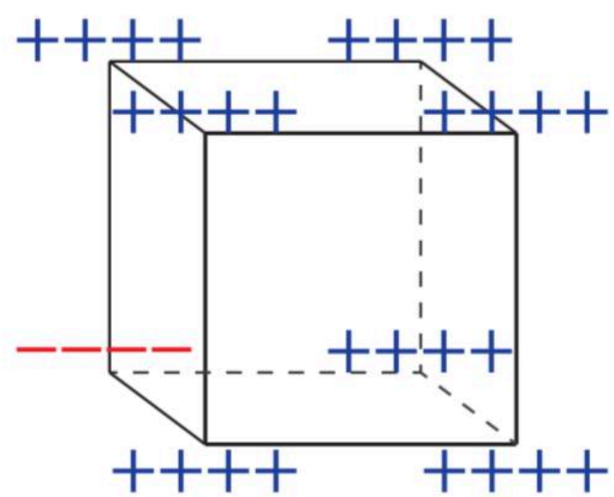
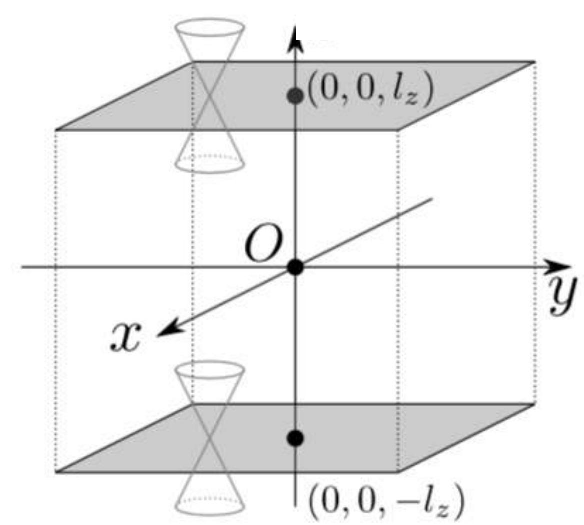
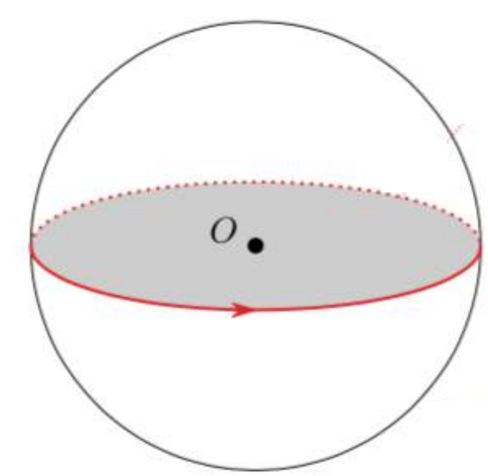
Sum of inversion parities

$$\kappa_1 \equiv \frac{1}{4} \sum_{K \in \mathrm{TRIMs}} (n_K^+ - n_K^-) \in \mathbb{Z}.$$





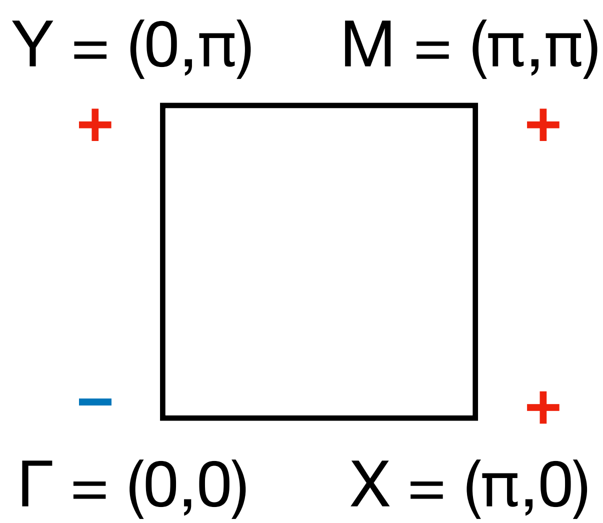




Po-Vishwanath-Watanabe, Nat. Commun. (2017)
Chen Fang & Liang Fu, arXiv:1709.01929
Khalaf-Po-Vsiwanath-Watanabe, PRX (2018)

Symmetry indicator for rotation symmetric systems in 2D

- *n*-fold rotation → Chern number *C* mod *n*



Fang-Gilbert-Bernevig PRB (2012)

$(-1)^C$ = product of rotation eigenvalues

In our language, $X = Z_n$

Symmetry indicator for rotation symmetric systems in 2D

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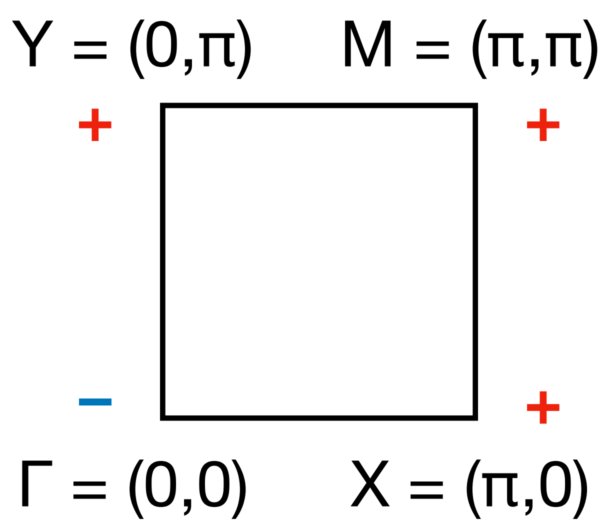

Fang-Gilbert-Bernevig PRB (2012)

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In our language, $X = Z_n$

- Extention to interacting systems using twisted boundary condition

Matsugatani-Ishiguro-Shiozaki-Watanabe PRL (2018)

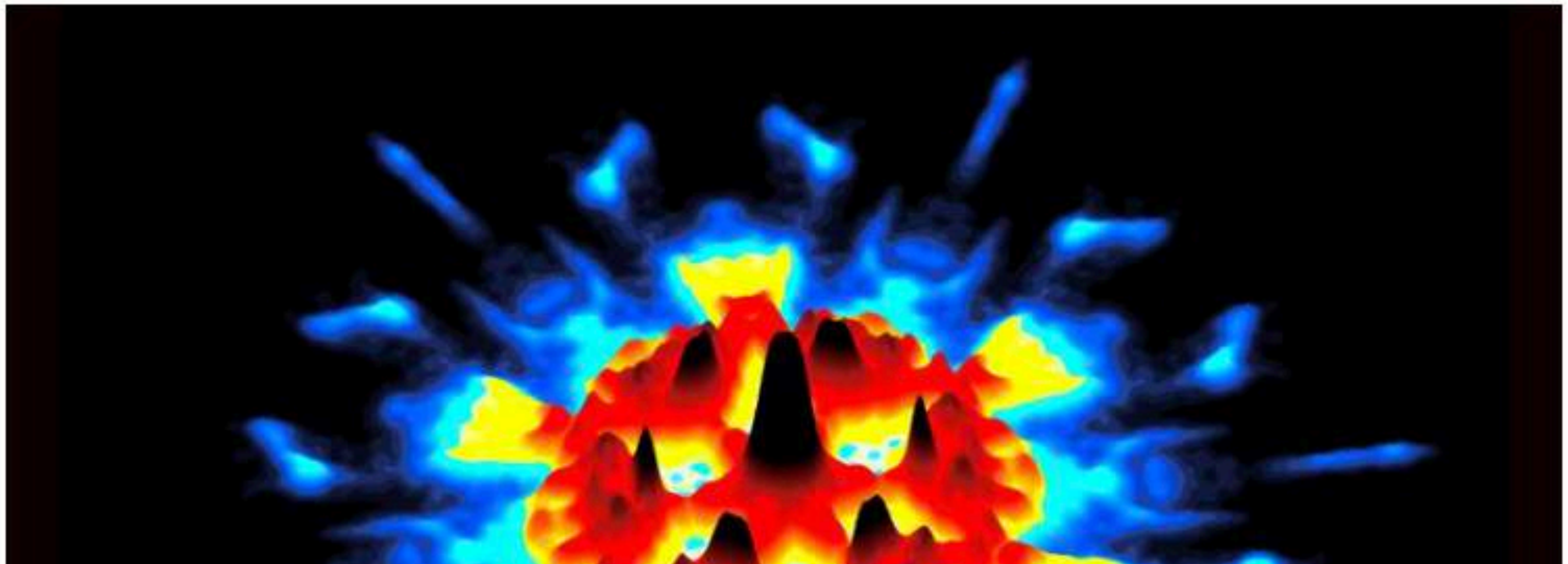
$$(\hat{T}_x)^{L_x} = e^{-i\theta_x \hat{N}}, \qquad (\hat{T}_y)^{L_y} = e^{-i\theta_y \hat{N}}$$

NEWS · 08 AUGUST 2018 · CORRECTION 17 AUGUST 2018

Thousands of exotic ‘topological’ materials discovered through sweeping search

Haul thrills physicists, who previously knew of just a few hundred of these peculiar materials.

Elizabeth Gibney



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Prof. Tay-Rong Chang’s talk

- Basics of symmetry indicators

- What can we “see” from it?

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2. Higher-order topological insulators
3. Weyl semimetals
4. Fragile topology
5. Topological superconductors

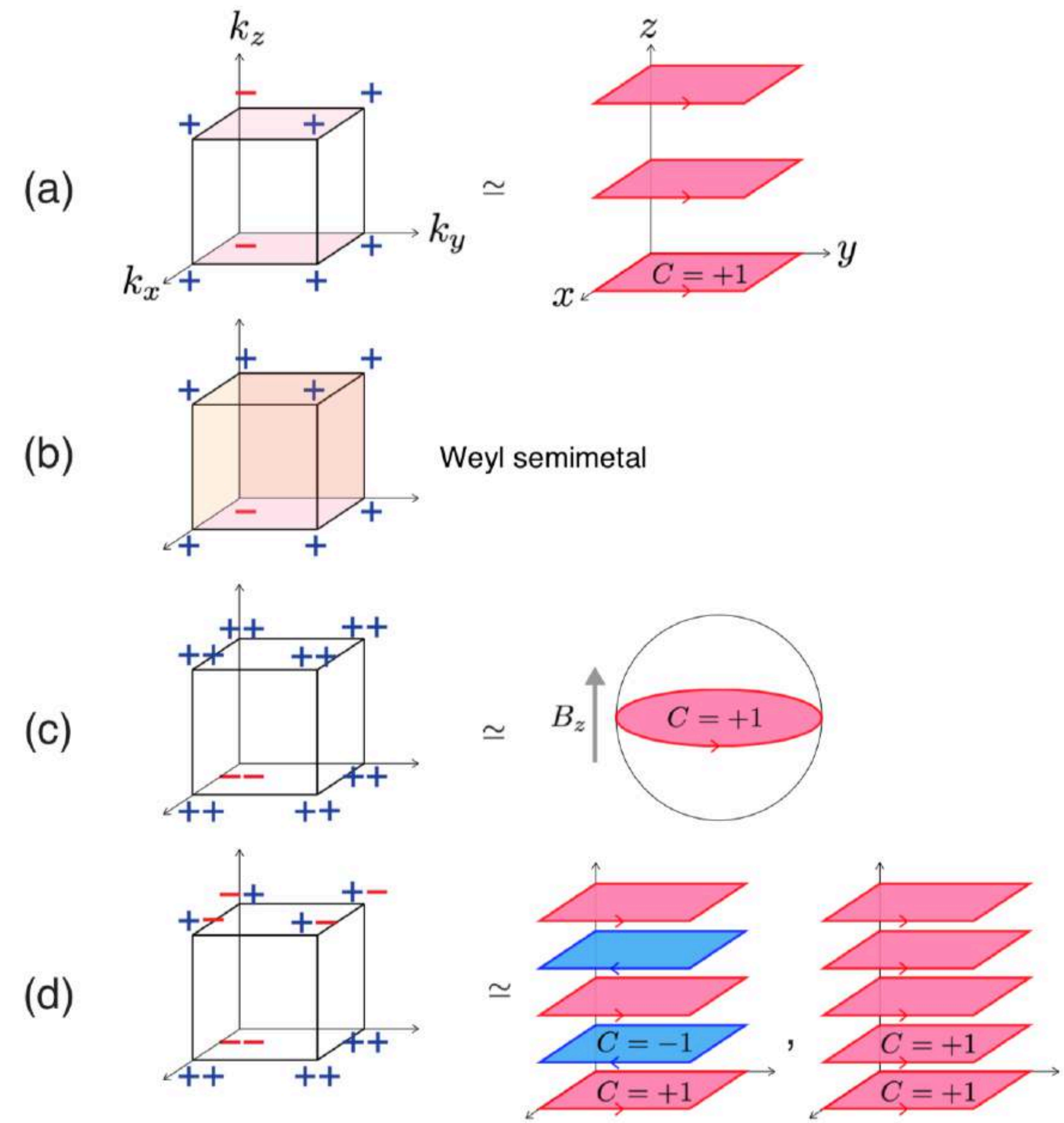
Symmetry indicator for TR breaking inversion symmetric system in 3D

$$X = Z_2 \times Z_2 \times Z_2 \times Z_4$$

Weyl SM

A. Turner, …, A. Vishwanath (2010)

{BS}: “band structure” can be semimetal
(band touching at generic points in BZ)



See also
Song-Zhang-Fang PRX (2018)
for nodal semimetals in the ansence of SOC

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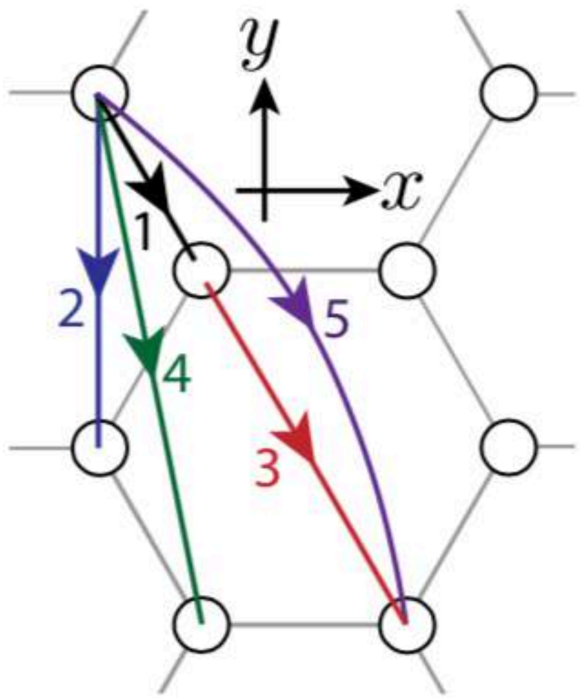
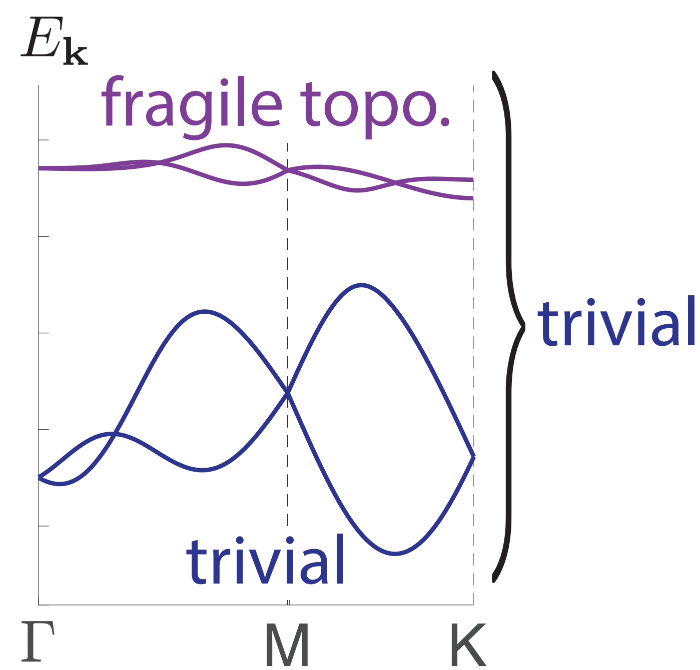
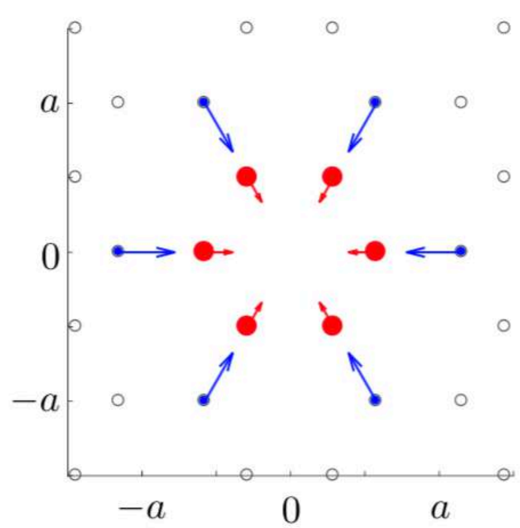
Honeycomb lattice with SOC

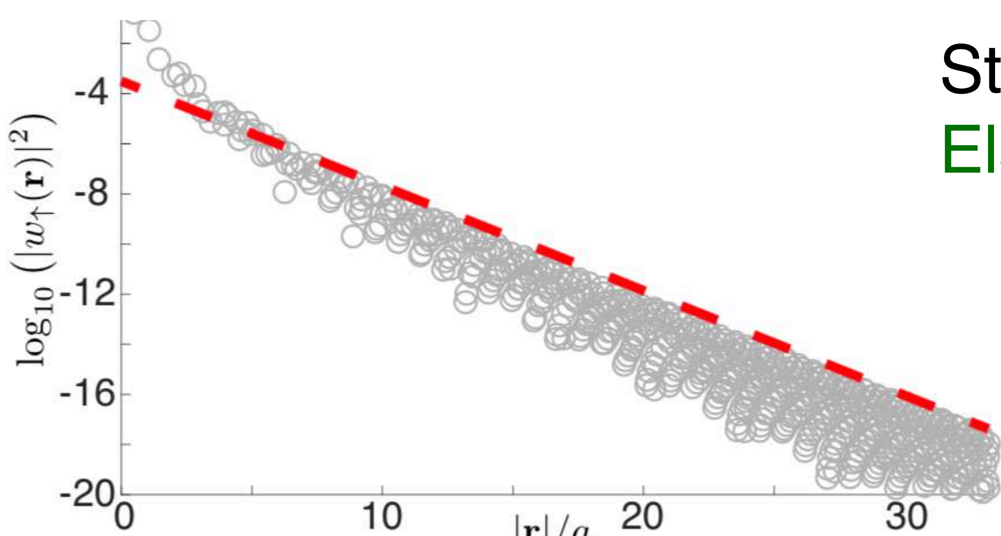




Po-Watanabe-Vishwanath PRL (2018)

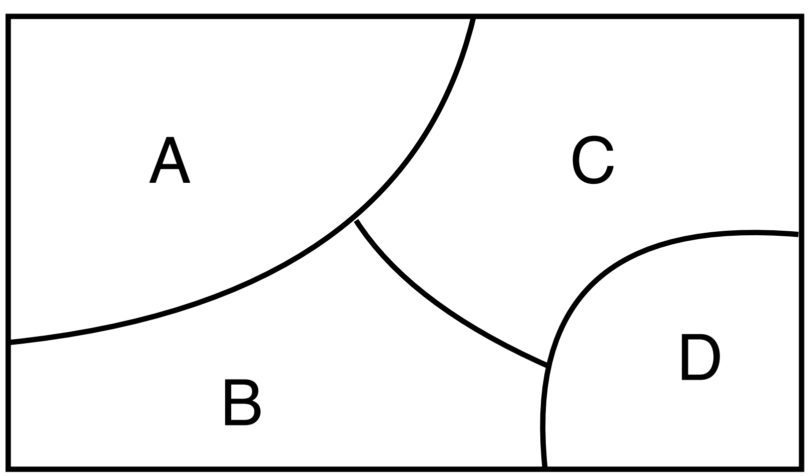


Wannier orbital



Exponential decay of Wannier

Stability against interaction:
Else-Po-Watanabe arXiv:1809.02128



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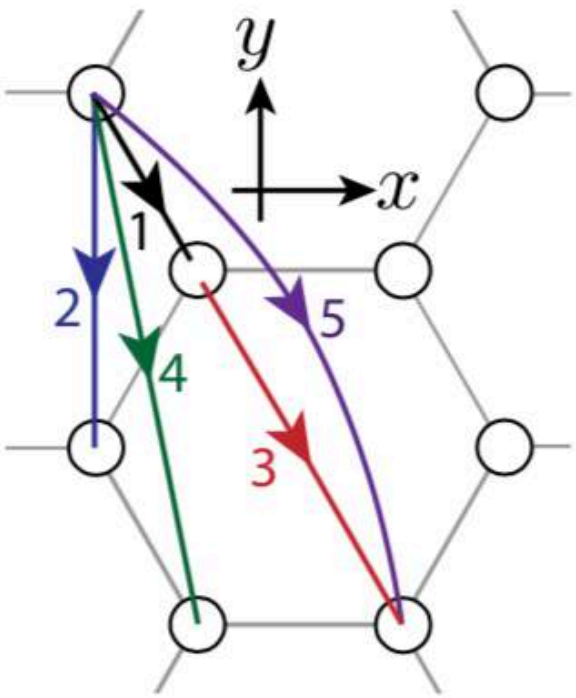
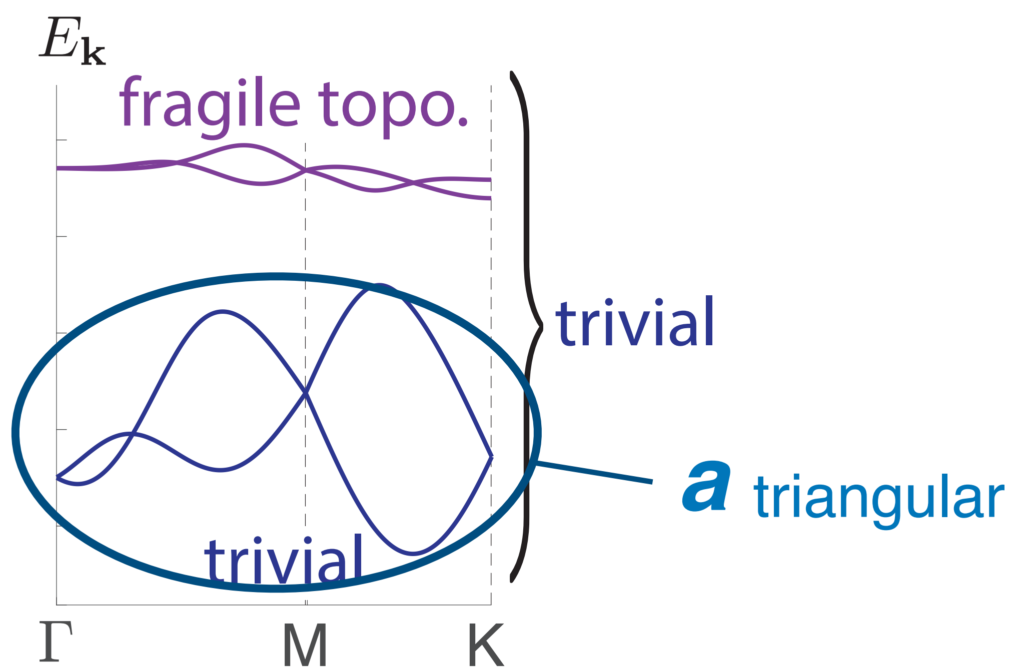
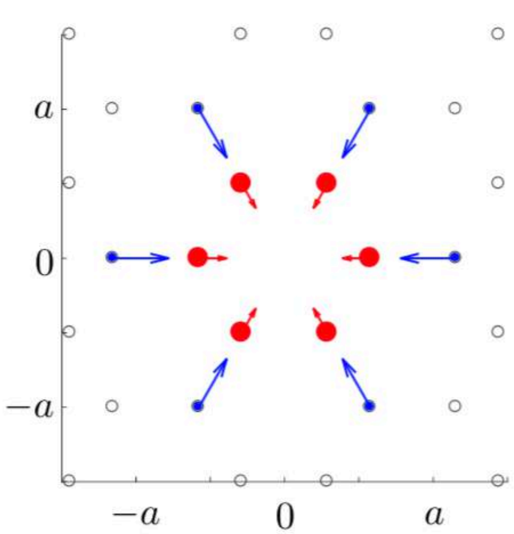




Po-Watanabe-Vishwanath PRL (2018)



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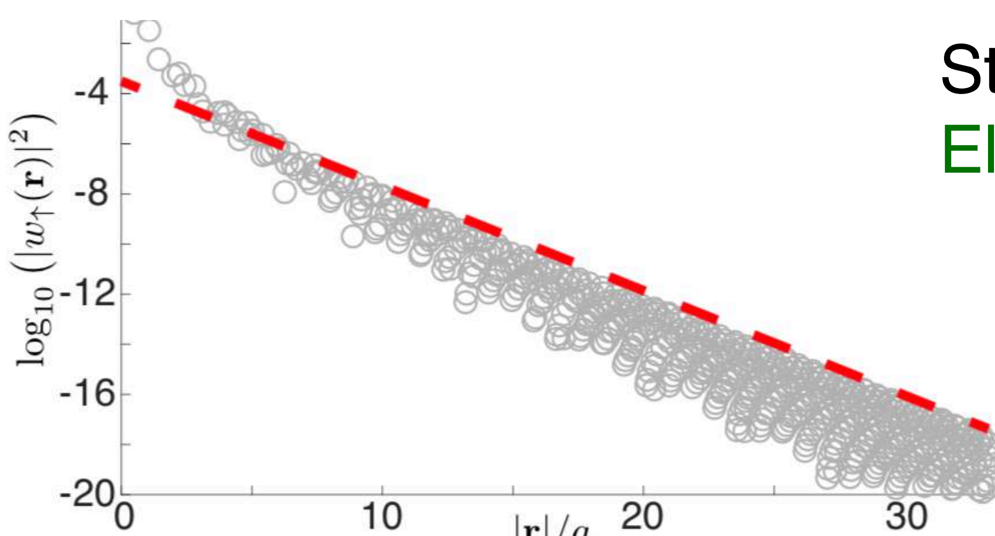

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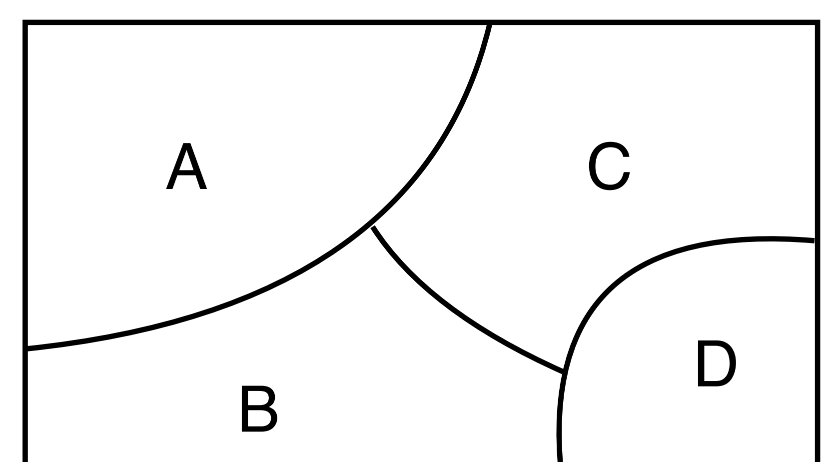

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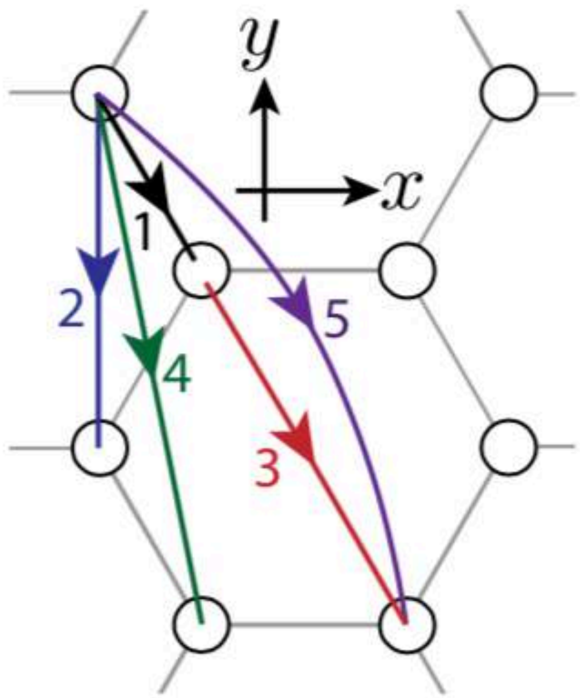

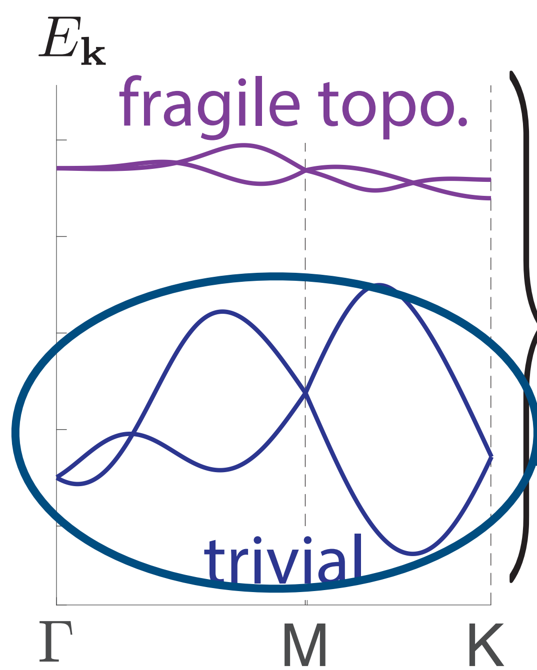

trivial ***a*** honeycomb

a triangular

Po-Watanabe-Vishwanath PRL (2018)

Stability against interaction:
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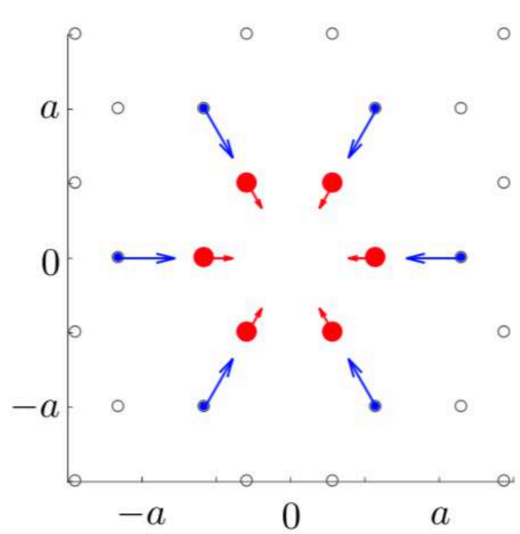
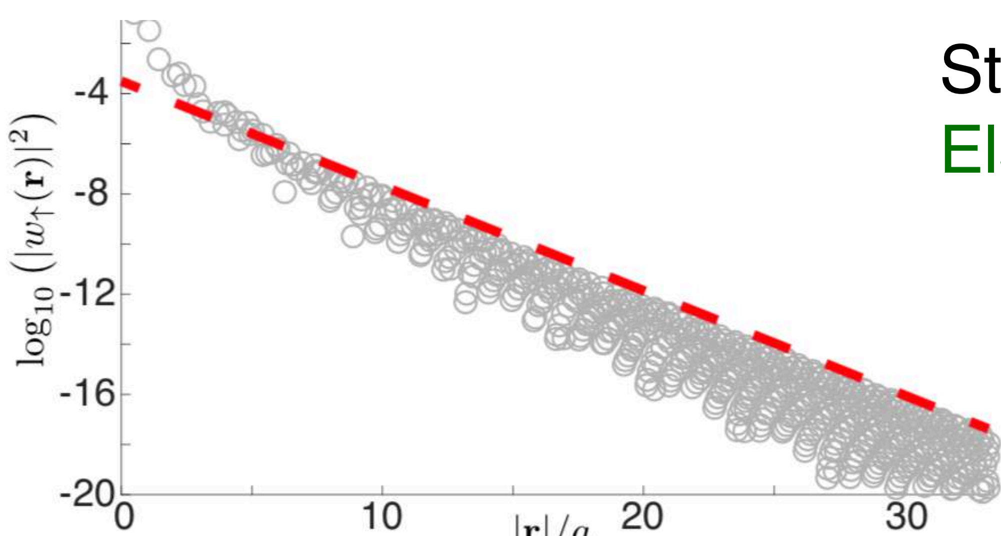


Wannier orbital



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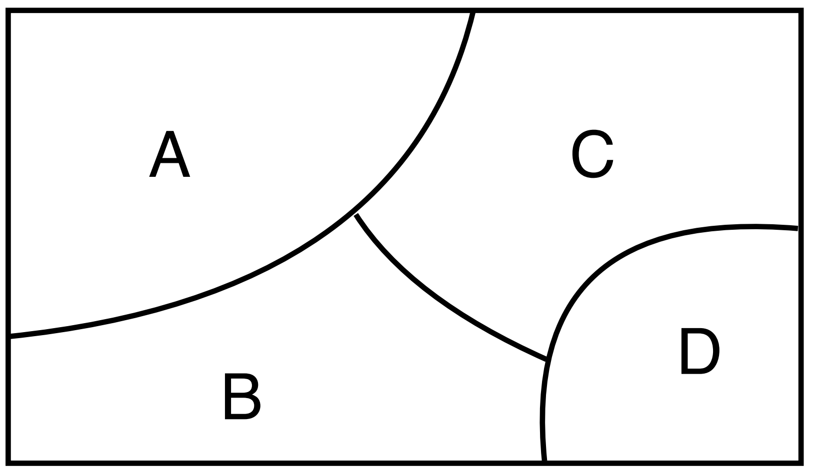

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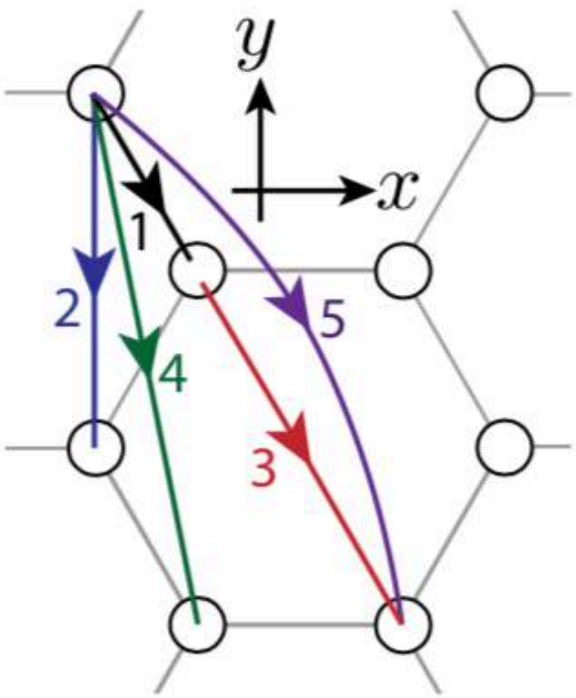

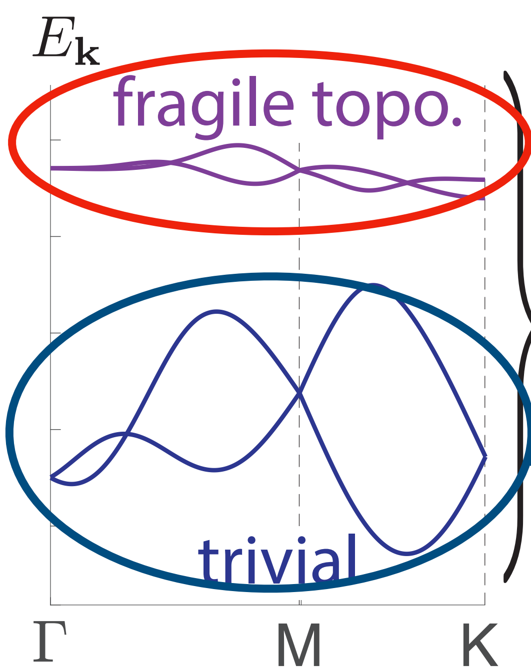

a honeycomb − ***a*** honeycomb

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Po-Watanabe-Vishwanath PRL (2018)

Stability against interaction:
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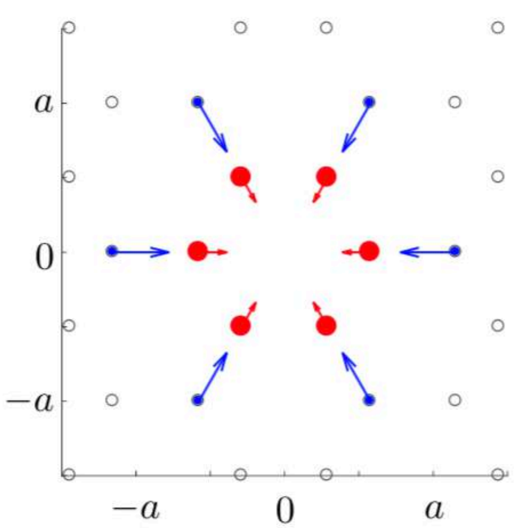
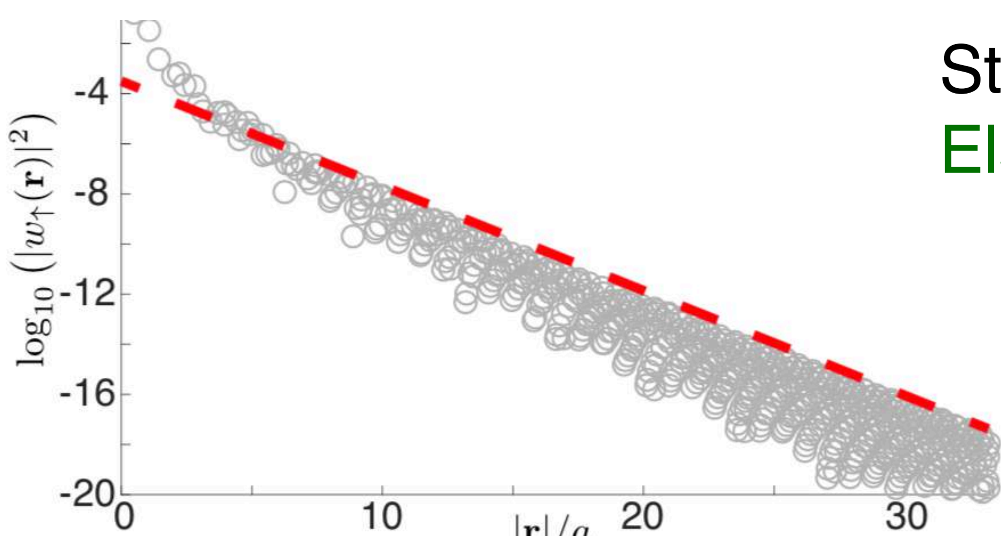


Wannier orbital



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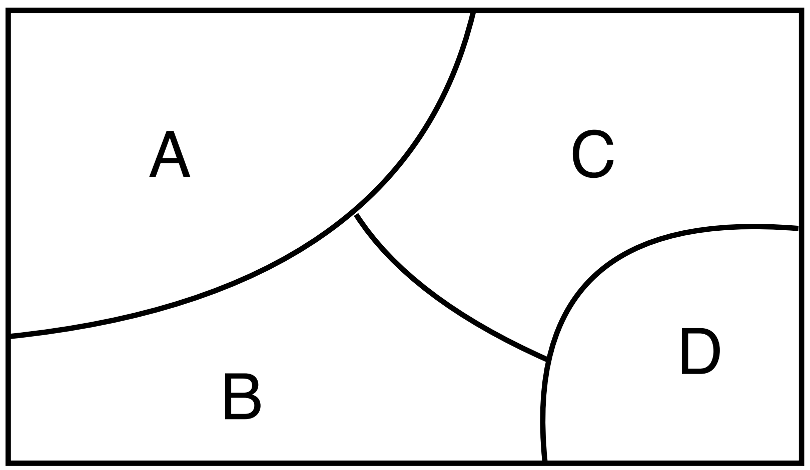

- Basics of symmetry indicators

- What can we “see” from it?

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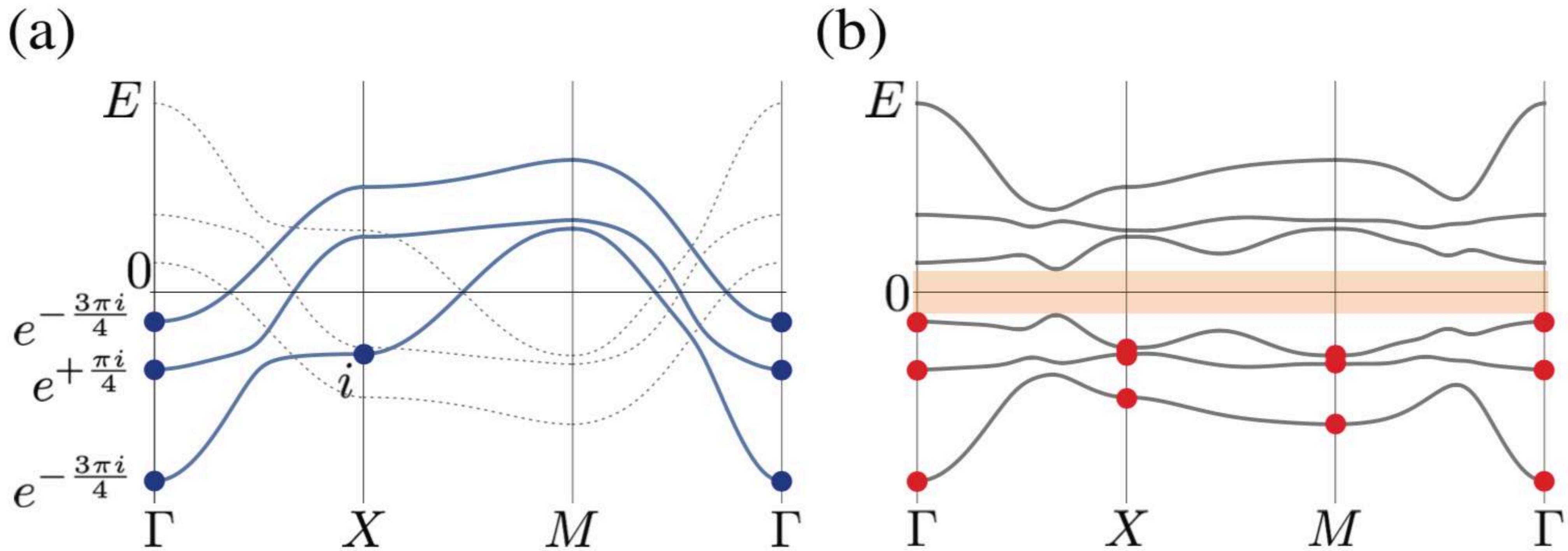
Weak-coupling assumption



$$H_{\boldsymbol{k}}^{\mathrm{BdG}} = \begin{pmatrix} H_{\boldsymbol{k}} & \Delta_{\boldsymbol{k}} \\ \Delta_{\boldsymbol{k}}^{\dagger} & -H_{-\boldsymbol{k}}^{*} \end{pmatrix}$$

$$U_{\boldsymbol{k}}^{\mathrm{BdG}}(g) = \begin{pmatrix} U_{\boldsymbol{k}}(g) & 0 \\ 0 & \chi_g U_{-\boldsymbol{k}}(g)^{*} \end{pmatrix}$$

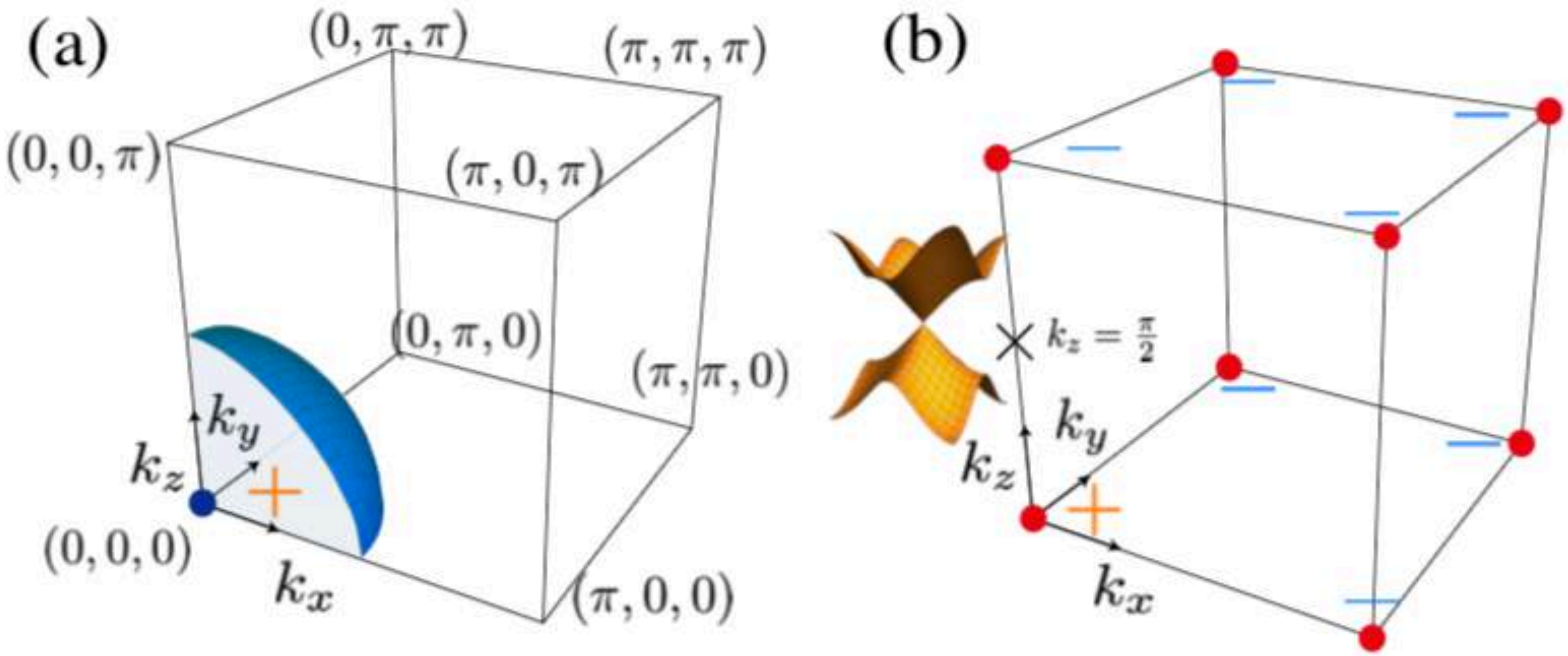
$$\begin{aligned} (n_{\boldsymbol{k}}^{\alpha})^{\mathrm{BdG}} &= n_{\boldsymbol{k}}^{\alpha}\Big|_{\mathrm{occ.}} + n_{-\boldsymbol{k}}^{f_{-\boldsymbol{k}}(\alpha)}\Big|_{\mathrm{unocc.}} \\ &= (n_{\boldsymbol{k}}^{\alpha} - n_{-\boldsymbol{k}}^{f_{-\boldsymbol{k}}(\alpha)})\Big|_{\mathrm{occ.}} + n_{-\boldsymbol{k}}^{f_{-\boldsymbol{k}}(\alpha)}\Big|_{\mathrm{all\ bands}}. \end{aligned}$$

We can extract indicators for SCs
from the band structure in the normal phase!

Ono-Yanase-Watanabe, arXiv:1811.08712

p+ip SC with nodes

(SC version of Weyl semimetal)



$$H_{\boldsymbol{k}} = t(3 - \cos k_x - \cos k_y - \cos k_z) - \mu,$$
$$\Delta_{\boldsymbol{k}} = \Delta(\sin k_x + i \sin k_y).$$

Ono-Yanase-Watanabe, arXiv:1811.08712

Summary

- Extract band topology by comparing {BS} and {AI}

  Nat. Commun. (2017) Sci. Adv. (2018)

- Applications include

1. Conventional topological insulators (Chern, Z2 TI, etc) PRL (2018)
2. Higher-order topological insulators PRX (2018)
3. Weyl semimetals
4. Fragile topology PRL (2018), arXiv:1809.02128
5. Topological superconductors PRB (2018), arXiv:1811.08712

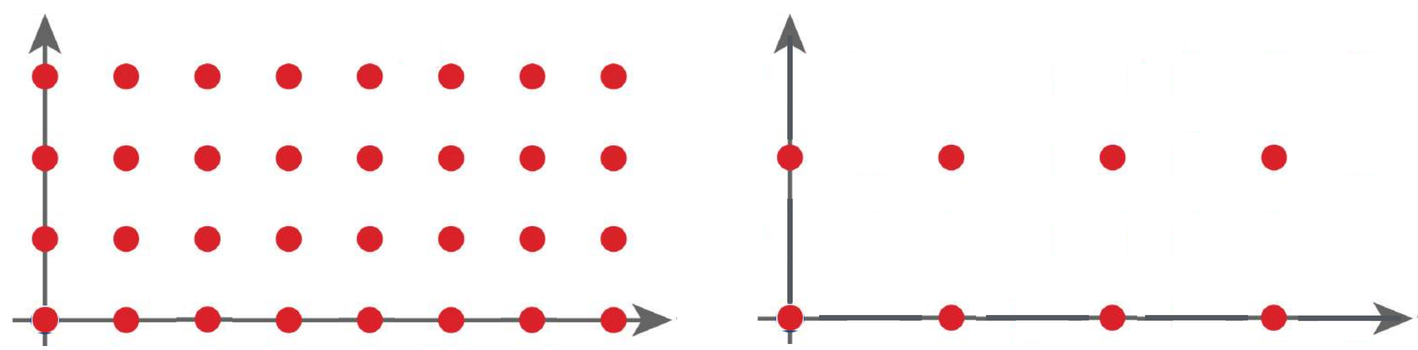
A useful fact

dBS = dAI

Po-Vishwanath-Watanabe
Nat. Commun. (2017)

{BS} = {$\boldsymbol{b}$ = {$n_{\boldsymbol{k}}^{\alpha}$} | satisfying compatibility rels.} ⊂ Z^{dBS}

{AI} = {$\boldsymbol{a}$ = {$n_{\boldsymbol{k}}^{\alpha}$} | corresoinding to AI} ⊂ Z^{dAI}



lattice of {BS} lattice of {AI}

$$\underset{\boldsymbol{b}}{(+,+,+,-)} = \mathbf{1/2}\left[\underset{\boldsymbol{a}_1}{(+,+,+,+)} + \underset{\boldsymbol{a}_2}{(+,+,-,-)} + \underset{\boldsymbol{a}_3}{(+,-,+,-)} - \underset{\boldsymbol{a}_4}{(+,-,-,+)}\right]$$

We do not have to solve compatibility relations to find out {BS}!