Are the standard-model parameters free?

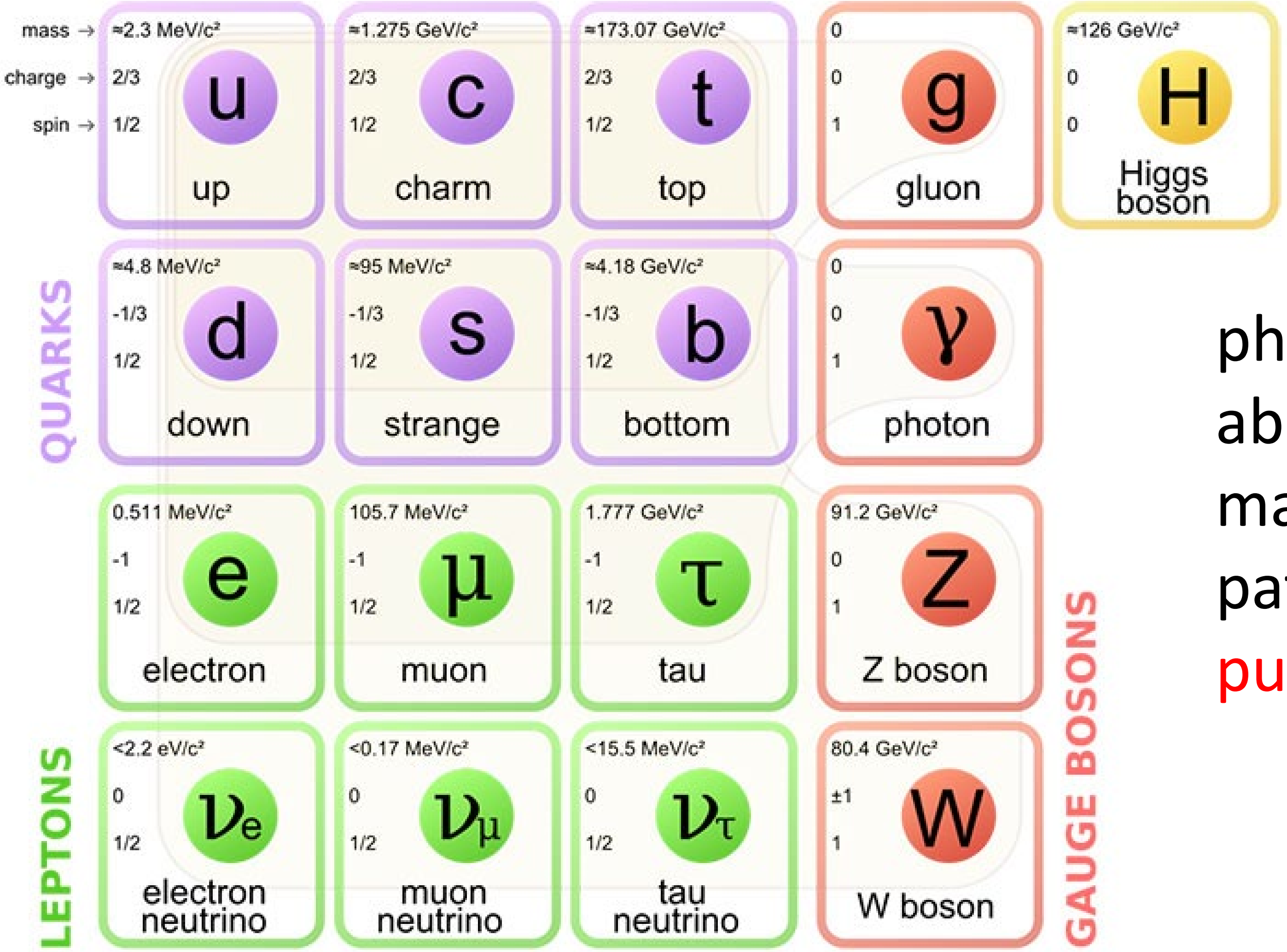
Hsiang-nan Li

Academia Sinica

Presented at NTU on Sep. 26, 2023

2304.05921

Standard Model



≈2.3 MeV/c² 2/3 1/2 u up	≈1.275 GeV/c² 2/3 1/2 c charm	≈173.07 GeV/c² 2/3 1/2 t top	0 0 1 g gluon	≈126 GeV/c² 0 0 H Higgs boson
≈4.8 MeV/c² -1/3 1/2 d down	≈95 MeV/c² -1/3 1/2 s strange	≈4.18 GeV/c² -1/3 1/2 b bottom	0 0 1 γ photon	
0.511 MeV/c² -1 1/2 e electron	105.7 MeV/c² -1 1/2 μ muon	1.777 GeV/c² -1 1/2 τ tau	91.2 GeV/c² 0 1 Z Z boson	
<2.2 eV/c² 0 1/2 ν_e electron neutrino	<0.17 MeV/c² 0 1/2 ν_μ muon neutrino	<15.5 MeV/c² 0 1/2 ν_τ tau neutrino	80.4 GeV/c² ±1 1 W W boson	

physicists are curious about flavor structure: mass hierarchy, mixing patterns,…

puzzles for decades

Textbooks:
Chapter 15 in Quarks & Leptons by Halzen and Martin:
…. the (fermion) masses depend on the arbitrary (Yukawa) couplings and cannot be predicted.

ChatGPT:
The Standard Model parameters are free in the sense that their values cannot be determined by the theory alone, and experimental measurements play a crucial role in determining their values.

To explain them, need new physics, but…

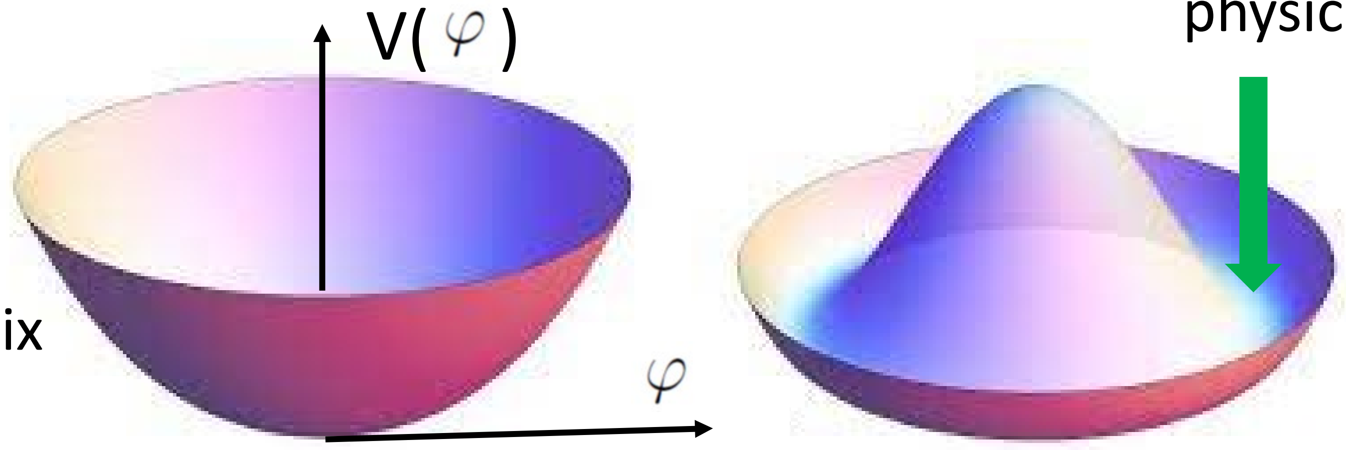
Today's talk:

At least some of the SM parameters are not free, but constrained dynamically via analyticity for internal consistency

Your discretion is advised

Higgs mechanism

- Before and after symmetry breaking



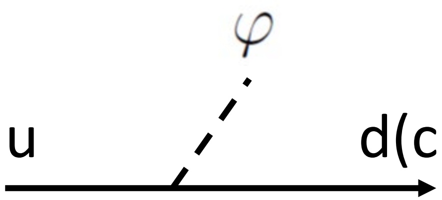
VEV v
physical world

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi^+ \\ \varphi^0\end{pmatrix}, \quad \varphi^+ \rightarrow 0, \quad \varphi^0 \rightarrow v$$

3X3 Yukawa matrix

left-handed doublet $\overline{Q_L} Y_u u_R \varphi + \overline{Q_L} Y_d d_R \tilde{\varphi}$

(u,c,t) (d,s,b)



- Massless particles ➡ massive particles
- Flavor changing via Yukawa couplings ➡ diagonal Yukawa matrices
- Quarks in weak eigenstates ➡ quark mixing

CKM matrix for weak int.

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = \begin{bmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix} .$$

$$\sum_k |V_{ik}|^2 = \sum_k |V_{ki}|^2 = 1$$

$$\sum_k V_{ik} V_{jk}^* = 0 .$$

unitarity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{bmatrix} .$$

Chau-Keung parametrization

Wolfenstein parametrization

$$\begin{bmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{bmatrix} + O(\lambda^4) .$$

Gauge interaction

- Gauge interaction like QED

4-vector potential generators

$$D_\mu \psi(x) = \left(\partial_\mu - ig\, A_\mu^a(x)\, t^a\right) \psi(x)$$

- Vector interaction, photon has spin 1
- Gauge group fixes t^a: 1 for QED, U(1) group; Pauli matrices for weak interaction, SU(2) group; Gell-Mann matrices for QCD, SU(3) group
- Generators describe basic transformation, e.g.

Red = (1,0,0)
Blue= (0,1,0)

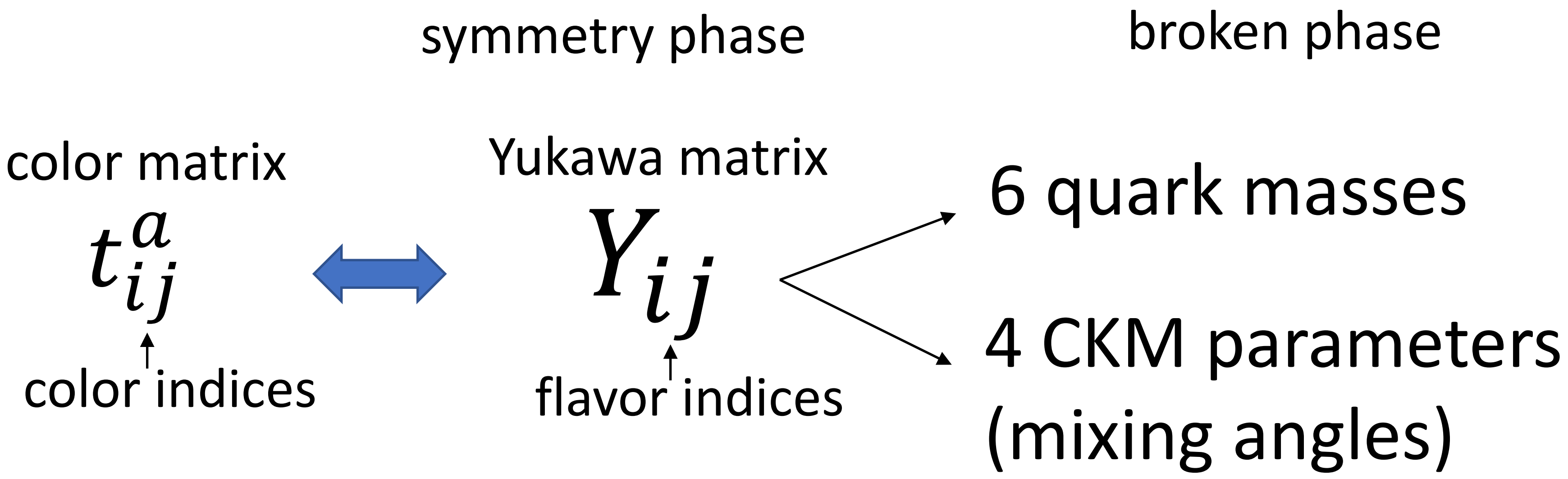
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Red → Blue

- Only overall coefficient is free, single coupling g

Scalar interaction

- No such symmetry constraint on scalar coupling



symmetry phase

broken phase

color matrix

t_{ij}^{a}

color indices

Yukawa matrix

Y_{ij}

flavor indices

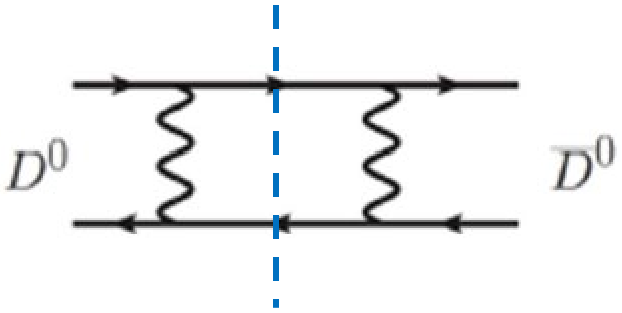
6 quark masses

4 CKM parameters (mixing angles)

9302302, Santamaria

- Yukawa couplings are arbitrary, so are quark masses, mixing matrix!
- But this observation made at Lagrangian level without considering dynamical interplay among various particles

Dispersion relation

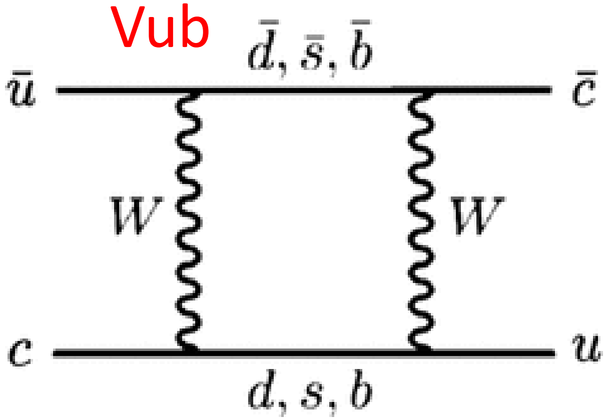


- Example: mixing of D meson with mass squared s

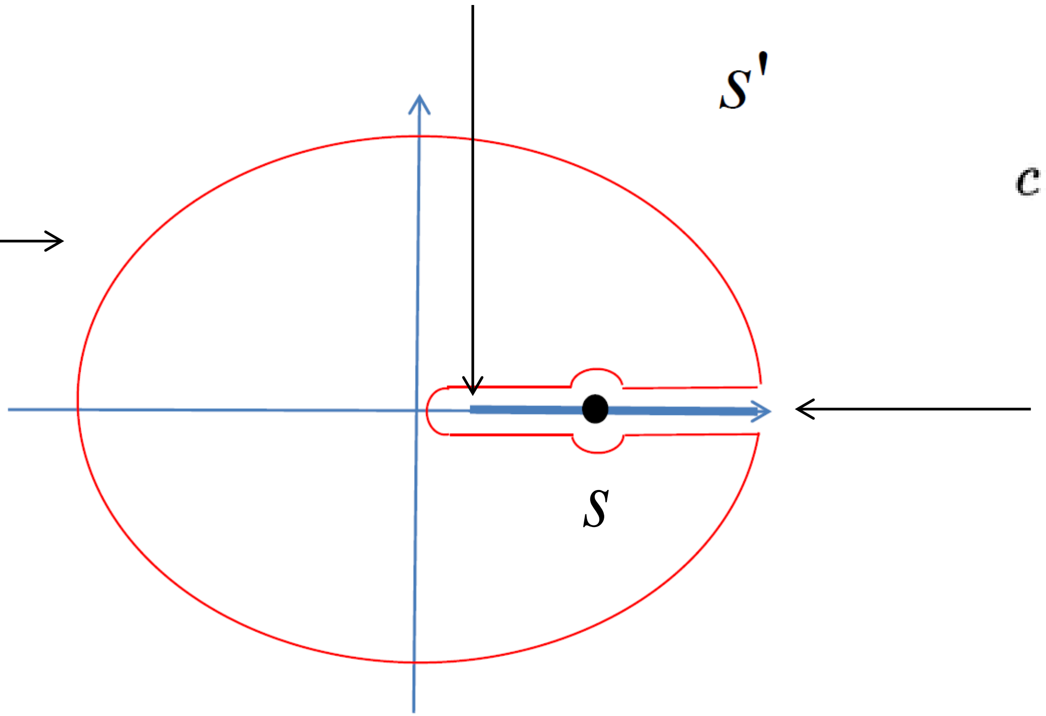
$$M_{12}(s) - \frac{i}{2}\Gamma_{12}(s) = \langle D^0|\mathcal{H}|\bar{D}^0\rangle$$

imaginary part

Vud
Vus
Vub



real part $$M_{12}(s) = \frac{P}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Gamma_{12}(s')}{s - s'}$$



large circle contribution suppressed

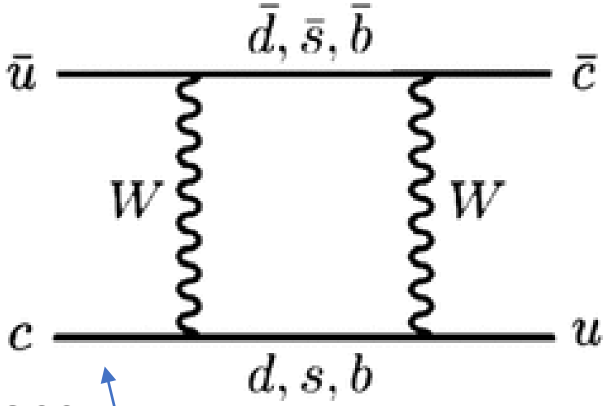
branch cut caused by intermediate real states

Observations

- Fundamental parameters in theory (like Standard Model) usually constrained by symmetries at Lagrangian level
- Analyticity is crucial property of physical observables
- Γ_{12} involves CKM matrix elements and fermion masses
- Additional dynamical constraints imposed by dispersion relations, if M_{12} is known ?
- Turn out that dispersive constraints are so strong that Yukawa couplings in SM are in fact not free parameters

Idea

- Neutral state mixing disappears at high energy, where electroweak symmetry is restored



EW symmetry broken
at low energy;
constrains fermion
masses and mixing angles

invariant mass squared

symmetry restored

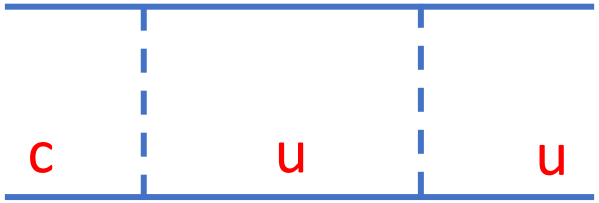
$$M_{12}(s) = \frac{1}{2\pi}\int ds' \frac{\Gamma_{12}(s')}{s-s'} \approx 0 \text{ at large } s$$

real part

imaginary part of box diagrams

Proof of $M_{12}(s) \approx 0$

- Consider mixing of $Q_L\bar{q}_L$, $\bar{Q}_L q_L$ neutral states
- Before breaking, all particles are massless, quarks in flavor eigenstates
- Mixing occurs via exchanges of charged or neutral scalars, whose strengths described by Yukawa matrices



neutral scalar exchanges

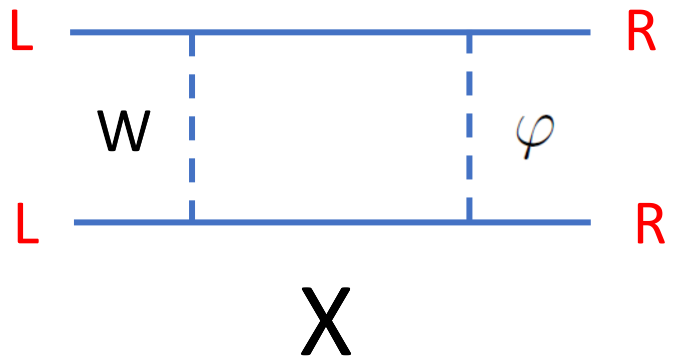
- After breaking, particles get masses, quarks turned to mass eigenstates
- Mixing occurs via W boson exchanges, whose strengths described by CKM matrix

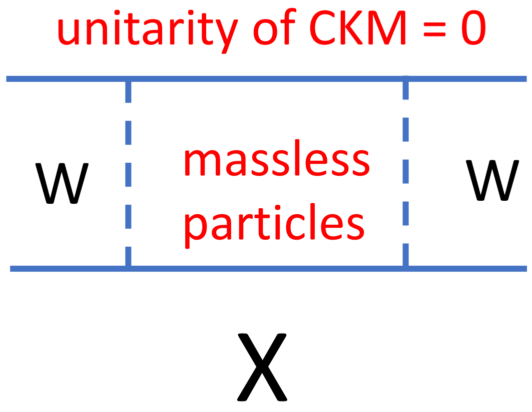
Mixing in symmetric phase

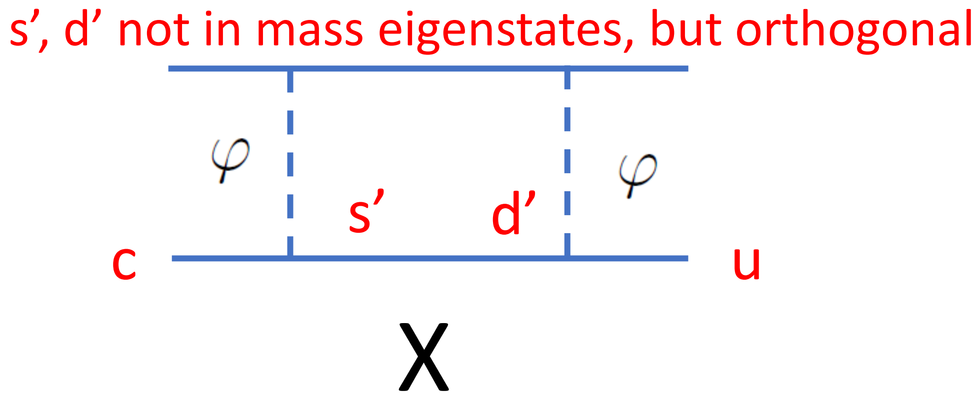
- Yukawa interaction $\overline{Q_L} Y_u u_R \varphi + \overline{Q_L} Y_d d_R \tilde{\varphi}$ $\qquad \varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix},$

left-handed doublet

- In symmetric phase, implement quark field transformation adopted in broken phase $u_L \to U_u u_L \quad u_R \longrightarrow V_u u_R \quad d_L \to U_d d_L \quad d_R \longrightarrow V_d d_R$
- Yukawa matrices diagonalized, but charged scalar currents exist
- down-type quarks, coupling to up-type quarks in mass eigenstates through charged scalar currents, are not in mass eigenstates







High-energy input

- Heavy quark Q provides large s in box diagrams. Symmetry restores and intermediate particles become massless, $M_{12}(s) \approx 0$
- s’ can be low, so $\Gamma_{12}(s')$ depends on CKM matrix elements associated with massive intermediate quarks in broken phase.
- Box-diagram contribution

Cheng 1982
Buras et al 1984

$$\begin{aligned}
\Gamma_{12}(s) &\propto \sum_{i,j} \lambda_i \lambda_j \Gamma_{ij}(s), \\
\Gamma_{ij}(s) &= \frac{1}{s^2} \frac{\sqrt{s^2 - 2s(m_i^2 + m_j^2) + (m_i^2 - m_j^2)^2}}{(m_W^2 - m_i^2)(m_W^2 - m_j^2)} \\
&\times \left\{ \left(m_W^4 + \frac{m_i^2 m_j^2}{4} \right) [2s^2 - 4s(m_i^2 + m_j^2) + 2(m_i^2 - m_j^2)^2] + 3 m_W^2 s (m_i^2 + m_j^2)(m_i^2 + m_j^2 - s) \right\}
\end{aligned}$$

for D mixing $i, j = d, s, b \qquad \lambda_i \equiv V_{ci}^* V_{ui}$

Constraints

- How to diminish dispersive integral $\int ds' \frac{\Gamma_{12}(s')}{s-s'}$?
- Asymptotic expansion

$$\Gamma_{ij}(s') \approx \Gamma_{ij}^{(1)} s' + \Gamma_{ij}^{(0)} + \frac{\Gamma_{ij}^{(-1)}}{s'} + \cdots$$

EW symmetry restoration scale

$$\Gamma_{ij}^{(1)} = \frac{4m_W^4 - 6m_W^2(m_i^2+m_j^2) + 4m_i^2 m_j^2}{2(m_W^2-m_i^2)(m_W^2-m_j^2)}, \quad \Rightarrow \quad \Lambda^2/s$$

$$\Gamma_{ij}^{(0)} = -\frac{3(m_i^2+m_j^2)\left[4m_W^4 - 4m_W^2(m_i^2+m_j^2) + m_i^2 m_j^2\right]}{2(m_W^2-m_i^2)(m_W^2-m_j^2)} \quad \Rightarrow \quad (m_i^2+m_j^2)\Lambda/s$$

$$\Gamma_{ij}^{(-1)} = \frac{3(m_i^4+m_j^4)\left[4m_W^4 - 2m_W^2(m_i^2+m_j^2) + m_i^2 m_j^2\right]}{2(m_W^2-m_i^2)(m_W^2-m_j^2)}. \quad \Rightarrow \quad (m_i^4+m_j^4)\ln\Lambda/s$$

to have finite integral

$$\boxed{\sum_{i,j} \lambda_i \lambda_j \Gamma_{ij}^{(m)} \approx 0, \quad m = 1, 0, -1}$$

$$\int ds' \frac{\Gamma_{12}(s')}{s-s'} \approx \frac{1}{s}\sum_{i,j}\lambda_i\lambda_j g_{ij} \qquad g_{ij} \equiv \int_{t_{ij}}^{\infty} ds' \left[\Gamma_{ij}(s') - \Gamma_{ij}^{(1)} s' - \Gamma_{ij}^{(0)} - \frac{\Gamma_{ij}^{(-1)}}{s'}\right]$$

to diminish integral

$$\boxed{\sum_{i,j} \lambda_i \lambda_j g_{ij} \approx 0}$$

Minimization

- Rewrite constrains $r^2 R_{dd}^{(m)} + 2r R_{ds}^{(m)} + 1 \approx 0, , \quad m = 1, 0, -1, i$

$$R_{dd}^{(m)} = \frac{\Gamma_{dd}^{(m)} - 2\Gamma_{db}^{(m)} + \Gamma_{bb}^{(m)}}{\Gamma_{ss}^{(m)} - 2\Gamma_{sb}^{(m)} + \Gamma_{bb}^{(m)}}, \quad R_{ds}^{(m)} = \frac{\Gamma_{ds}^{(m)} - \Gamma_{db}^{(m)} - \Gamma_{sb}^{(m)} + \Gamma_{bb}^{(m)}}{\Gamma_{ss}^{(m)} - 2\Gamma_{sb}^{(m)} + \Gamma_{bb}^{(m)}} \qquad m = 1, 0, -1$$

- Expression for $m = i$ similar, but with g_{ij}
- Ratio of CKM elements

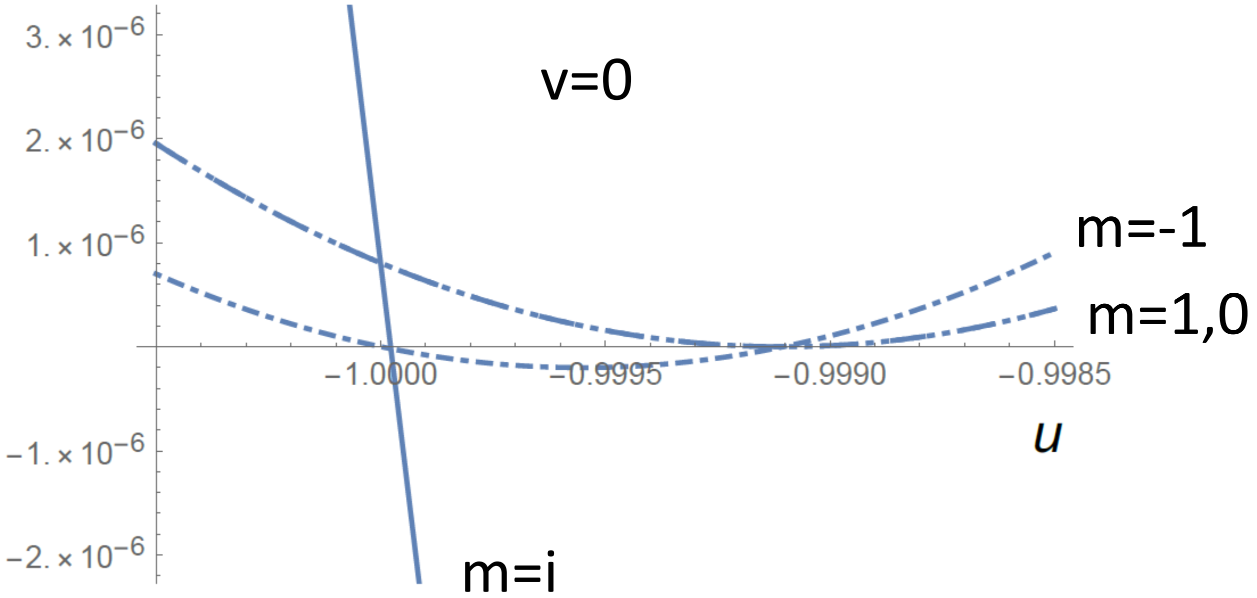
$$r = \frac{\lambda_d}{\lambda_s} = \frac{V_{cd}^* V_{ud}}{V_{cs}^* V_{us}} \equiv u + iv,$$

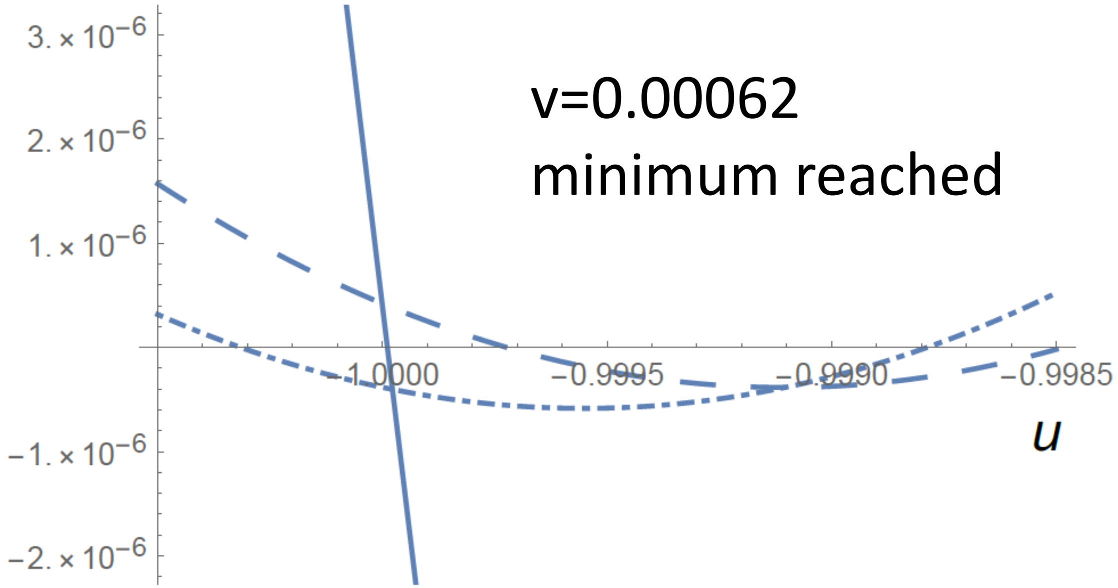
- Tune u and v to minimize the sum (real parts of constraints)

$$\sum_{m=1,-1,i} \left[(u^2 - v^2) R_{dd}^{(m)} + 2u R_{ds}^{(m)} + 1 \right]^2$$

Results

$$m_d = 0.005 \text{ GeV} \quad m_s = 0.12 \text{ GeV} \quad m_b = 4.0 \text{ GeV} \quad m_W = 80.377 \text{ GeV}$$





$$r = \frac{V_{cd}^* V_{ud}}{V_{cs}^* V_{us}} = -1.0 + (6.2^{+1.2}_{-1.0}) \times 10^{-4} i$$

variation of ms by 0.01 GeV

PDG

$$u = -1.00029 \pm 0.00002, \quad v = 0.00064 \pm 0.00002$$

they agree well

Analytical solution

- Insert u=-1 into m=1 constraint to get analytical expression of v

$$v \approx \frac{(m_W^2 - m_b^2)(m_s^2 - m_d^2)}{(m_W^2 - m_s^2)(m_b^2 - m_d^2)} \approx \frac{m_s^2}{m_b^2}$$

- In terms of Wolfenstein parameters $v = A^2\lambda^4\eta$

Ahn et al, 2011

- Produce well-known numerical relation

$$\lambda = V_{us} \approx (A^2\eta)^{-1/4}\sqrt{\frac{m_s}{m_b}} \approx \sqrt{\frac{m_s}{m_b}}$$

$$A \approx 0.826 \qquad \eta \approx 0.348$$

$$(A^2\eta)^{-1/4} \approx 1.43 \sim O(1)$$

Belfatto et al, 2023

Cheng, Sher 1987

Lepton mixing

Pontecorvo–Maki–Nakagawa–Sakata

- Apply the same formalism to lepton $\mu^- e^+$-$\mu^+ e^-$ mixing through similar box diagrams with intermediate neutrino channels
- Correspondence $m_{d,s,b} \longleftrightarrow m_{1,2,3}$ $\quad V_{cd}^* V_{ud}/(V_{cs}^* V_{us}) \longleftrightarrow r = U_{\mu 1}^* U_{e1}/(U_{\mu 2}^* U_{e2})$
- Normal hierarchy (NH) $\quad m_1^2 = 10^{-6}\ \text{eV}^2$

de Salas et al, 2018

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2 = (7.55^{+0.20}_{-0.16}) \times 10^{-5}\ \text{eV}^2 \qquad \Delta m_{32}^2 \equiv m_3^2 - m_2^2 = (2.424 \pm 0.03) \times 10^{-3}\ \text{eV}^2$$

- Predict

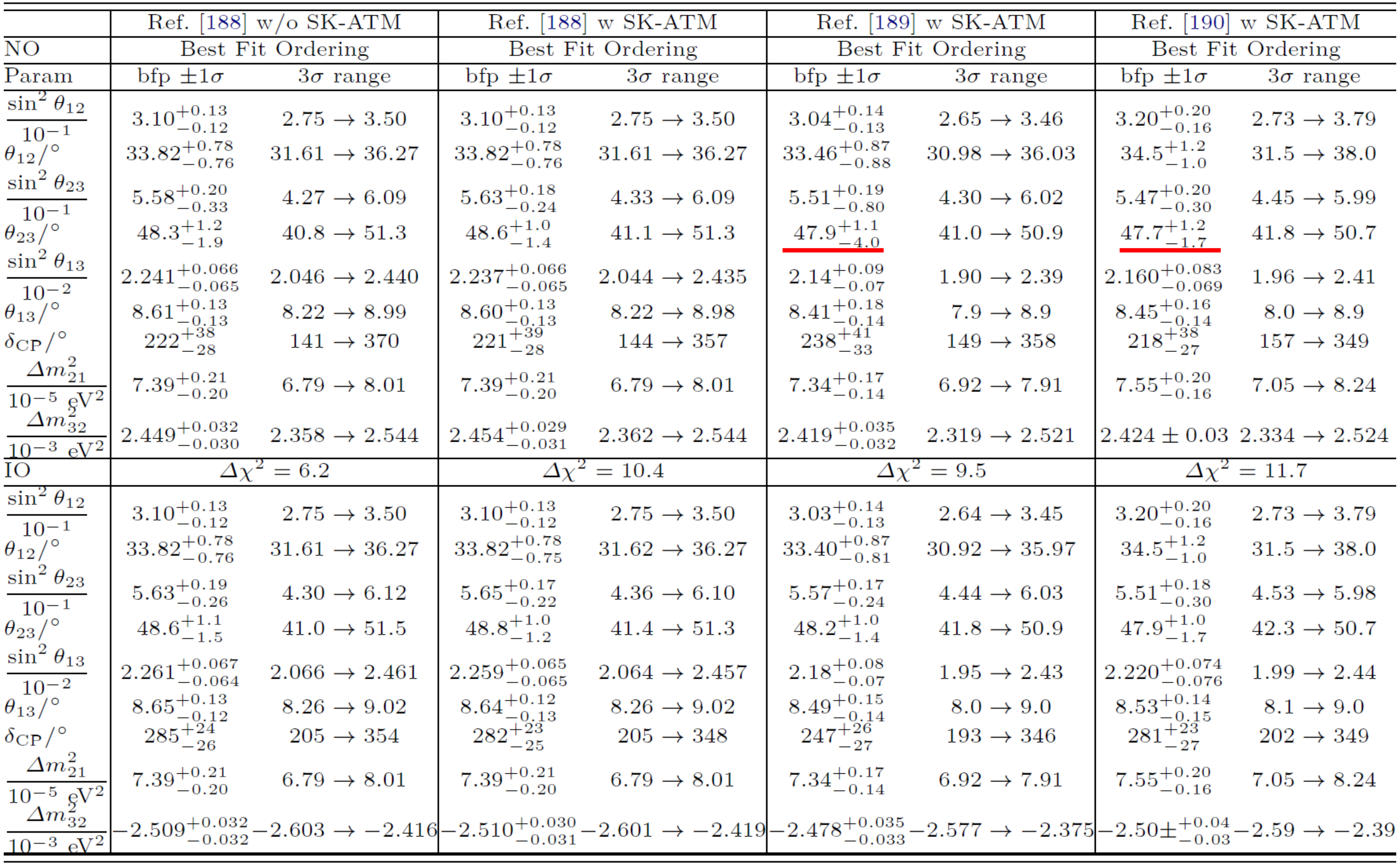
$$r = \frac{U_{\mu 1}^* U_{e1}}{U_{\mu 2}^* U_{e2}} \approx -1.0 - 0.02i$$

global fit

$$r = -(0.738^{+0.050}_{-0.048}) - (0.179^{+0.136}_{-0.125})i$$

- Inverted hierarchy (IH) $\quad r \approx -1.0 - O(10^{-5})i \qquad r = -(1.03^{+0.05}_{-0.16}) - (0.356^{+0.015}_{-0.048})i$
- Quasi-degenerate $\quad r \approx -0.97 - O(10^{-5})i$
- NH and observed PMNS matrix satisfy constraint at order of magnitude

PDG



	Ref. [188] w/o SK-ATM		Ref. [188] w SK-ATM		Ref. [189] w SK-ATM		Ref. [190] w SK-ATM	
NO	Best Fit Ordering		Best Fit Ordering		Best Fit Ordering		Best Fit Ordering	
Param	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.04^{+0.14}_{-0.13}$	$2.65 \to 3.46$	$3.20^{+0.20}_{-0.16}$	$2.73 \to 3.79$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.46^{+0.87}_{-0.88}$	$30.98 \to 36.03$	$34.5^{+1.2}_{-1.0}$	$31.5 \to 38.0$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.58^{+0.20}_{-0.33}$	$4.27 \to 6.09$	$5.63^{+0.18}_{-0.24}$	$4.33 \to 6.09$	$5.51^{+0.19}_{-0.80}$	$4.30 \to 6.02$	$5.47^{+0.20}_{-0.30}$	$4.45 \to 5.99$
$\theta_{23}/^\circ$	$48.3^{+1.2}_{-1.9}$	$40.8 \to 51.3$	$48.6^{+1.0}_{-1.4}$	$41.1 \to 51.3$	$47.9^{+1.1}_{-4.0}$	$41.0 \to 50.9$	$47.7^{+1.2}_{-1.7}$	$41.8 \to 50.7$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.241^{+0.066}_{-0.065}$	$2.046 \to 2.440$	$2.237^{+0.066}_{-0.065}$	$2.044 \to 2.435$	$2.14^{+0.09}_{-0.07}$	$1.90 \to 2.39$	$2.160^{+0.083}_{-0.069}$	$1.96 \to 2.41$
$\theta_{13}/^\circ$	$8.61^{+0.13}_{-0.13}$	$8.22 \to 8.99$	$8.60^{+0.13}_{-0.13}$	$8.22 \to 8.98$	$8.41^{+0.18}_{-0.14}$	$7.9 \to 8.9$	$8.45^{+0.16}_{-0.14}$	$8.0 \to 8.9$
$\delta_{\rm CP}/^\circ$	222^{+38}_{-28}	$141 \to 370$	221^{+39}_{-28}	$144 \to 357$	238^{+41}_{-33}	$149 \to 358$	218^{+38}_{-27}	$157 \to 349$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.34^{+0.17}_{-0.14}$	$6.92 \to 7.91$	$7.55^{+0.20}_{-0.16}$	$7.05 \to 8.24$
$\frac{\Delta m^2_{32}}{10^{-3}\ \text{eV}^2}$	$2.449^{+0.032}_{-0.030}$	$2.358 \to 2.544$	$2.454^{+0.029}_{-0.031}$	$2.362 \to 2.544$	$2.419^{+0.035}_{-0.032}$	$2.319 \to 2.521$	2.424 ± 0.03	$2.334 \to 2.524$
IO	$\Delta\chi^2 = 6.2$		$\Delta\chi^2 = 10.4$		$\Delta\chi^2 = 9.5$		$\Delta\chi^2 = 11.7$	
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.10^{+0.13}_{-0.12}$	$2.75 \to 3.50$	$3.03^{+0.14}_{-0.13}$	$2.64 \to 3.45$	$3.20^{+0.20}_{-0.16}$	$2.73 \to 3.79$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$	$33.82^{+0.78}_{-0.75}$	$31.62 \to 36.27$	$33.40^{+0.87}_{-0.81}$	$30.92 \to 35.97$	$34.5^{+1.2}_{-1.0}$	$31.5 \to 38.0$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.63^{+0.19}_{-0.26}$	$4.30 \to 6.12$	$5.65^{+0.17}_{-0.22}$	$4.36 \to 6.10$	$5.57^{+0.17}_{-0.24}$	$4.44 \to 6.03$	$5.51^{+0.18}_{-0.30}$	$4.53 \to 5.98$
$\theta_{23}/^\circ$	$48.6^{+1.1}_{-1.5}$	$41.0 \to 51.5$	$48.8^{+1.0}_{-1.2}$	$41.4 \to 51.3$	$48.2^{+1.0}_{-1.4}$	$41.8 \to 50.9$	$47.9^{+1.0}_{-1.7}$	$42.3 \to 50.7$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.261^{+0.067}_{-0.064}$	$2.066 \to 2.461$	$2.259^{+0.065}_{-0.065}$	$2.064 \to 2.457$	$2.18^{+0.08}_{-0.07}$	$1.95 \to 2.43$	$2.220^{+0.074}_{-0.076}$	$1.99 \to 2.44$
$\theta_{13}/^\circ$	$8.65^{+0.13}_{-0.12}$	$8.26 \to 9.02$	$8.64^{+0.12}_{-0.13}$	$8.26 \to 9.02$	$8.49^{+0.15}_{-0.14}$	$8.0 \to 9.0$	$8.53^{+0.14}_{-0.15}$	$8.1 \to 9.0$
$\delta_{\rm CP}/^\circ$	285^{+24}_{-26}	$205 \to 354$	282^{+23}_{-25}	$205 \to 348$	247^{+26}_{-27}	$193 \to 346$	281^{+23}_{-27}	$202 \to 349$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$	$7.34^{+0.17}_{-0.14}$	$6.92 \to 7.91$	$7.55^{+0.20}_{-0.16}$	$7.05 \to 8.24$
$\frac{\Delta m^2_{32}}{10^{-3}\ \text{eV}^2}$	$-2.509^{+0.032}_{-0.032}$	$-2.603 \to -2.416$	$-2.510^{+0.030}_{-0.031}$	$-2.601 \to -2.419$	$-2.478^{+0.035}_{-0.033}$	$-2.577 \to -2.375$	$-2.50\pm^{+0.04}_{-0.03}$	$-2.59 \to -2.39$

Other observations

- Chau-Keung parametrization $\mathrm{Im}(r) \propto \frac{s_{13}s_{23}}{s_{12}}$
- Larger mixing angles in lepton sector due to $\frac{m_2^2}{m_3^2} \approx 3.1 \times 10^{-2} \gg \frac{m_s^2}{m_b^2} \approx 9.0 \times 10^{-4}$
- How about $\tau^- e^+$-$\tau^+ e^-$ or $t\bar{u}$-$\bar{t}u$ mixing? Same intermediate channels, so same constraints. Additional solutions?
- Two possibilities: first, small $\lambda_i \lambda_j$, so that constraints met automatically, happening to quark sector $|V_{cs}^* V_{us}|^2 = \lambda^2 \approx 5 \times 10^{-2}$ $|V_{ts}^* V_{us}|^2 = A^2 \lambda^6 \approx 9 \times 10^{-5}$
- Second, two solutions of v with opposite signs: one for $\mu^- e^+$-$\mu^+ e^-$ another for $\tau^- e^+$-$\tau^+ e^-$
- Check data

$U_{\tau 1}^* U_{e1}/(U_{\tau 2}^* U_{e2})$	$r = U_{\mu 1}^* U_{e1}/(U_{\mu 2}^* U_{e2})$	
$-(1.231^{+0.078}_{-0.186}) + (0.204^{+0.085}_{-0.138})i$	$r = -(0.738^{+0.050}_{-0.048}) - (0.179^{+0.136}_{-0.125})i$	de Salas et al, 2018
$-(1.139^{+0.139}_{-0.207}) + (0.266^{+0.050}_{-0.124})i$	$r = -(0.801^{+0.219}_{-0.097}) - (0.265^{+0.090}_{-0.145})i$	Capozzi et al, 2018

Constraint on θ_{23}

$c_\delta \equiv \cos\delta \qquad s_\delta \equiv \sin\delta$

CP phase

- Ratios in CK parametrization

roughly equal

$$\frac{U^*_{\mu 1}U_{e1}}{U^*_{\mu 2}U_{e2}} = -\frac{c_{12}}{s_{12}}\,\frac{c_{12}s_{12}(c_{23}^2 - s_{13}^2 s_{23}^2) + c_{23}s_{13}s_{23}c_\delta(c_{12}^2 - s_{12}^2) - c_{23}s_{13}s_{23}s_\delta i}{(c_{12}c_{23} - s_{12}s_{13}s_{23})^2 + 2c_{12}c_{23}s_{12}s_{13}s_{23}(1 - c_\delta)}$$

$$\frac{U^*_{\tau 1}U_{e1}}{U^*_{\tau 2}U_{e2}} = -\frac{c_{12}}{s_{12}}\,\frac{c_{12}s_{12}(s_{23}^2 - c_{23}^2 s_{13}^2) - c_{23}s_{13}s_{23}c_\delta(c_{12}^2 - s_{12}^2) + c_{23}s_{13}s_{23}s_\delta i}{(c_{12}s_{23} + c_{23}s_{12}s_{13})^2 - 2c_{12}c_{23}s_{12}s_{13}s_{23}(1 - c_\delta)}$$

roughly equal

- The two ratios differ only by sign of Im
- Relation among mixing angles

$$(c_{12}^2 + s_{12}^2 s_{13}^2)(c_{23}^2 - s_{23}^2) \approx 0 \quad \Longleftarrow \quad \begin{aligned} &(c_{12}^2 - s_{12}^2 s_{13}^2)(c_{23}^2 - s_{23}^2) - 4c_{12}c_{23}s_{12}s_{13}s_{23}c_\delta \approx 0, \\ &c_{12}s_{12}(1 + s_{13}^2)(c_{23}^2 - s_{23}^2) + 2(c_{12}^2 - s_{12}^2)c_{23}s_{13}s_{23}c_\delta \approx 0 \end{aligned}$$

- Indicate $c_{23} \approx s_{23}$, i.e., $\theta_{23} \approx 45^\circ$ .... maximal lepton mixing!

Summary and conjecture

- Dispersion relations for observables impose stringent constraint
- Fermion masses and mixing angles in both quark and lepton sectors constrained by SM dynamics itself
- Different mixing patterns due to different fermion mass ratios

$$\frac{m_2^2}{m_3^2} \approx 3.1 \times 10^{-2} \gg \frac{m_s^2}{m_b^2} \approx 9.0 \times 10^{-4}$$

- Normal hierarchy favored by dispersive constraint
- Maximal lepton mixing demanded by solutions for mixing between generations 1, 2 and generations 1, 3
- It is likely that SM has only three fundamental (gauge) parameters
- Scalar sector, coupling various generations, is not free