The Romance Between Maths and Physics

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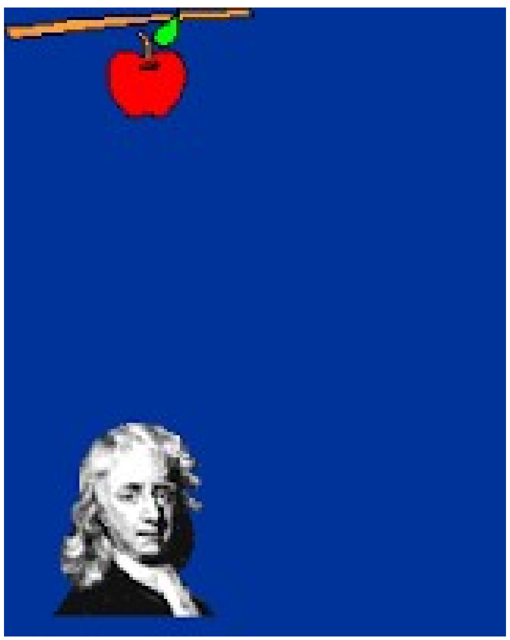
Netherlands Organisation for Scientific Research

European Research Council
Established by the European Commission

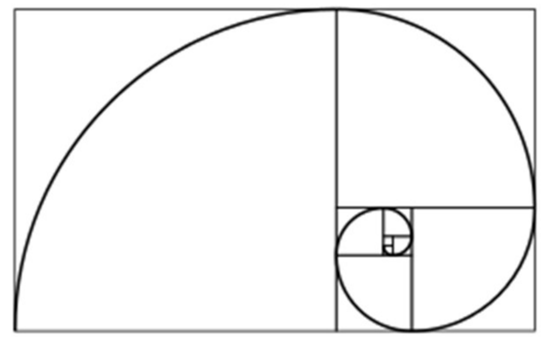
Very happy to be back in NTU indeed!

Question 1:
Why is Nature predictable at all (to some extent)?



Question 2:
Why are the predictions in the form of mathematics?

$$F = G\frac{m_1 m_2}{r^2}$$



the *unreasonable effectiveness* of mathematics in natural sciences.
Eugene Wigner (1960)

First we resorted to gods and spirits to explain the world , and then there were ….. *mathematicians*?!

Nicolaus Copernicus

Mathematician

Nicolaus Copernicus was a Renaissance-era mathematician, astronomer, and Catholic clergyman who formulated a model of the universe that placed the Sun rather than Earth at the center of the universe. Wikipedia

Born: February 19, 1473, Torun, Poland

Died: 1543, Frombork, Poland

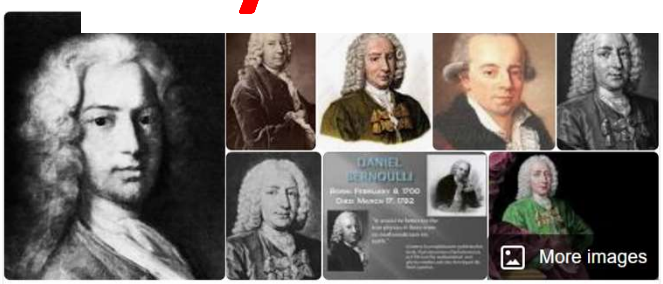
Isaac Newton

Mathematician

Sir Isaac Newton PRS was an English mathematician, physicist, astronomer, theologian, and author who is widely recognised as one of the most influential scientists of all time and as a key figure in the scientific revolution. Wikipedia

Born: January 4, 1643, Woolsthorpe Manor House, United Kingdom

Physicists or Mathematicians?



Daniel Bernoulli

Mathematician

Daniel Bernoulli FRS was a Swiss mathematician and physicist and was one of the many prominent mathematicians in the Bernoulli family from Basel. He is particularly remembered for his applications of mathematics to mechanics, especially fluid mechanics, and for his pioneering work in probability and statistics. Wikipedia

Born: February 8, 1700, Groningen, Netherlands

Died: March 17, 1782, Basel, Switzerland

Joseph-Louis Lagrange

Mathematician

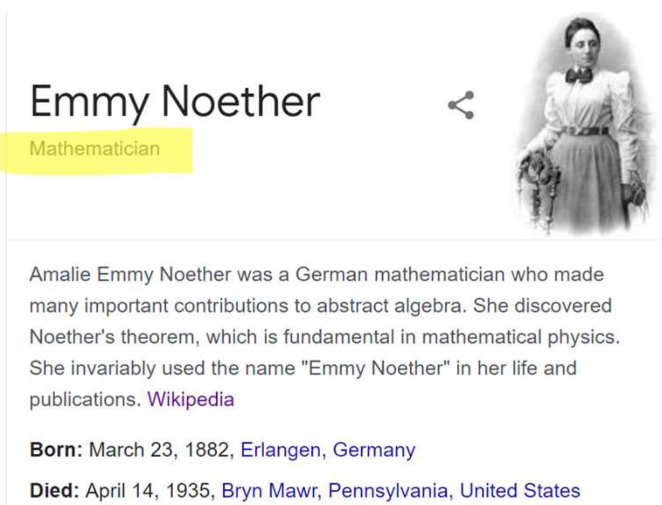
Joseph-Louis Lagrange, also reported as Giuseppe Luigi Lagrange or Lagrangia, was an Italian mathematician and astronomer, later naturalized French. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics. Wikipedia

Born: January 25, 1736, Turin, Italy

Died: April 10, 1813, Paris, France

Until the 19th century, the relation between physical sciences and mathematics is so close that there was hardly any distinction made between “physicists” and “mathematicians”.

Even after the specialisation starts to be made, the two maintain an extremely close relation and cannot live without one another.



Some of the love declarations …

Quantum mechanics requires the introduction into physical theory of a vast new domain of pure mathematics - the whole domain connected with non-commutative multiplication. This, coming on top of the introduction of the new geometries by the theory of relativity, indicates a trend which we may expect to continue. We may expect that in the future further big domains of pure mathematics will have to be brought in to deal with the advances in fundamental physics.

Dirac (1938)

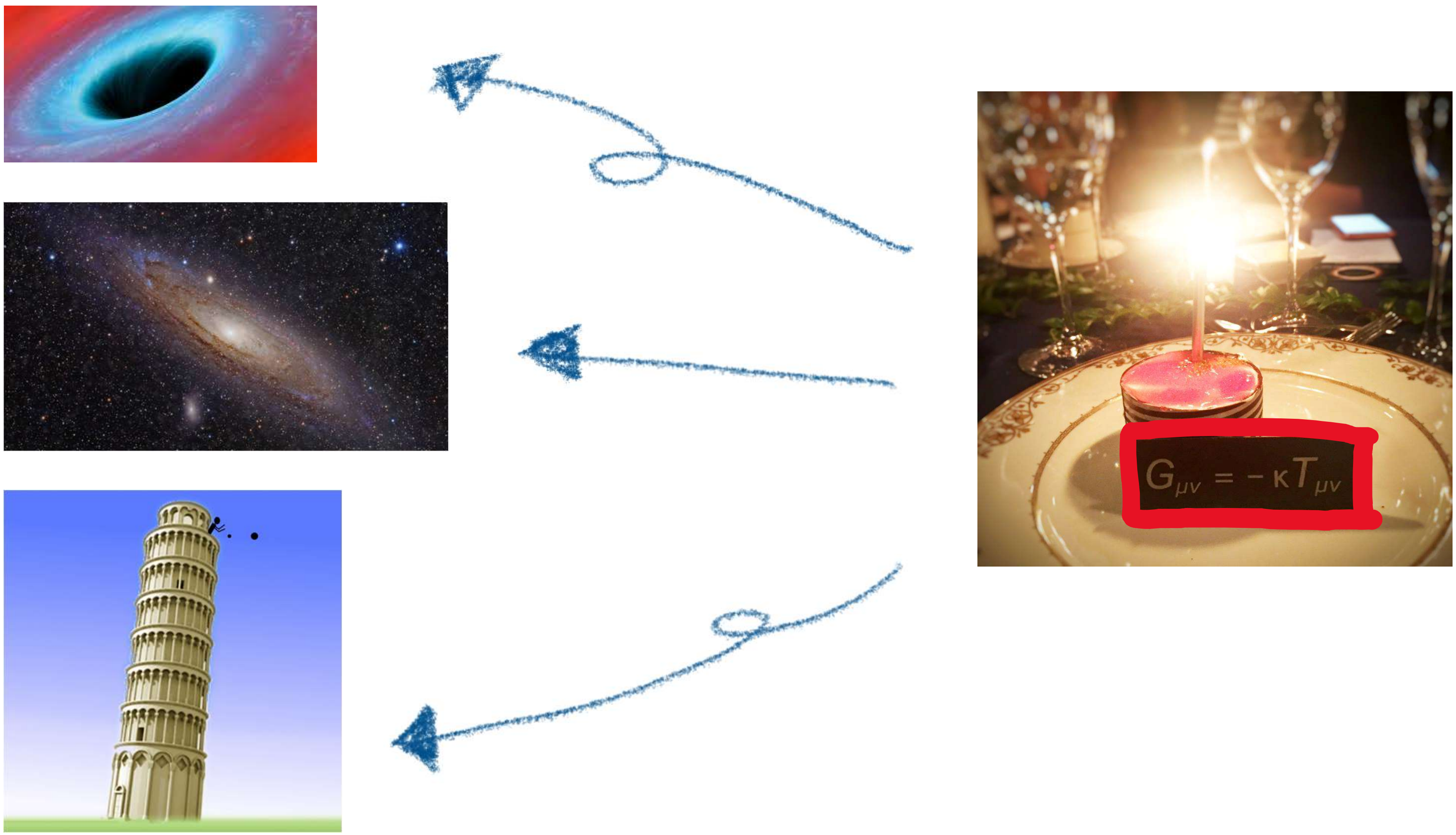
If you want to be a physicist, you must do three things—
first, study mathematics, second, study more mathematics,
and third, do the same.

Sommerfeld (1934)

Our experience up to date justifies us in feeling sure that in Nature is actualized the ideal of mathematical simplicity. It is my conviction that pure mathematical construction enables us to discover the concepts and the laws connecting them, which gives us the key to understanding nature… In a certain sense, therefore, I hold it true that pure thought can grasp reality, as the ancients dreamed.

Einstein (1934)

Indeed, the most irresistible ***reductionistic*** charm of physics, could not have been possible without mathematics …



Love or Hate?
It's Complicated…

In the era when Physics seemed invincible (think about the standard model), they thought they didn't need each other anymore.

As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.

Dyson (1972)

If all of mathematics disappeared, physics would be set back by exactly one week.

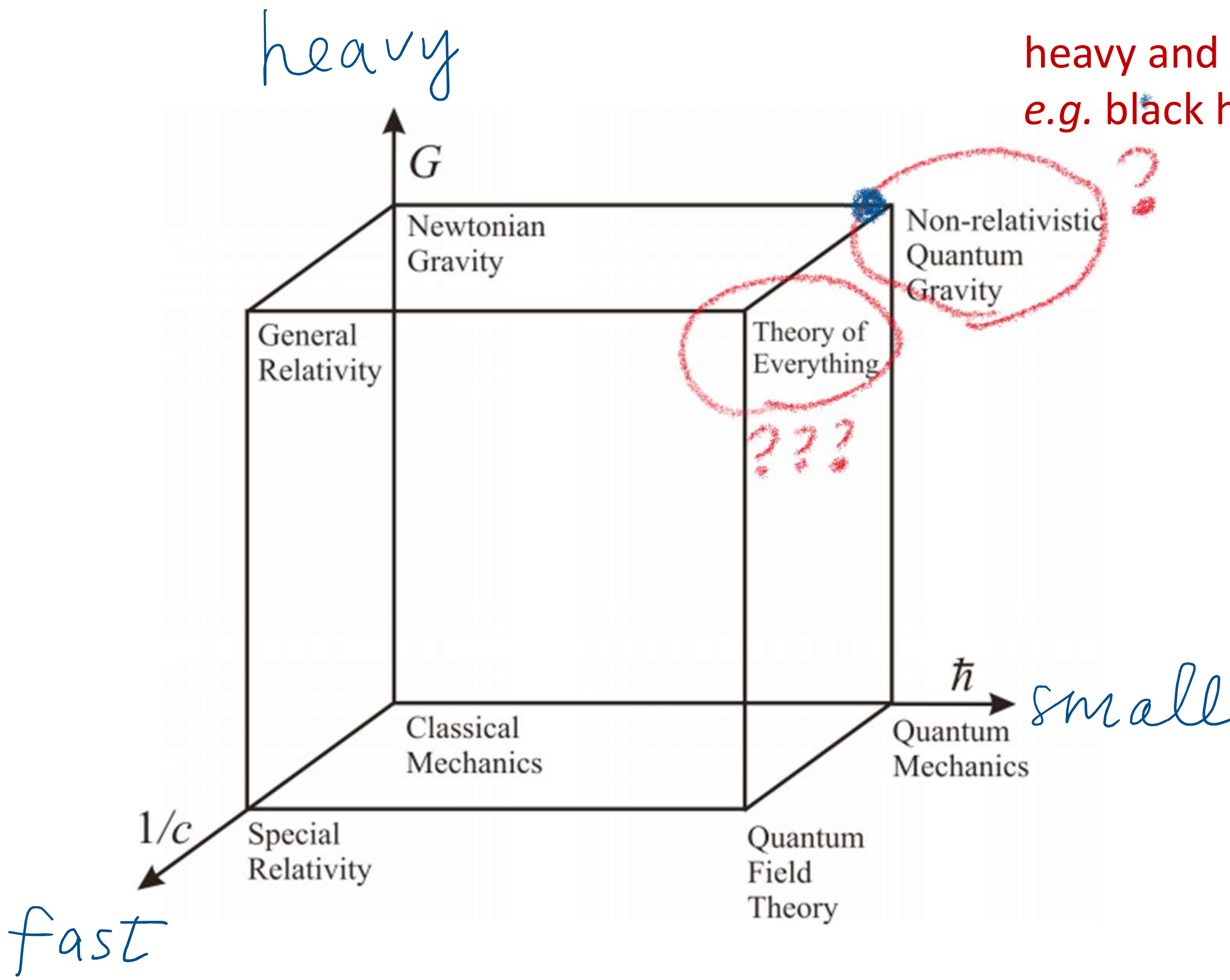
Richard Feynman

Sir Michael Atiyah

That was the week that God created the world.

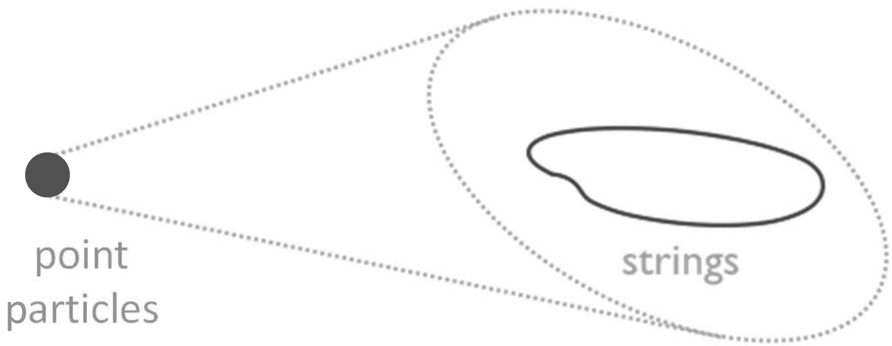
(An anecdote recounted by Robbert Dijkgraaf.)

But, is physics really so invincible? What about these problems?

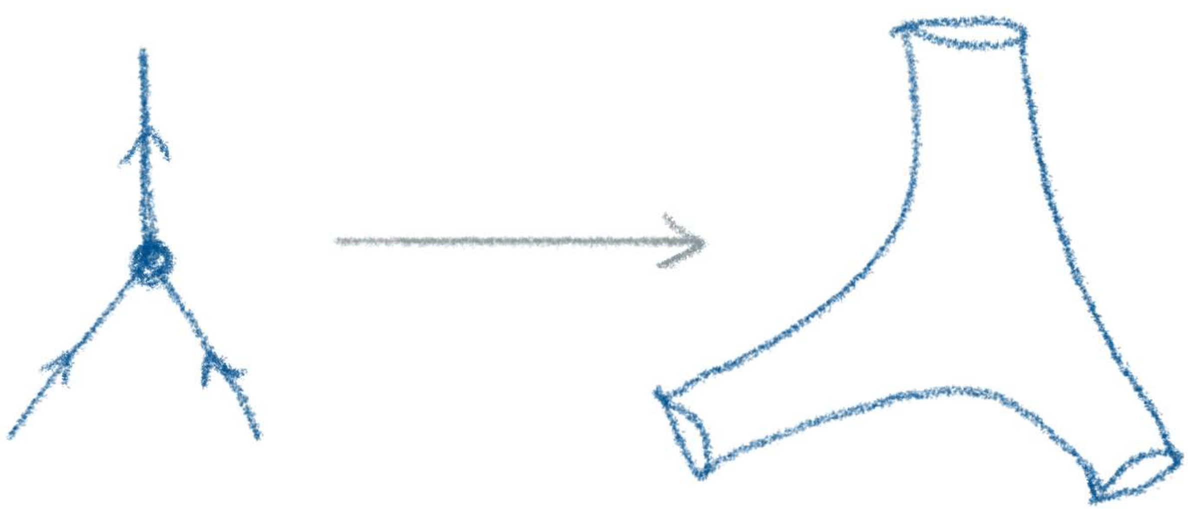


heavy and small
e.g. black holes

String Theory: solving quantum gravity



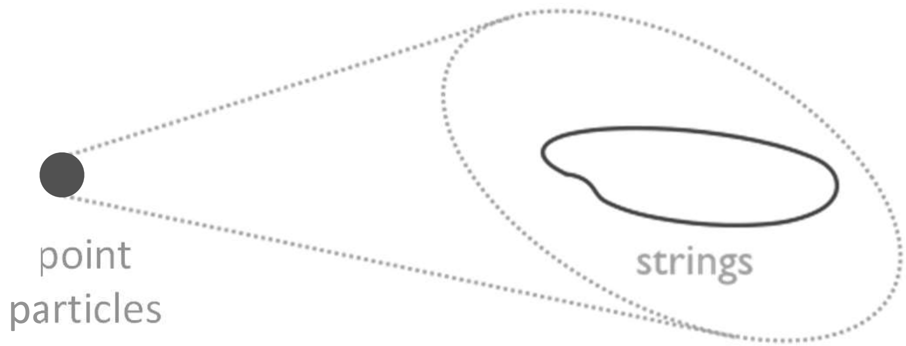
unifies
Quantum Mechanics
and
General Relativity



conventional field theory
singularities

string theory
no distinguished singular point

String Theory: the ultimate dream of reductionism.



unifies
Quantum Mechanics
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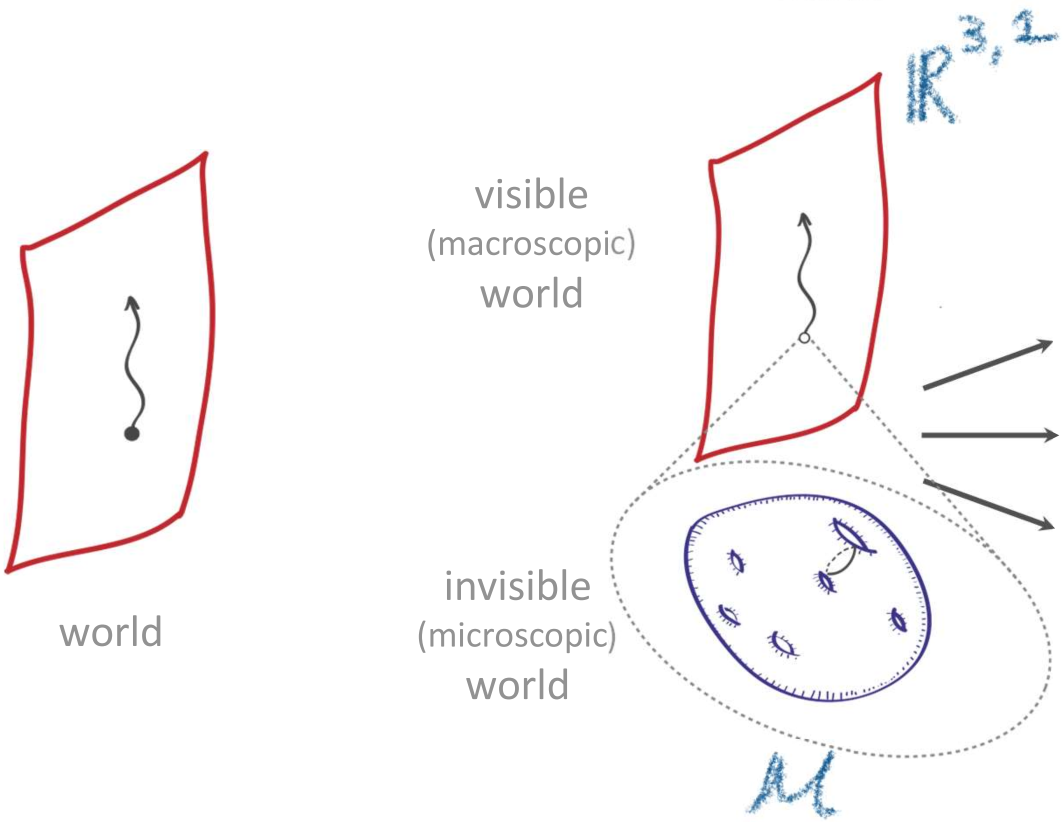
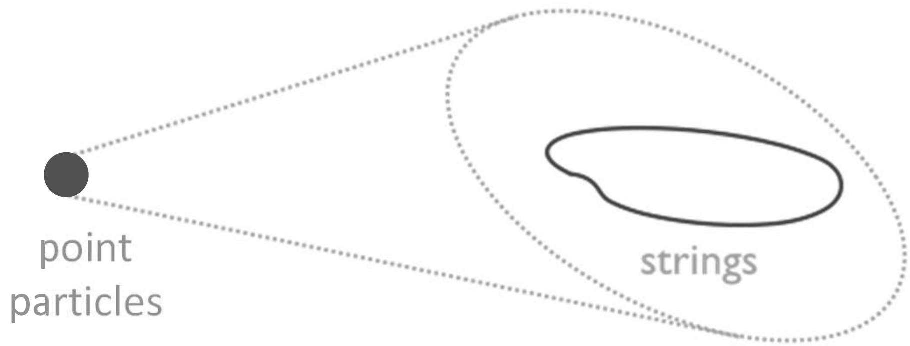
General Relativity



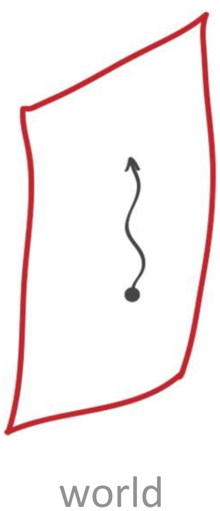
The invisible determines the physics we see!

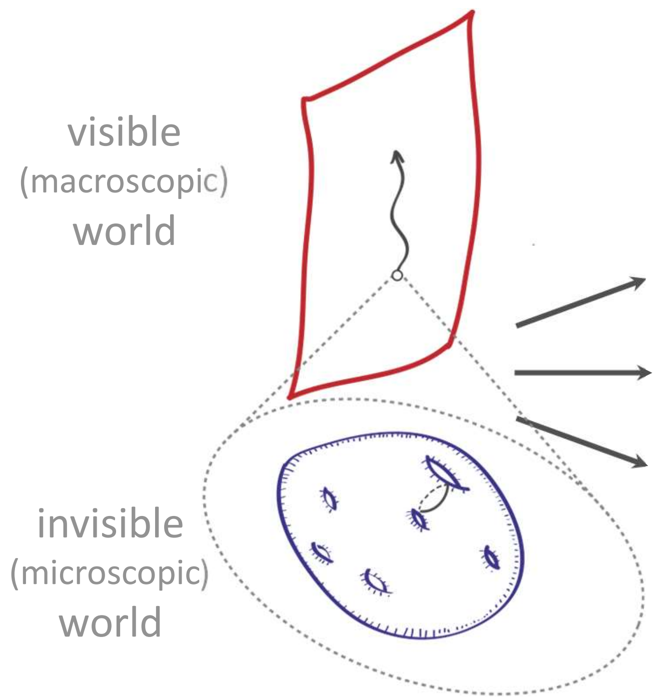
(Quantum) Gravity
Standard model
Beyond SM physics
⋮

String Theory: a looming conundrum.



unifies
Quantum Mechanics
and
General Relativity



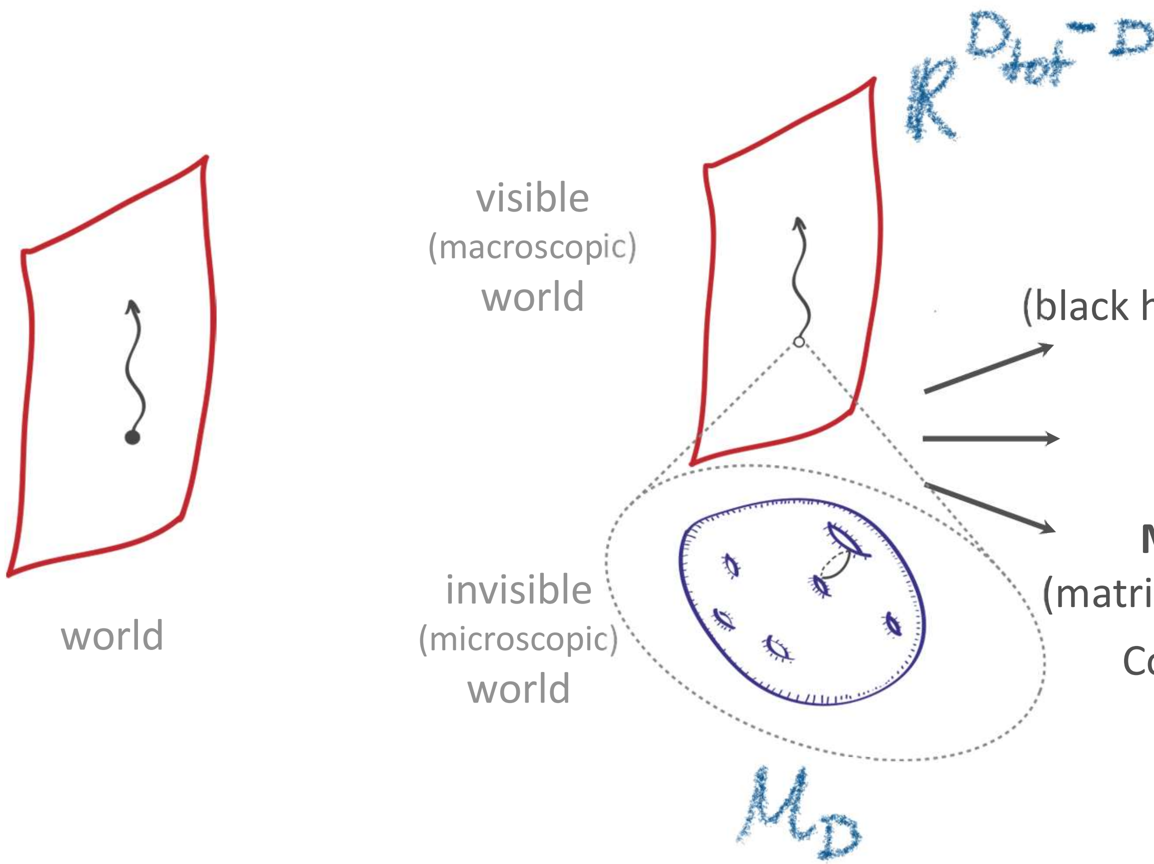


The invisible determines the physics we see!

Different choices of the compactification manifolds lead to different "worlds" with different physics.

String Theory: as a model for all sorts of physics.

or, turning weakness into strength.

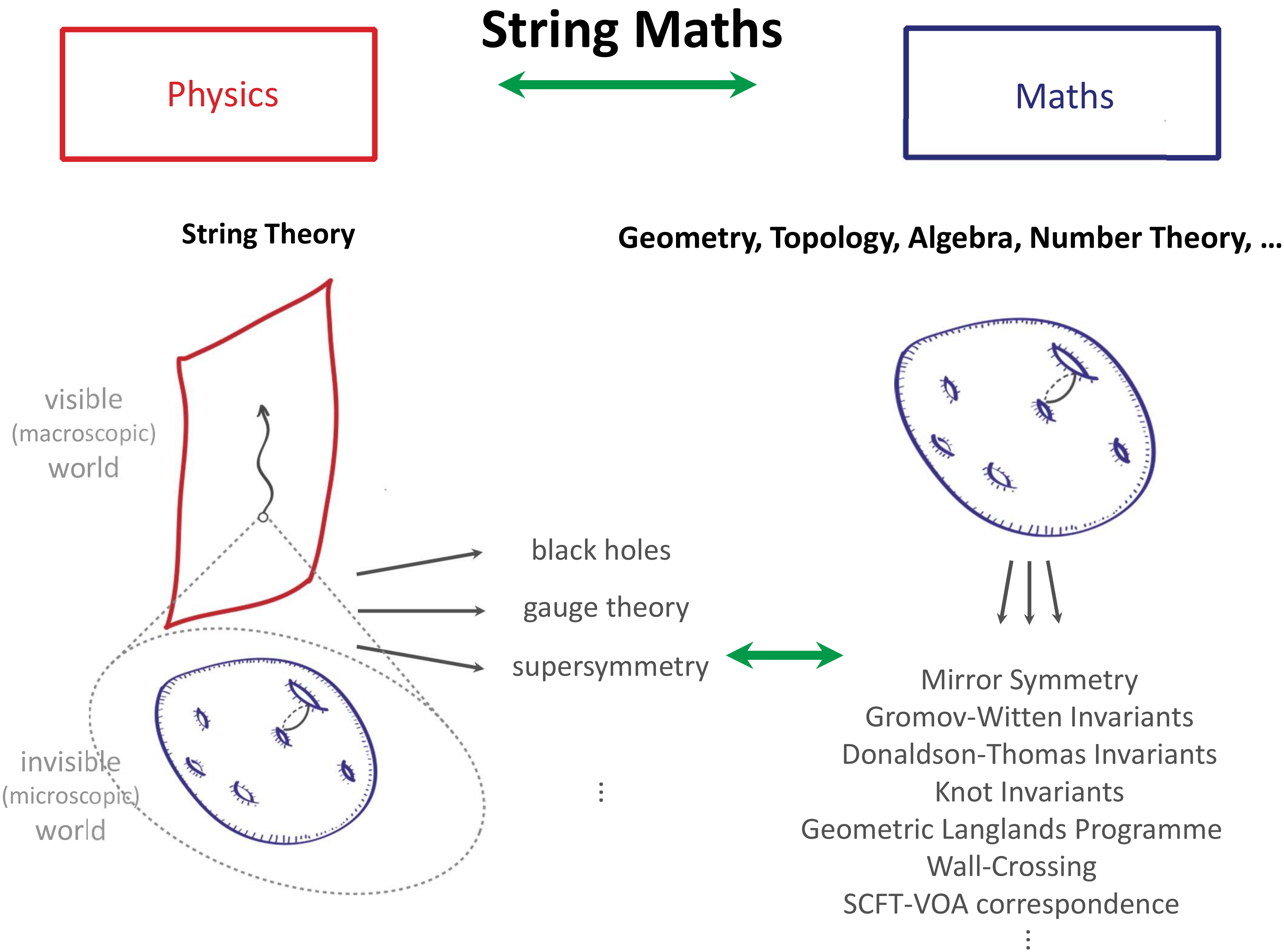


Quantum Gravity
(black holes, holographic principle)

Gauge Theory

Mathematical Physics
(matrix models, integrability, ...)

Condensed Matter Physics
Particle Physics
Cosmology
⋮


String Maths
Physics
Maths
String Theory
Geometry, Topology, Algebra, Number Theory, ...
visible
(macroscopic)
world
invisible
(microscopic)
world
black holes
gauge theory
supersymmetry
⋮
Mirror Symmetry
Gromov-Witten Invariants
Donaldson-Thomas Invariants
Knot Invariants
Geometric Langlands Programme
Wall-Crossing
SCFT-VOA correspondence
⋮

In the form of "String Math", "Physical Mathematics"*, "Physmatics"**, ….

*: from Greg Moore, **: from Eric Sharpe

I. Moonshine


Finite
Groups
Moonshine
Modular
Forms
functions with special
symmetries

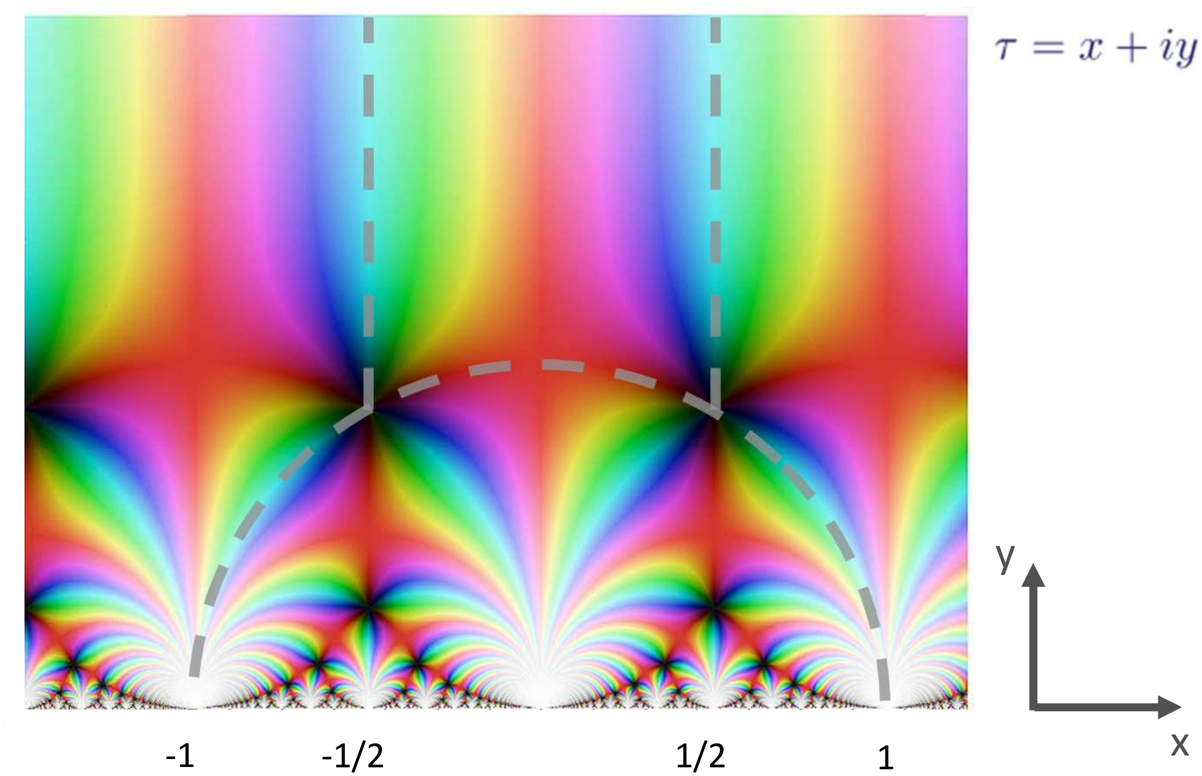
Modular Forms

= A holomorphic function on the upper-half plane
that transforms "nicely" under SL(2,***Z***).

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau+b}{c\tau+d}$

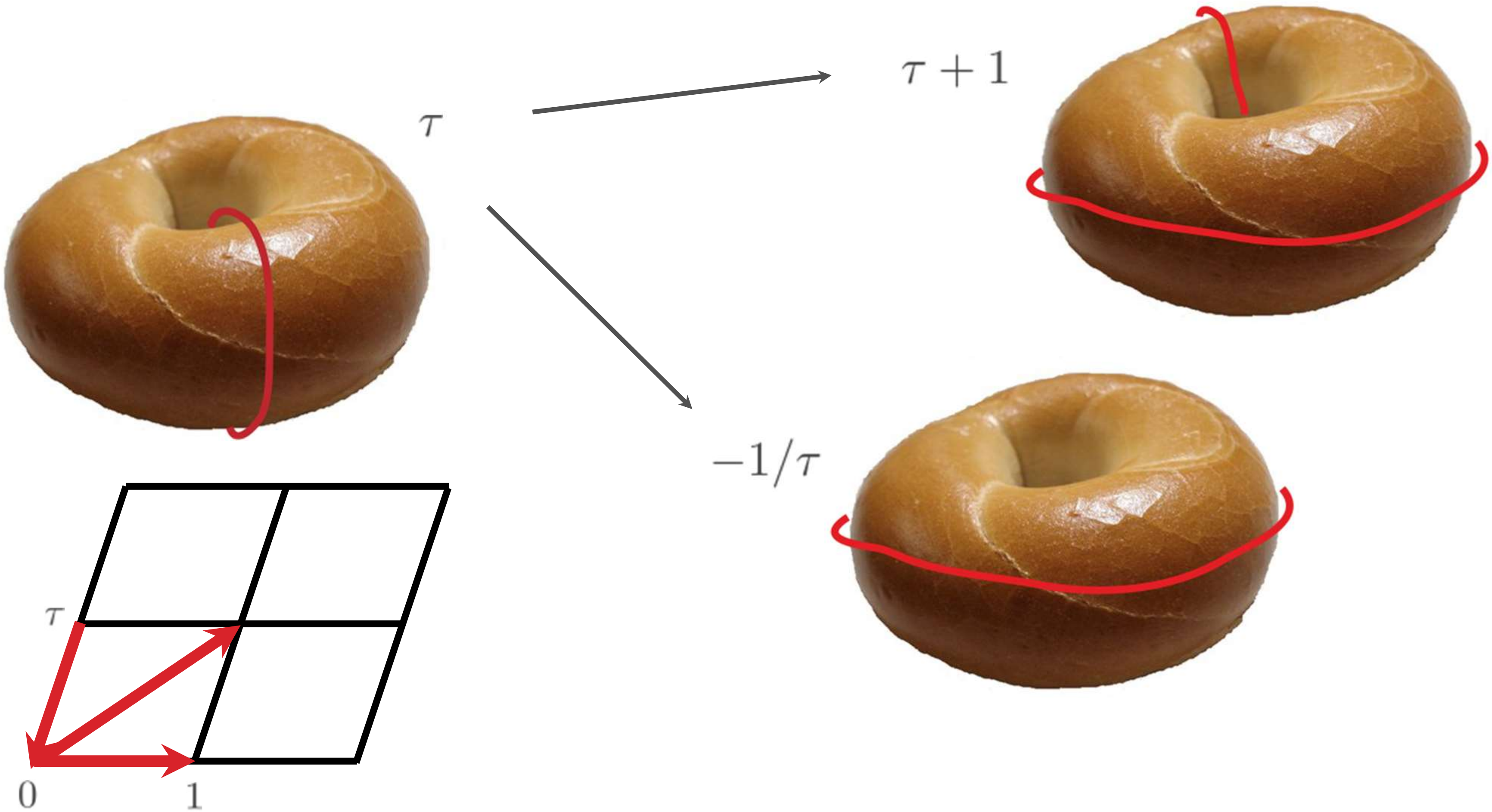
Example: the *J*-function

$$J(\tau) = J(\tau+1) = J(-1/\tau)$$



Modular Forms

reflect the symmetry of a torus.



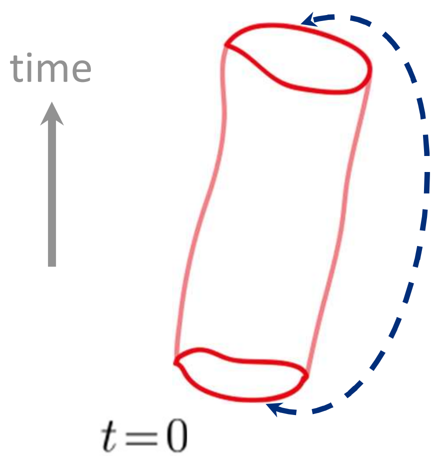
The modular action leaves the torus the same,
just a different way to parametrise it.

Modular Forms

are natural products of string theory.

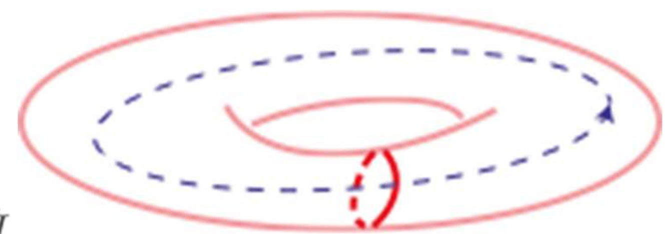
A string moving in time = a cylinder.

$$t = \tau / i = 1/k_B T$$



partition function

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}} e^{2\pi i \tau \hat{H}}$$



The partition functions are computed by identifying the initial and final time. This turns the cylinder into a torus. As a result the string partition functions are modular forms!

String theory is good at producing functions with symmetries!

More generally, there can be space-time symmetries (such as E-M, S-, T-dualities) as well as world-sheet symmetries (such as SL(2,Z)).

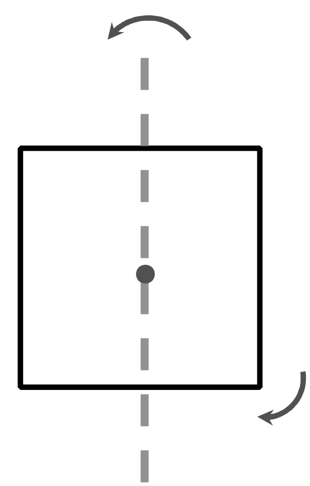
All symmetries have to be reflected in suitable partition functions.

Finite Groups $\xleftrightarrow{\text{Moonshine}}$ Modular Forms

Finite Groups

Discrete Symmetries of Objects

Example 1. Symmetry of a square = the dihedral group *"Dih_4"*.



Finite Groups

Discrete Symmetries of Objects

Example 2. Close Packing Lattices: considering the most efficient way to stack up identical balls.

Face-Centered Cubic (fcc) Lattices
e.g. Cu, Ag, Au

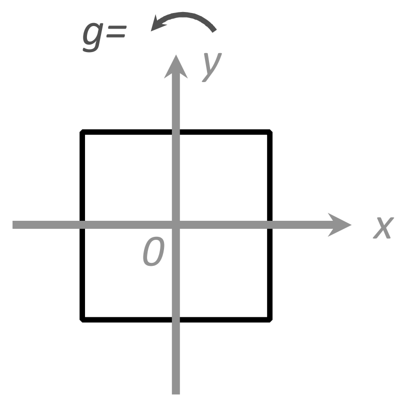
Finite Groups

the Representations

A representation V of a finite group G is a space that G acts on.

$$\begin{matrix} V \\ \circlearrowleft \\ G \end{matrix}$$

e.g. $G = Dih_4$ =Symmetry group of a square.



2-dimensional representation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathrm{ch}_V g := \mathrm{Tr}_V g = 0$$

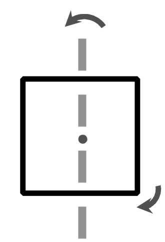
Some Interesting Examples

the “*Sporadic Groups*”

The only 26 groups that cannot be studied systematically.
They are usually symmetries of very interesting objects.

Example 1. ***M_{24}*** = the oldest sporadic *“Mathieu Groups”.* [Mathieu 1860]

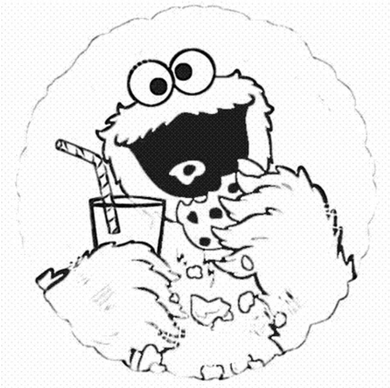
$|M_{24}|$ = the number of elements in *M_{24} = 244,823,040.* *cf.* *$|Dih_4| = 8$*



Example 2. *“The **Monster**”* = the largest sporadic groups. [Fischer, Griess 1970-80]

*|**M**| ~ 10^{54} ~ the number of atoms in the solar system.*

The smallest non-trivial representation has 196,883 dimensions.




Finite Groups
Moonshine
Modular Objects

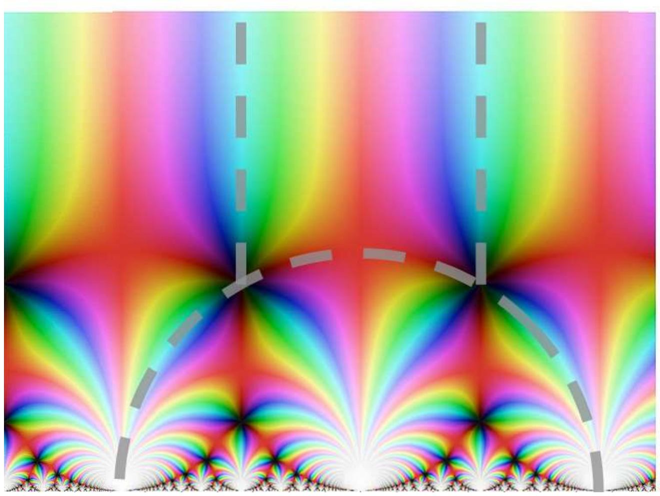
Moonshine

relates modular forms and finite groups.

$$\begin{aligned} J(\tau) &= J(\tau+1) = J(-\tfrac{1}{\tau}) \\ &= q^{-1} + \underbrace{196884}_{= 1+196883} q + \underbrace{21493760}_{= 1+196883+21296876} q^2 + \ldots \qquad (q = e^{2\pi i \tau}) \end{aligned}$$

[McKay late 70's]

dim irreps of Monster

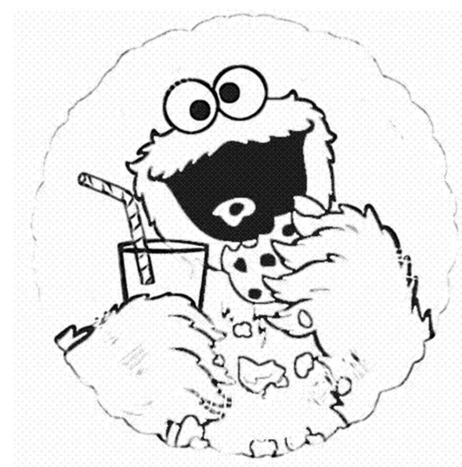


The Most Natural Modular Function

??

Monstrous Moonshine Conjecture

[Conway–Norton '79]

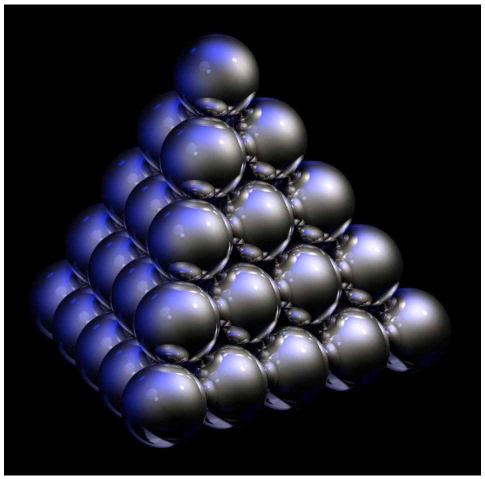


The Largest Sporadic Group

String Theory

explains Monstrous Moonshine.

Q: What's the most efficient way to stack up *24-dimensional* identical balls?



A: It's given by the *Leech lattice* $\boldsymbol{\Lambda_{Leech}}$. [Leech 1967]

$\boldsymbol{\Lambda_{Leech}}$ has very interesting sporadic symmetries.

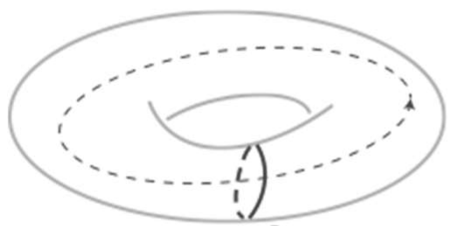
Q: But it's *24-dimensional!* What can we do with it?

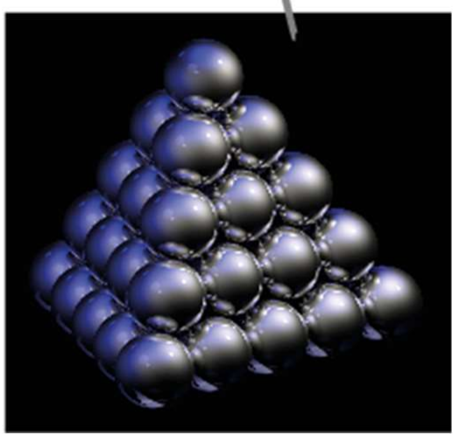
A: Just the right number of dimensions for string theory!

String Theory

explains Monstrous Moonshine.

Strings in the Leech lattice background:

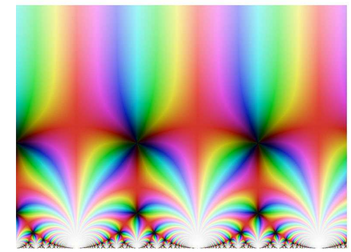




R^{24}/Λ_{Leech}

Modular Symmetry

The Partition Function = $J(\tau)$



Monstrous Moonshine

Sporadic Symmetry

Λ_{Leech}

$Co1 \xrightarrow{Z/2}$ ***Monster***

Monstrous Moonshine

A Meeting Place of Different Subjects

To prove the Monstrous Moonshine Conjecture

- Use ideas shared with string theory

(orbifold conformal field theory, no-ghost theorem, ...)

- Led to important developments in algebra and representation theory

(vertex operator algebra, Borcherds–Kac–Moody algebra, ...)

[Frenkel–Lepowsky–Meurman, 80's]

[Borcherds, 80-90's]

Umbral Moonshine

('13 w. J. Duncan, J. Harvey)

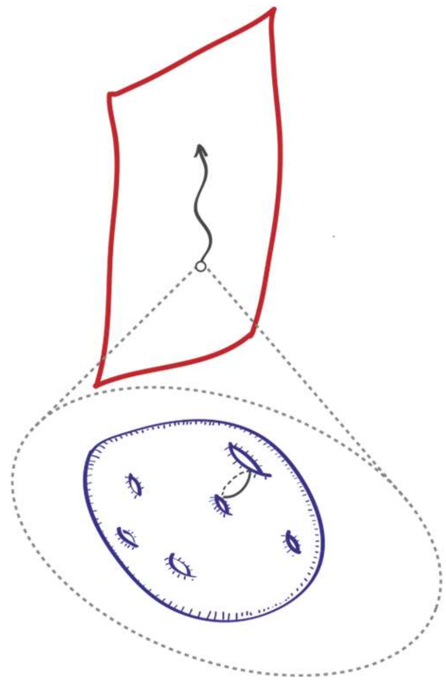
"Niemeier" Finite Groups

Umbral Moonshine

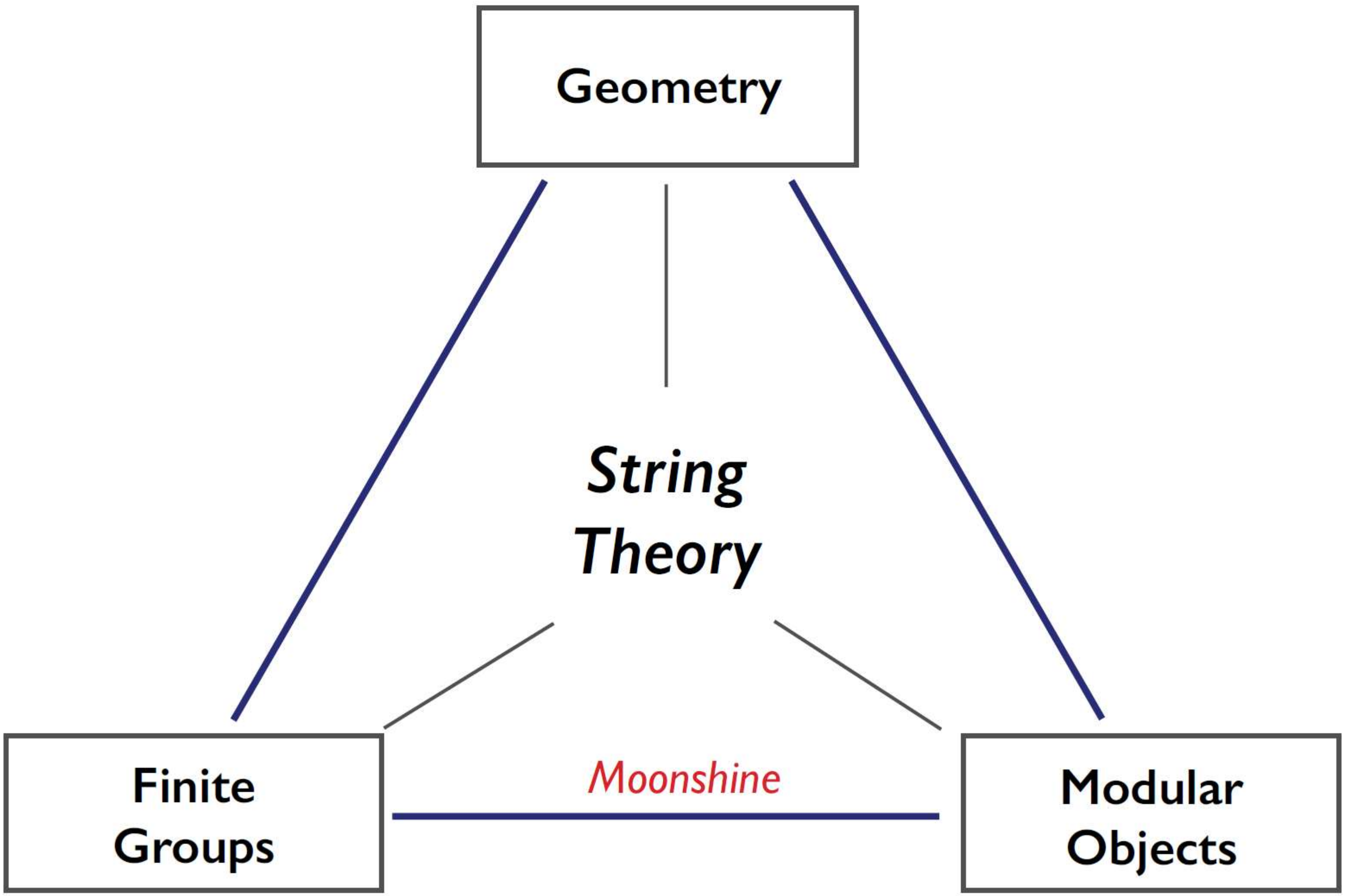
"Mock" Modular Forms

Physical Origin:
superstring theory on *K3 surface.*

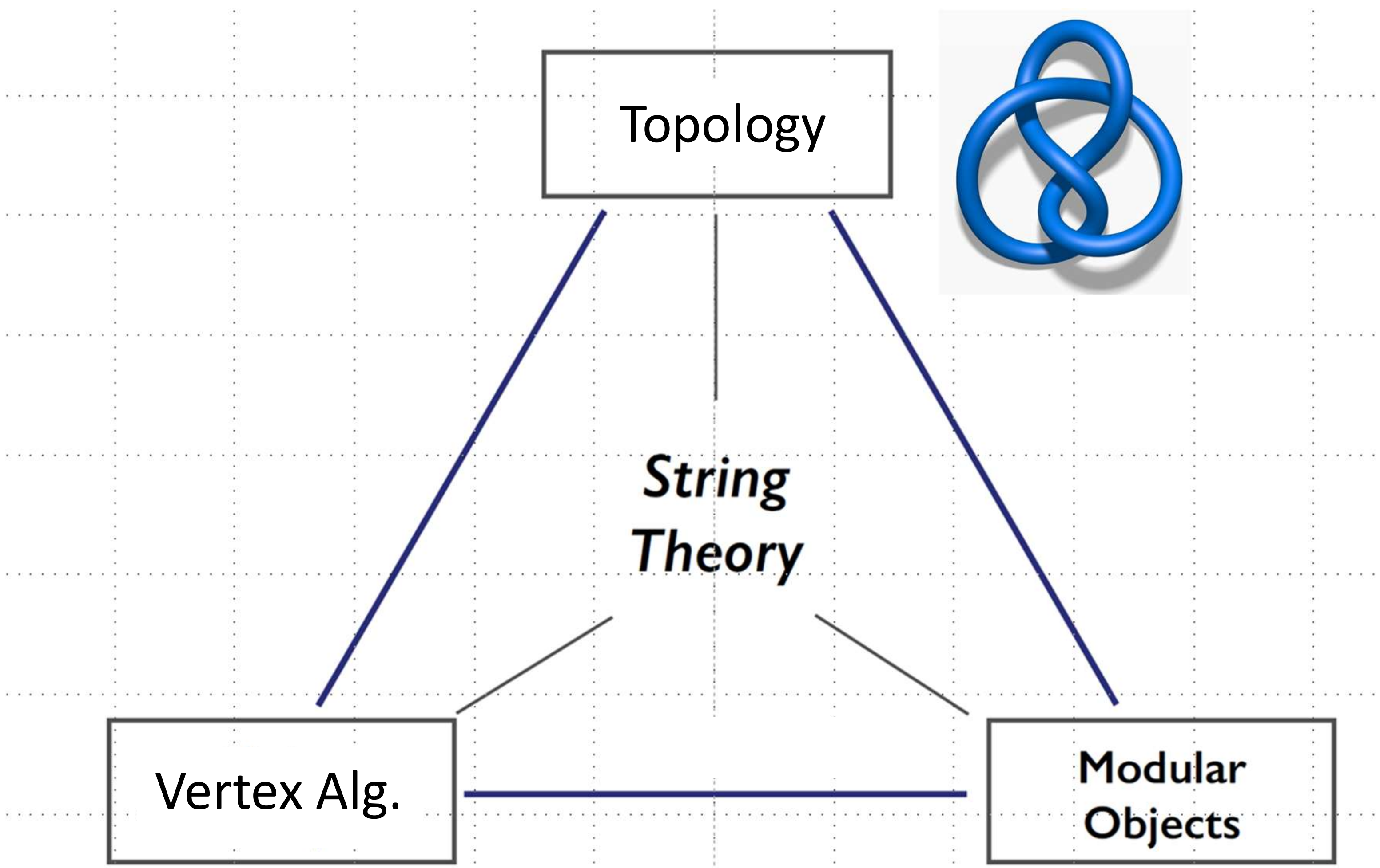
(unique non-trivial CY 2-fold, with great mathematical importance)



A Mysterious Story About

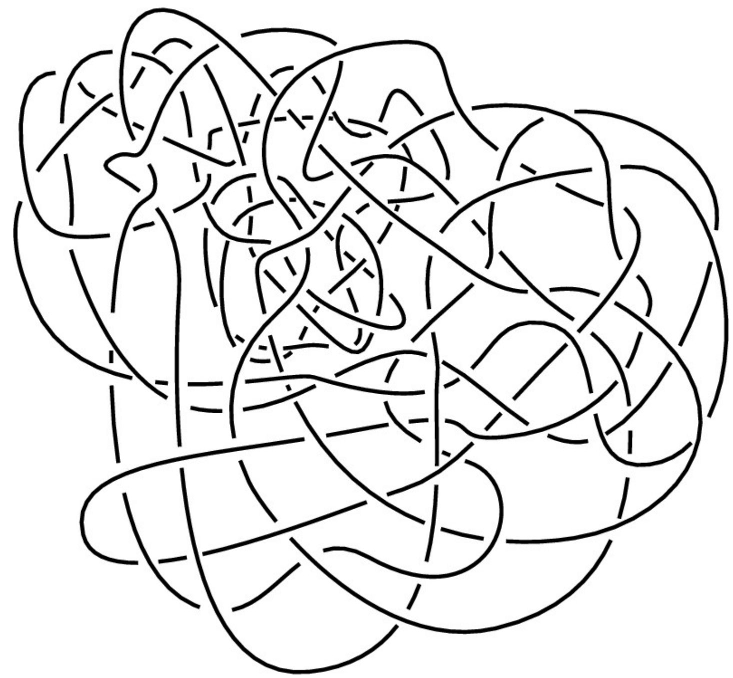


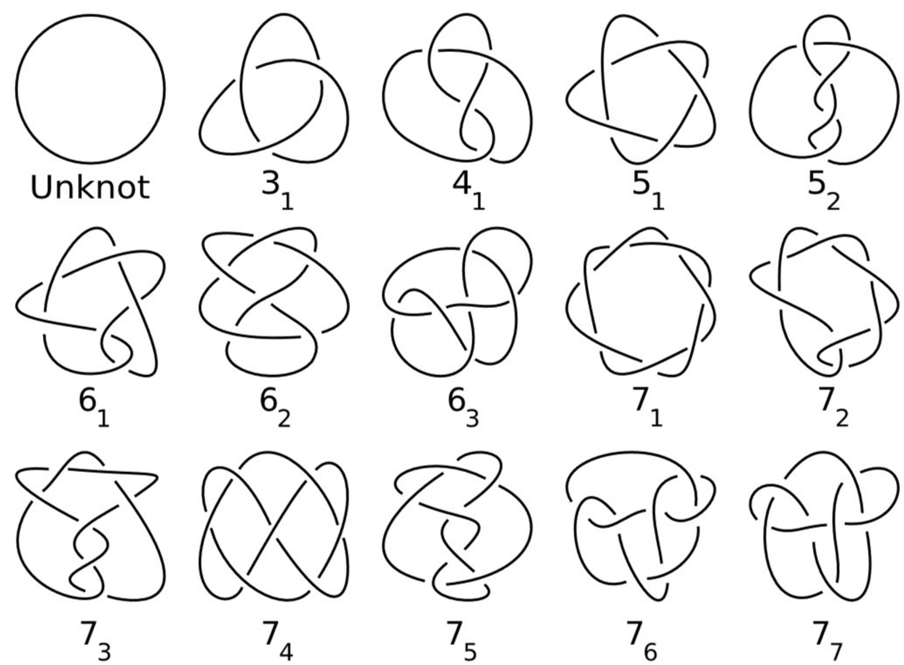
II. Topological Invariants, and Quantum Modular Forms



Q: How to distinguish one knot from another?

A: **Topological Invariants**.



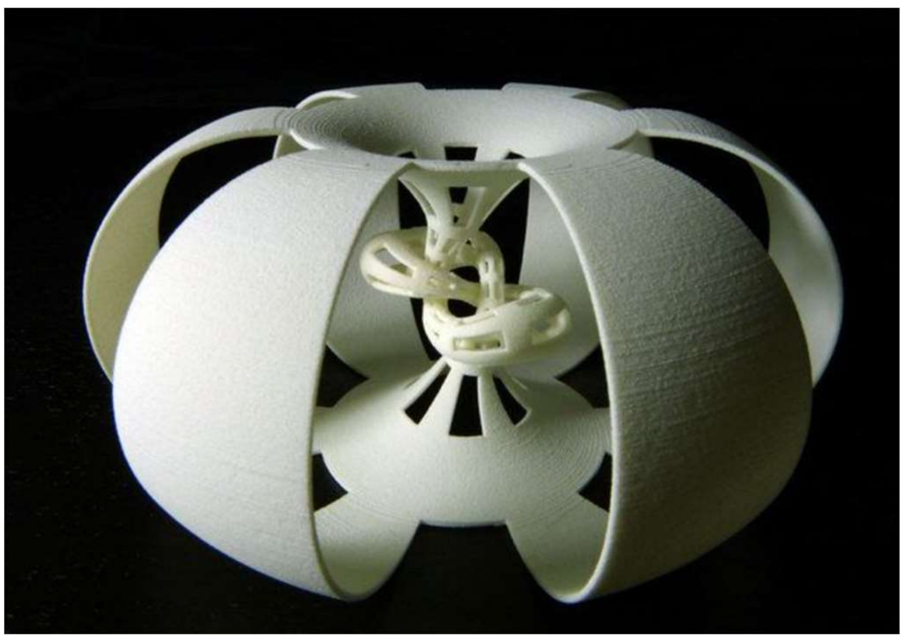


A closely related question:

Q: How to distinguish one 3-manifold from another?

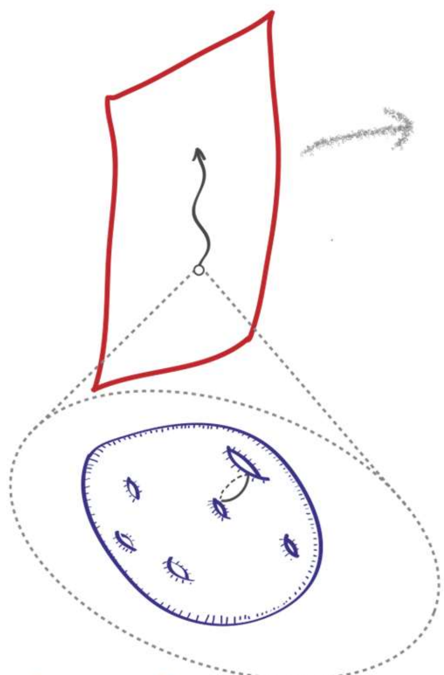
A: **Topological Invariants**.

Knot Complement:= 3-manifold – (the nbhd of a) knot



Can string theory help?

A: Yes. Via compactification.



$M = M_d$

the quantum states protected by supersymmetry do not depend on the detail of M (eg. metric), and are topological inv. of M as a result.

M-theory: non-pert formulation of string theory.
M5 brane: a 6-dim object in M-theory.

6d M5 Brane-theory on $Cigar \times$ time $\times M_3$

3d SQFT $\mathcal{T}[M_3]$

susy indices

susy B.C. $\mathcal{B}_b$

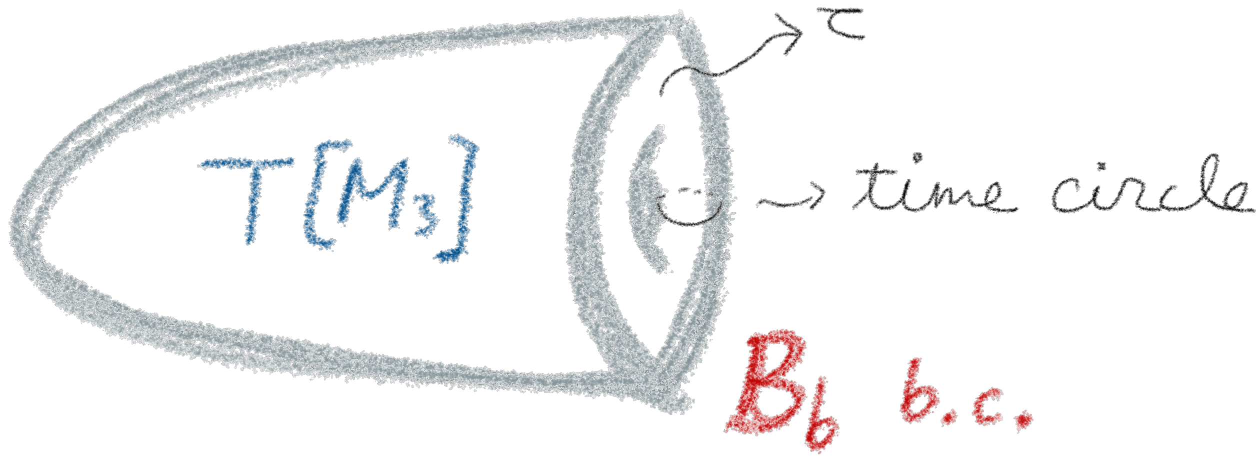
M_3 Topology

top. inv.

Ab. G **flat connections**

[Gukov–Pei–Putrov–Vafa, 17]

From the point of view of the 3d SQFT:



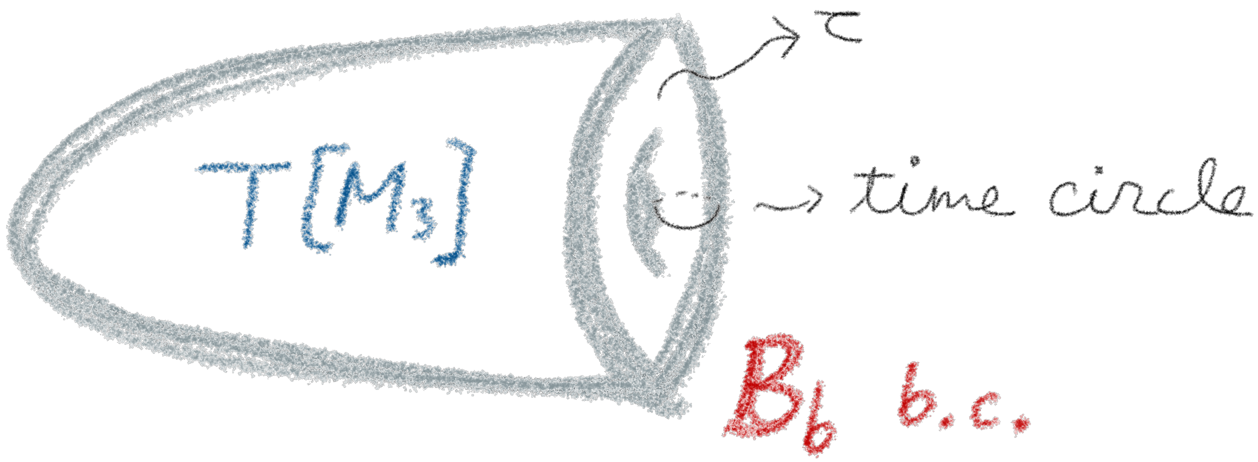
$\hat{Z}_b(\tau; M_3) :=$ susy partition fn on this backgrnd w. B_b b.c.

= "half-index" counting susy states

= Top. inv. of M_3

(inf.) q-series

Modularity from the boundary theory.



$T[M_3]$	$\hat{Z}_b(\tau; M_3)$
trivial	modular
non-trivial but gapped	somewhat modular
not even gapped	??

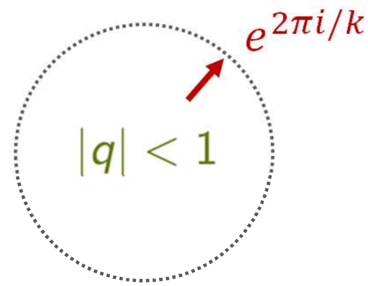
$\widehat{Z}_b(\tau; M_3)$ and $WRT(k; M_3)$

$=$ Chern-Simons partition fn.

$k \in \mathbb{Z}$ level

Question: Can we go from $\mathbb{Z}$ to $\mathbb{H}$, and have an inv. extending WRT?

Idea: q-series $\xrightarrow[q \to e^{2\pi i/k}]{\text{radial limit}} WRT(k)$ $\qquad (*)$



Remarks: 1. cf. previous work by Habiro. 2. $(*)$ is not sufficient to fix the q-series. e.g. "universal" WRT

3-man. inv. : $\widehat{Z}_a(\tau; M_3)$ and $\mathrm{WRT}(k; M_3)$

$$\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\text{radial limit}} \mathrm{WRT}$$

WRT inv.

$$\mathrm{WRT}(M_3; k) = (i\sqrt{2k})^{b_1(M_3)-1} \sum_{a,b \in H_1(M_3,\mathbb{Z})} e^{2\pi i k \mathrm{CS}(a)} \left(\lim_{\tau \to \frac{1}{k}} S_{ab}^{(A)} \widehat{Z}_b(\tau) \right)$$

CS level

3-man. inv. : $\widehat{Z}_a(\tau; M_3)$ and $\mathrm{WRT}(k; M_3)$

$$\widehat{Z}_a(\tau) \xrightarrow[\text{summed over } a]{\text{radial limit}} \mathrm{WRT}$$

WRT inv.

$$\mathrm{WRT}(M_3; \underbrace{k}_{\text{CS level}}) = (i\sqrt{2k})^{b_1(M_3)-1} \sum_{a,b \in H_1(M_3,\mathbb{Z})} e^{2\pi i k \mathrm{CS}(a)} \left(\lim_{\tau \to \frac{1}{k}} S^{(A)}_{ab} \widehat{Z}_b(\tau) \right)$$

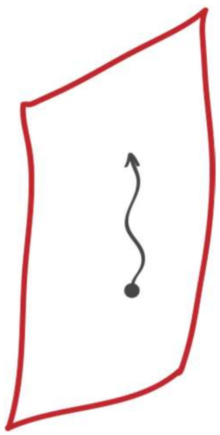
1. Physics gives $\hat{Z}_b(\tau; M_3)$, a more powerful top. inv. than Chern-Simons.

2. The mathematical (quantum) modular properties helps to get new information on physics .

The mathematics involved in string theory … in subtlety and sophistication vastly exceeds previous uses of mathematics in physical theories. … String theory has led to a whole host of amazing results in mathematics in areas that seem far removed from physics.

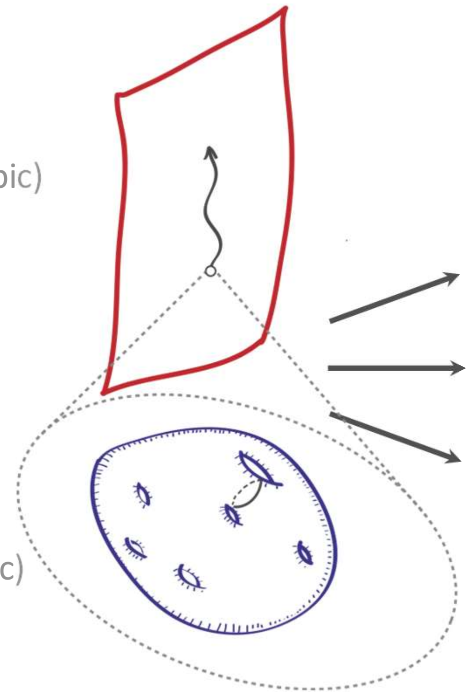
2005, Sir Michael Atiyah

But, how about physics?



world

visible
(macroscopic)
world



invisible
(microscopic)
world

The invisible determines the physics we see!

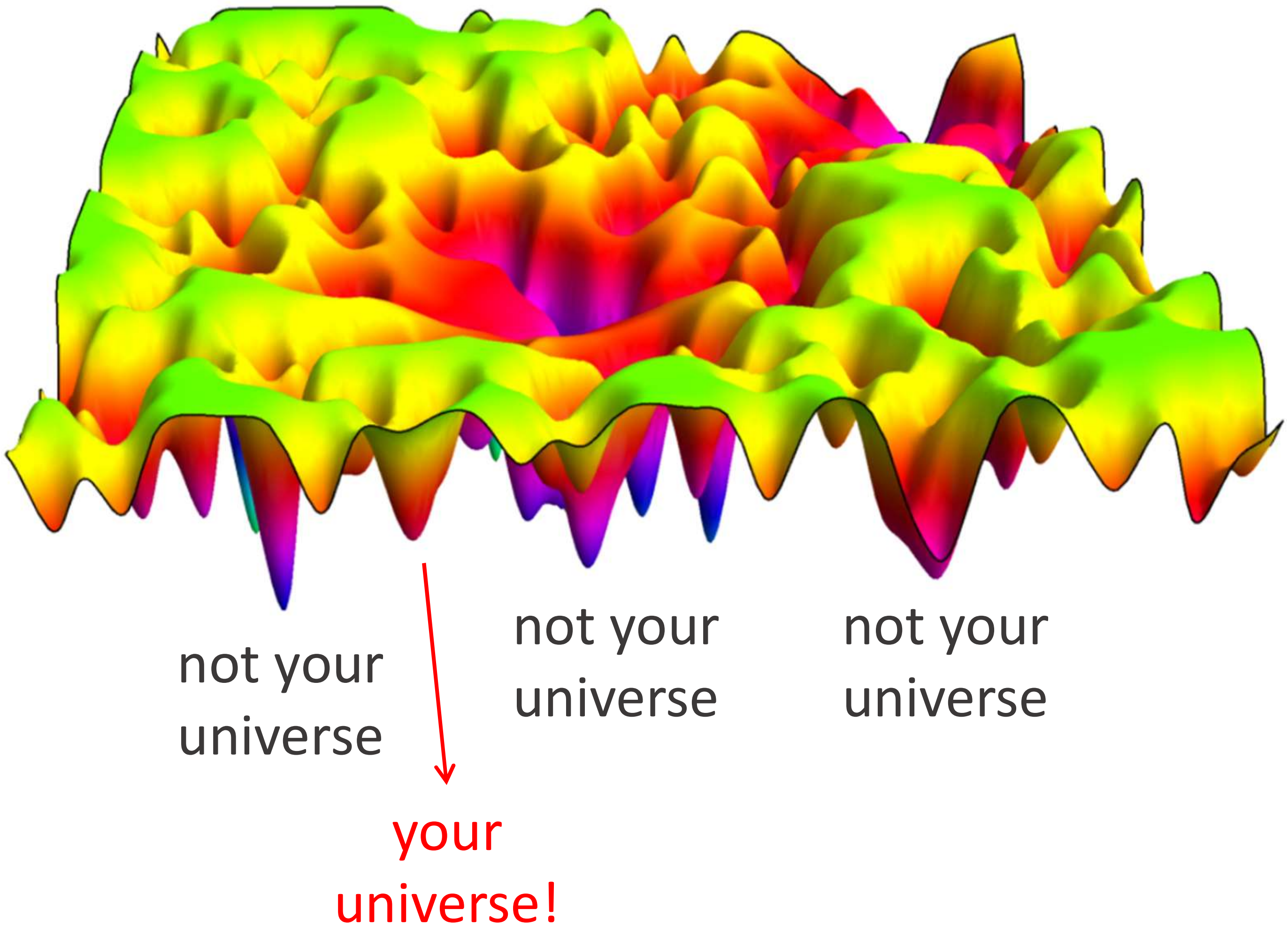
Different choices of the compactification manifolds lead to different “worlds” with different physics.

Which internal manifold is *ours? ? ?*

How many options to choose from?

$\sim 10^{500}$ *?!?!?*

String Theory Landscape



Why? LIVE WHERE YOU CAN (?).

String Theory Landscape:

a blessing that provides the ultimate answer?

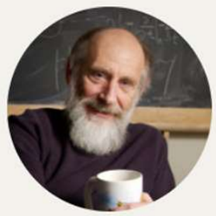
of possible configurations. Unless one can find a reason to reject all but a few of the string theory vacua, we will have to accept that much of what we had hoped to calculate are environmental parameters, like the distance of the earth from the sun, whose values we will never be able to deduce from first principles.

We lose some, and win some. The larger the number of possible values of physical parameters provided by the string landscape, the more string theory legitimates anthropic reasoning as a new basis for physical theories: Any scientists who study nature must live in a part of the landscape where physical parameters take values suitable for the appearance of life and its evolution into scientists.

Living in the multiverse, 2015, **S. Weinberg**

Makes sense! But at the same time it relies on a ***probabilistic*** *view on fundamental physics* and is distinctively nontheistic and non-anthropocentric.

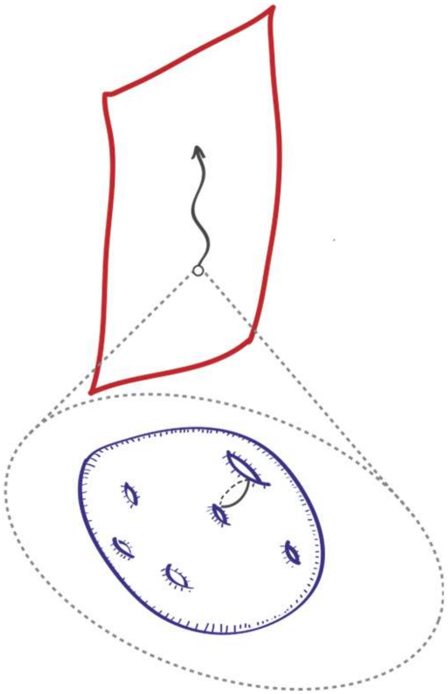
Not feeling comfortable? You are not alone.



"It probably bothered the first human beings who realized that the world was not just their local valley. It probably terrified them a little bit, but by now we're used to the world getting bigger and bigger. The String Theory Landscape just says it's way bigger than we thought."

—LEONARD SUSSKIND

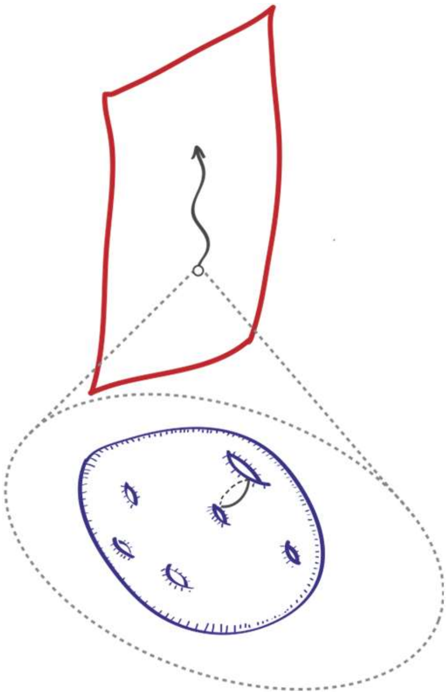
The invisible determines the physics we see!



Q: Can we at least find the exact string theory background that gives *our* universe?

A: A daunting task.
Involving a series of NP-hard and undecidable tasks.

The invisible determines the physics we see!



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In recent years, *artificial intelligence* has been employed to look for a string universe like ours.

at the moment, it's especially good for solving things *approximately* and *probabilistically*.

AI as the new physicist: are we out of job?

MIT Technology Review

Who needs Copernicus if you have machine learning?

It took humanity centuries to decide that Earth orbits the sun. Now a neural network has come to the same conclusion, using the same data, in just a few hours.

Astrophysics > Astrophysics of Galaxies

[Submitted on 16 Oct 2019]

Newton vs the machine: solv networks

Physics > Computational Physics

[Submitted on 27 May 2019 (v1), last revised 15 Ap

AI Feynman: a Physics-I

AI as a new colleague.

So far, AI has been lots of help in astro-, particle, and material physics, among other things.

Machine learning and the physical sciences

arXiv:1903.10563v2 [physics.comp-ph]

AI as a new helper.

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Machine learning and the physical sciences
arXiv:1903.10563v2 [physics.comp-ph]

Some argue that the purpose of physics is to make reliable predictions about the future, given data about the present and the past.

If AI can produce *approximately correct* prediction *almost all the time* (say within 1% error 99.78% of the time), has it then learned the physics?

AI as the new physicist: learn math first*.

Dogs and babies understand that objects fall instead up flying upwards (they act surprised or frightened if objects behave abnormally). In this sense they too can predict the future motion of a moving object and hence “understand physics”. However, I won’t say it is physics unless they show me

$$\boxed{F = m\, a}$$

*: work in progress.

To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ...
If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.

Richard Feynman (1967)

We can eavesdrop on nature not only by paying attention to experiments but also by trying to understand how their results can be explained with the deepest mathematics. You could say that the universe speaks to us in numbers.

Nima Arkani-Hamed (2019)

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Thank You For Your Attention!

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Nima Arkani-Hamed (2019)