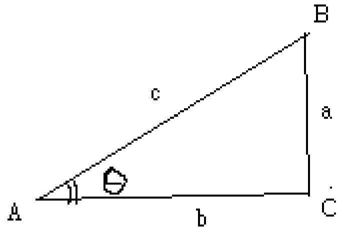
三角函數



$\sin\theta = \dfrac{a}{c}$ $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{a}{b}$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\cos\theta = \dfrac{b}{c}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{b}{a}$ $\csc\theta = \dfrac{1}{\sin\theta}$

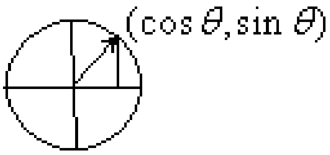
$\cos^2\theta + \sin^2\theta = 1$

$\sin(90-\theta) = \cos\theta$

$\tan(90-\theta) = \cot\theta$

$\sec(90-\theta) = \csc\theta$

廣義三角函數



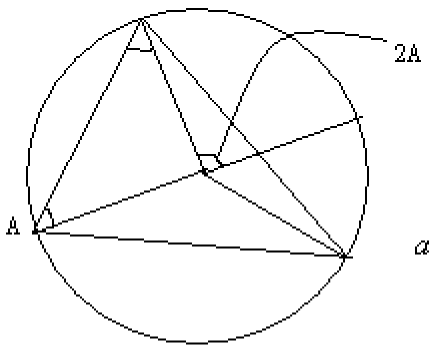
$\sin(180-\theta) = \sin\theta$

$\sin(270+\theta) = \cos\theta$

三角形面積公式和正弦定理

Δ面積 $= \dfrac{1}{2}ab\sin C = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = 2R$



$a = R\sin A + R\sin A$

投影定理及餘弦定理

a=bcosC+ccosB

b=acosC+ccosA

c=acosB+bcosA

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{a(b^2 + c^2 - a^2)}{2abc} \quad \Rightarrow 2bc\cos A = b^2 + c^2 - a^2$$

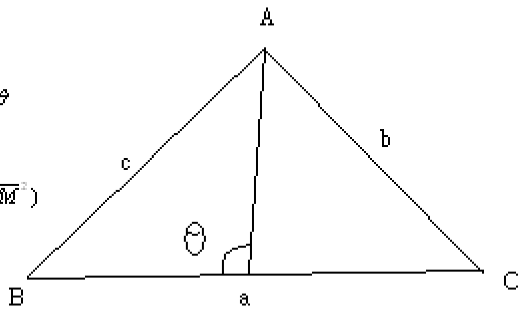
$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \begin{cases} A = 0 & a = |b - c| \\ A = 180^0 & a = b + c \end{cases}$$

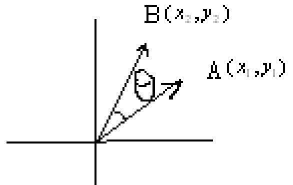
三角形中線定理

$$c^2 = \overline{AM}^2 + \overline{BM}^2 - 2\overline{AM}\ \overline{BM}\cos\theta$$

$$b^2 = \overline{AM}^2 + \overline{CM}^2 + 2\overline{AM}\,\overline{CM}\cos\theta$$

$$\overline{BM} = \overline{CM} \Rightarrow b^2 + c^2 = 2(\overline{AM}^2 + \overline{BM}^2)$$





內積

$$\overrightarrow{OA} = (x_1, y_1) \qquad \overrightarrow{OB} = (x_2, y_2)$$

$$\overrightarrow{OA}\overrightarrow{OB}\cos\theta = \frac{1}{2}\left\{\overline{OA}^2 + \overline{OB}^2 - \overline{AB}^2\right\} = \frac{1}{2}\left\{\left(x_1^2 + y_1^2\right) + \left(x_2^2 + y_2^2\right) - \left[\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2\right]\right\}$$

$$= x_1x_2 + y_1y_2$$

$$\left|\vec{r_1}-\vec{r_2}\right|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$$

$$\begin{cases} 1^0 = \dfrac{2\pi}{360}\text{弧度} \\ \text{弧長}S = r\theta \\ \text{扇形面積}A = \dfrac{1}{2}r^2\theta \end{cases}$$

和角公式 $(x_1, y_1) = r_1(\cos\theta_1, \sin\theta_1)$

$(x_2, y_2) = r_2(\cos\theta_2, \sin\theta_2)$

$$\overline{OA}\,\overline{OB}\cos\theta = r_1r_2\cos(\theta_1-\theta_2) = r_1r_2(\cos\theta_1 cso\theta_2 + \sin\theta_1\sin\theta_2)$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$\sin(\theta_2-\theta_1) = \cos\left[\frac{\pi}{2}-(\theta_2-\theta_1)\right] = \cos\left[\left(\frac{\pi}{2}-\theta_2\right)+\theta_1\right]$$

$$= \cos\left(\frac{\pi}{2}-\theta_2\right)\cos\theta_1 - \sin\left(\frac{\pi}{2}-\theta_2\right)\sin\theta_1 = \sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1$$

$$\sin(\theta_2+\theta_1) = \sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1$$

In sum,

$$\sin(\theta_2+\theta_1) = \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1$$
$$\sin(\theta_2-\theta_1) = \sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1$$
$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$

$$\text{積化和差}\begin{cases} \sin\theta_1\cos\theta_2 = \dfrac{1}{2}[\sin(\theta_1+\theta_2)+\sin(\theta_1-\theta_2)] \\ \cos\theta_1\sin\theta_2 = \dfrac{1}{2}[\sin(\theta_1+\theta_2)-\sin(\theta_1-\theta_2)] \\ \cos\theta_1\cos\theta_2 = \dfrac{1}{2}[\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)] \\ -\sin\theta_1\sin\theta_2 = \dfrac{1}{2}[\cos(\theta_1+\theta_2)-\cos(\theta_1-\theta_2)] \end{cases}$$

$$\text{和差化積}\begin{cases} \sin\alpha+\sin\beta = 2\sin(\dfrac{\alpha+\beta}{2})\cos(\dfrac{\alpha-\beta}{2}) \\ \sin\alpha-\sin\beta = 2\cos(\dfrac{\alpha+\beta}{2})\sin(\dfrac{\alpha-\beta}{2}) \\ \cos\alpha+\cos\beta = 2\cos(\dfrac{\alpha+\beta}{2})\cos(\dfrac{\alpha-\beta}{2}) \\ \cos\alpha-\cos\beta = -2\sin(\dfrac{\alpha+\beta}{2})\sin(\dfrac{\alpha-\beta}{2}) \end{cases}$$

兩倍角

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

半角

$$\cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\tan(\frac{\theta}{2}) = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{1+2\cos^2(\frac{\theta}{2})-1}$$

$$\left\{\begin{aligned} \cos(3\theta) &= \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta) \\ &= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta) \\ &= 2\cos^3\theta - \cos\theta - 2\cos(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta \\ \sin(3\theta) &= 3\sin\theta - 4\sin^3\theta \end{aligned}\right.$$

複數

$$Z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$Z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$Z_1Z_2 = r_1r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_1)\big)$$

$$= r_1r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$$

$$f(\theta) = \cos\theta + i\sin\theta$$

$$f(\theta_1)f(\theta_2) = f(\theta_1+\theta_2)$$

f : 指數函數 $\qquad f(\theta) = e^{i\theta}$

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{n=0}^{\infty}\frac{1}{n!}$$

$$e^x = \lim_{n\to\infty}(1+\frac{1}{n})^{nx} = \lim_{n'\to\infty}(1+\frac{x}{n'})^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{i\theta} = (1-\frac{\theta^2}{2}+\frac{\theta^4}{4!}.............) + i(\theta - \frac{\theta^3}{3!}+\frac{\theta^5}{5!}.........) = \cos\theta + i\sin\theta$$

Taylor expansion

$$\frac{d}{dx}(e^x) = \sum_{n=0}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n'=0}^{\infty}\frac{x^{n'}}{n'!} = e^x$$

$y = e^x \qquad e^y = x \Rightarrow y = g(x) \qquad$ *Inverse function of* exp

$$\frac{dy}{dx}\cdot e^{y} = 1 \Rightarrow (\frac{dy}{dx})\cdot x = 1$$
$$\frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \int \frac{1}{x}dx$$
$$g(x) = \ln x \equiv \log_e(x)$$
$$\ln x = \int_1^x \frac{1}{x'}dx'$$
$$\ln(x_1 \cdot x_2) = \ln(e^{y_1}e^{y_2}) = \ln(e^{(y_1+y_2)}) = y_1 + y_2 = \ln(x_1) + \ln(x_2)$$
$$\log(x) = \log_{10}(x)$$
$$\sum_{k=1}^{n}\cos(k\theta) = \sum_{k=1}^{n}\mathrm{Re}\{e^{ik\theta}\} = \mathrm{Re}\{\frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}}\}$$
$$= \mathrm{Re}\{\frac{e^{i\theta}(1-e^{in\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}\} = \mathrm{Re}\{\frac{e^{i\theta}+1-e^{i(n+1)\theta}+e^{i(n-1)\theta}}{2-2e^{i\theta}}\}$$
$$= \frac{(\cos\theta-1)+\{\cos(n\theta)-\cos((n+1)\theta)\}}{2-2\cos\theta}$$

if n θ =2m π $\Rightarrow \sum_{k=1}^{n}\cos(k\theta)=0$

求極值 y=asinx+bcosx+c

$$=\sqrt{a^2+b^2}\{\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\} + c$$

令 $\cos x_0 = \frac{a}{\sqrt{a^2+b^2}}$ $\sin x_0 = \frac{b}{\sqrt{a^2+b^2}}$

$$y = \sqrt{a^2+b^2}\sin(x+x_0) + c$$
$$-\sqrt{a^2+b^2} + c \le y \le \sqrt{a^2+b^2} + c$$

$$\frac{d}{dx}e^{ix} = ie^{ix} = \frac{d}{dx}(\cos x + i\sin x) = i(\cos x + i\sin x)$$

$$\Rightarrow \begin{cases} \dfrac{d}{dx}\cos x = -\sin x \quad ; \quad \dfrac{d}{dx}\tan x = \dfrac{\cos x}{\cos x} + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} \\ \qquad\qquad \dfrac{d}{dx}\sin x = \cos x \end{cases}$$

$$\begin{cases} \dfrac{d^2}{dx^2}\cos x = -\cos x \\ \dfrac{d^2}{dx^2}\sin x = -\sin x \end{cases}$$

簡諧運動

$$f = ma \qquad -kx = m\ddot{x} \qquad \ddot{x} \equiv \frac{d^2}{dt^2}x \qquad \Rightarrow m\ddot{x} + kx = 0$$

$$k = m\omega^2 \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\ddot{x} + \omega^2 x = 0$$

$$x = A\cos(\omega x) + B\sin(\omega x)$$

A B fixed by initial condition

$$e^{i2\pi} = 1 \qquad e^{i2\pi m} = 1$$

$$if \quad Z^n = 1 \Rightarrow Z = e^{\frac{i2\pi m}{n}}, m = 0,1\ldots.n-1$$

反三角函數

$$x = \sin(y)$$

$$y = \sin^{-1}(x) \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$y = \cos^{-1}(x) \qquad 0 \le y \le \pi$$

$$y = \tan^{-1}(x) \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad ; \quad \cos^{-1}(-x) = \pi - \cos^{-1}x$$

$$\tan^{-1}(-x) = -\tan^{-1}(x) \qquad \sin^{-1}(x) + \cos^{-1}x = \frac{\pi}{2}$$

$$\begin{cases} 1 = \dfrac{dy}{dx}\cos(y) \qquad \dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}} \\ 1 = \dfrac{dy}{dx}\sin(y) \qquad \dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-1}{\sqrt{1-x^2}} \\ 1 = \dfrac{dy}{dx}\dfrac{1}{\cos^2 y} = \dfrac{dy}{dx}(1+x^2) \quad \dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2} \end{cases}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2}$$

$$\cos^2 x = \frac{e^{2ix} + e^{-2ix} + 2}{4} = \frac{2 + 2\cos(2x)}{4} = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{e^{2ix} + e^{-2ix} - 2}{2} = \frac{-2 + 2\cos(2x)}{4} = \frac{-1 + \cos(2x)}{2}$$

$$\cos^3 x = \frac{e^{3ix} + e^{-3ix} + 3e^{ix} + 3e^{-ix}}{8} = \frac{\cos(3x) + 3\cos(x)}{4}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2}$$

$$\cos^2 x = \frac{e^{2ix} + e^{-2ix} + 2}{4} = \frac{2 + 2\cos(2x)}{4} = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{e^{2ix} + e^{-2ix} - 2}{2} = \frac{-2 + 2\cos(2x)}{4} = \frac{-1 + \cos(2x)}{2}$$

$$\cos^3 x = \frac{e^{3ix} + e^{-3ix} + 3e^{ix} + 3e^{-ix}}{8} = \frac{\cos(3x) + 3\cos(x)}{4}$$

週期函數

時間週期

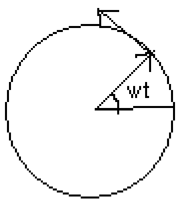
f(t+T)=f(t),T 週期

例如 $\cos(\frac{2\pi}{T}t)$, $\sin(\frac{2\pi}{T}t)$ *角頻率* $\omega = \frac{2\pi}{T}$

圓周運動 $(x(t), y(t)) = (R\cos(\omega t), R\sin(\omega t))$

$$\vec{V} = (\frac{dx}{dt}, \frac{dy}{dt}) = (-R\omega\sin(\omega t), R\omega\cos(\omega t))$$

$$\vec{a} = (\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}) = (-R\omega^2\cos(\omega t), -R\omega^2\sin(\omega t))$$ *方向向內*

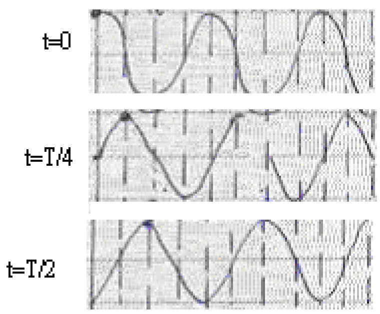


空間週期

f(x+λ)=f(x)， λ 波長

例如 $\cos(\frac{2\pi}{\lambda}x)$ $\sin(\frac{2\pi}{\lambda}x)$ *波數* $k \equiv \frac{2\pi}{\lambda}$

行進波 ： $A_0\cos(kx - \omega t)$



$$kx - \omega t = 0 \Rightarrow x = \frac{\omega}{k}t = Vt \qquad V = \frac{\lambda}{T}$$

指數函數　$a^x\,(a>0)$　以a為底的指數函數

指數律 $\begin{cases} a^x \cdot a^y = a^{(x+y)} \\ (a^x)^y = (a^y)^x \\ a^x b^x = (ab)^x \end{cases}$　$a^{-x} = \dfrac{1}{a^x}$　$a^x / a^y = a^{x-y}$

$a>1$　a^x 遞增函數　$x_1 > x_2 \Rightarrow a^{x_1} > a^{x_2}$

$a<1$　a^x 遞減函數　$x_1 > x_2 \Rightarrow a^{x_1} < a^{x_2}$

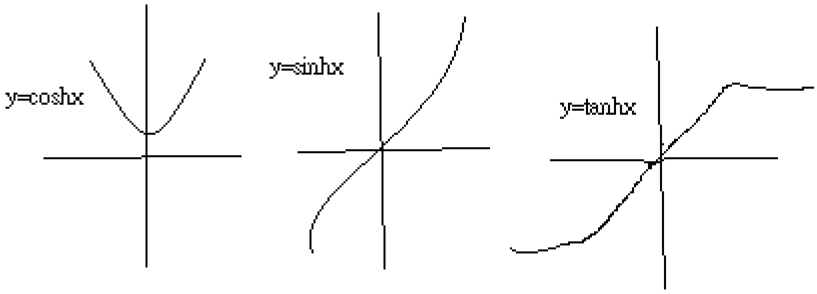
$a^x > 0 \quad -\infty < x < \infty$

$e, e = 2.71828....$

雙曲線三角函數

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$$\begin{cases} \cosh^2 x - \sinh^2 x = 1 \\ 1 - \tanh^2 x = \sec h^2 x \end{cases}$$

$$\begin{cases} \dfrac{d}{dx}\cosh x = \sinh x \\ \dfrac{d}{dx}\sinh x = \cosh x \end{cases} \qquad \begin{cases} \dfrac{d^2}{dx^2}\cosh x = \cosh x \\ \dfrac{d^2}{dx^2}\sinh x = \sinh x \end{cases}$$

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$$

橢圓 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$　雙曲線 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases} \qquad \begin{cases} x = a\cosh\theta \\ y = b\sinh\theta \end{cases}$$

對數函數 ： 指數函數的反函數

若 $y = a^x \Rightarrow \log_a y = x$ *以a為底的對數函數*

$\ln x \equiv \log_e x$ *自然對數*

$\log x \equiv \log_{10} x$ *常用對數*

$\log_a x \quad a > 0 \quad x > 0$

$\log_a a = 1 \quad \log_a 1 = 0$

對數定律

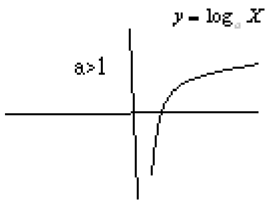
$$\begin{cases} \log_a(x \cdot y) = \log_a x + \log_a y \\ \log_a(x)^y = y \log_a x \end{cases}$$

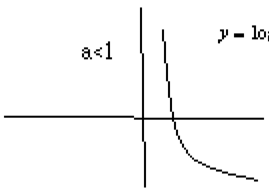
$\log_a \frac{1}{x} = -\log_a x \qquad \log_a(x/y) = \log_a x - \log_a y$

換底公式 $\log_a b = \dfrac{\log_c b}{\log_c a}$ *or* $\log_c b \log_b a = \log_c a$

$a > 1$ $\log_a x$ *遞增函數* $x_2 > x_1$ $\log_a x_2 > \log_a x_1$

$a < 0$ $\log_a x$ *遞減函數* $x_2 > x_1$ $\log_a x_2 < \log_a x_1$





反函數 $y = f(x) \Rightarrow x = g(y)$ *則f與g互為反函數*

$f \circ g(x) = x \qquad g \circ f(x) = x$

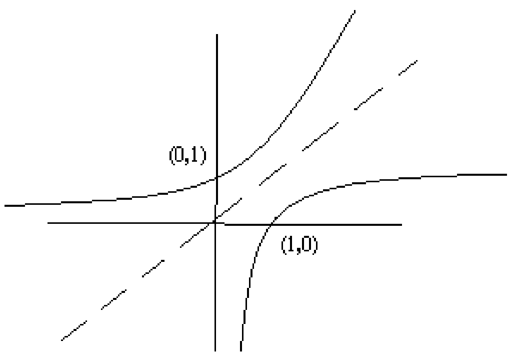
例如 $y = a^x \Rightarrow x = \log_a y$

$f(x) = a^x$ *與* $g(x) = \log_a x$ *互為反函數*

$\log_a a^x = x \qquad a^{\log_a x} = x$

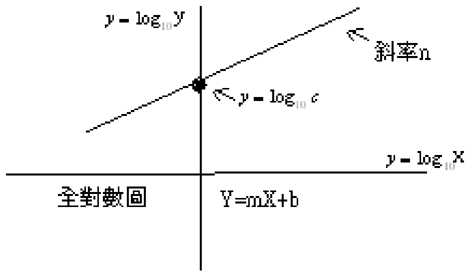
$y = f(x) \qquad y = f^{-1}(x) \Rightarrow x = f(y)$

$(x_0, y_0) \to (y_0, x_0)$



對數的應用

$$\begin{cases}\log_{10} 2 = 0.3010\\ \log_{10} 3 = 0.4771\\ \log_{10} 7 = 0.8451\end{cases}$$



估計 2^{100} *的數值*

$\log_{10} 2^{100} = 100\log_{10} 2 = 30.1 \Rightarrow 2^{100} \sim 10^{30}$

電腦2進位　32位元　　$\log_{10} 2^{32} = 9.652$

$$2^{32} \sim 5\times 10^{9}$$

$$\frac{1}{2^{32}} \sim 2\times 10^{-10}$$

$y = c\cdot x^{n}$

$\log_{10} y = \log_{10} c + n\times \log_{10} x$

內插法 ： 直線近似法

估計 $\log_{10}3.3$

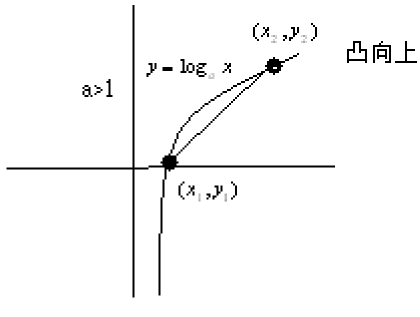
$\log_{10} 3 = 4.771 \quad \log_{10} 4 = 2\log_{10} 2 = 0.6020$

令 $x = 3.3$　y　*未知*　$x_1 = 3 \quad y_1 = 0.4771$

$x_2 = 4 \quad y_2 = 0.6020$

$$\frac{x - x_1}{y - y_1} = \frac{x_2 - x_1}{y_2 - y_1} \Rightarrow \frac{3.3 - 3}{y - 0.4771} = \frac{4 - 3}{0.6020 - 0.4771}$$

$y - 0.4771 = 0.3\times 0.1249 \approx 0.5144$　　*精確值* 0.5185



$x_1, x_2 > 0$

$$\log_a\left(\frac{x_1 + x_2}{2}\right) \geq \frac{1}{2}(\log_a x_1 + \log_a x_2)$$

當等號成立時 $x_1 = x_2$

$$\Rightarrow \frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}$$

算數平均數　　*幾何平均數*

求$x+\dfrac{1}{x}$的極值

$x>0$ $\quad \dfrac{x+\frac{1}{x}}{2} \ge \sqrt{x\cdot\frac{1}{x}} \Rightarrow x+\dfrac{1}{x} \ge 2 \qquad x=1$時為極小

$x<0$ $\quad \dfrac{(-x)+(-\frac{1}{x})}{2} \ge \sqrt{(-x)\cdot(-\frac{1}{x})}$

$-\dfrac{(x+\frac{1}{x})}{2} \ge 1 \Rightarrow x+\dfrac{1}{x} \le -2 \qquad , \ x=1$ 時為極大

$$\ln x = \int_1^x \frac{du}{u}$$

$$\ln(1+x) = \int_0^x \frac{du}{1+u} = \int_0^x \sum_{n=0}^{\infty} (-1)^n u^n du = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{4} ......$$

習題

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} ....$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} .... - 2(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} .....)$$

求$x^3+\dfrac{1}{x^2}$之極值(不用微分)

求 $\cosh x$及 $\tanh x$的的反函

求 $\sinh^{-1} x$及 $\tanh^{-1} x$的的微

$\tanh x = mx$

若$m \ge m_0$則上方程式只有0這個根,求m_0