線性變換：$\begin{cases} \vec{e}_i{}' = S_{ij}\vec{e}_j \\ x_i = x'_j S_j i \end{cases}$ 或 $x_i = (\mathrm{S}^T)ij\ x'_j$

例題 1. 求 $|x+y|+|x-2y| \le 3$

(a)令 $x' = x+y \quad \Rightarrow (x',y') = (x,y)\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$

$y' = x-2y$

$|x'|+|y'| \le 3 \quad \mathrm{S}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \ \mathrm{S} = \begin{pmatrix} \frac{2}{3} & , & \frac{1}{3} \\ \frac{1}{3} & & -\frac{1}{3} \end{pmatrix}$

$A' = |\det(\mathrm{S}^{-1})|\, A$

$\frac{1}{2} \times 6 \times 6 = 3 \cdot A \quad \Rightarrow A = 6$

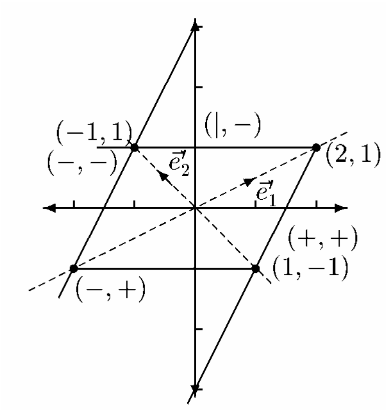
(b) $(+,+): 2x - y = 3$

$(+,-): 3y = 3$

$(-,+): -3y = 3$

$(-,-): -2x + y = 3$

$A = \begin{vmatrix} -3 & , & 0 \\ -1 & & -2 \end{vmatrix} = 6$



矩陣的對角化

$A_{ij} x_j = d_j$

$\mathrm{AX} = \mathrm{D}$

$(\mathrm{SAS}^{-1})(\mathrm{SX}) = (\mathrm{SD})$

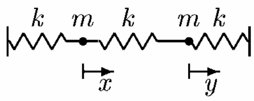
若 $\mathrm{A} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & 0 & \\ & & \ddots & \\ & 0 & & \lambda_n \end{pmatrix}$

為對角矩陣則 $x'_i = \frac{1}{\lambda_i} d'_i$

若向量 $\vec{v}$ 滿足 $\mathrm{A}\vec{v} = \lambda\vec{v}$，則 $\vec{v}$ 為 A 的本徵向量(eigenvector)，而 λ 為其對應的本徵值(eigenvalue)。

特徵方程式 $(\mathrm{A} - \lambda \mathrm{I})\vec{v} = \vec{0}$

$\vec{v} \neq \vec{0} \quad \Rightarrow \det(\mathrm{A} - \lambda \mathrm{I}) = 0$

例題 2. 

$$m\ddot{x} = -kx + k(y-x)$$

$$m\ddot{y} = -k(y-x) - ky$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{-k}{m}\begin{pmatrix} +2 & -1 \\ -1 & +2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathrm{S}\begin{pmatrix} x \\ y \end{pmatrix}$, S 與時間無關 $\begin{pmatrix} \ddot{x}' \\ \ddot{y}' \end{pmatrix} = \mathrm{S}\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \ddot{x}' \\ \ddot{y}' \end{pmatrix} = \frac{-k}{m}\left\{\mathrm{S}\begin{pmatrix} +2 & -1 \\ -1 & +2 \end{pmatrix}\mathrm{S}^{-1}\right\}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

若可將 $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ 對角化成 $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$，則上式可簡化成 $\begin{cases} \ddot{x}' = -\lambda_1 \dfrac{k}{m} x' \\ \ddot{y}' = -\lambda_2 \dfrac{k}{m} y' \end{cases}$

變成兩個完全獨立的簡諧振子，稱為簡正模式(normal mocle)。

特徵方程式：$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1 = 0 \quad \lambda = 1$ 或 $+3$

(a) $\lambda_1 = 1 \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow v_{1x} - v_{1y} = 0$$

$$\overline{v_1} = v_{1x}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

歸一化：$|\overline{v_1}| = 1 \quad \Rightarrow \overline{v_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

(b) $\lambda_2=3\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}v_{2x}\\ v_{2y}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$\Rightarrow v_{2x}+v_{2y}=0$

$\overline{v_2}=v_{2x}\begin{pmatrix}1\\ -1\end{pmatrix}$

歸一化：$|\overline{v_2}|=1 \quad \Rightarrow \overline{v_2}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

(c)求 S $\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}=(1)\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$

$\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}=(3)\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$

$\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$

$\underbrace{\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}^{-1}}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$

$S \quad \Rightarrow S=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$

＊ $S^{-1}=S^{T}$

對稱矩陣的特殊性質：

(a)一定可以被對角化。

(b)本徵值是實數。

(c)對應到不同本徵值的本徵向量彼此垂直。

正交(歸一)座標系(ortho-normal)

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

正交座標系間的變換

$\vec{e}_i{}' = S_{ik}\vec{e}_k$

$\vec{e}_j{}' = S_{jl}\vec{e}_l$

$\vec{e}_i{}' \cdot \vec{e}_j{}' = S_{ik}S_{jl}(\vec{e}_k \cdot \vec{e}_l) = S_{ik}S_{jl}\delta_{kl} = S_{ik}S_{jl} = (SS^T)_{ij}$

若 $\vec{e}_i{}' \cdot \vec{e}_j{}' = \delta_{ij} \quad \Rightarrow SS^T = I$

即 $S^T = S^{-1}$ 則 $S^T S = I$

若 S 滿足上式則 S 稱為正交矩陣(orthogonal matrix)

若 $\det(S) = 1$ 則稱為特殊正交矩陣(special orthogonal)

(a)當 S 為特殊正交矩陣，則新舊座標系之間只差一個旋轉

(b)當 S 為正交秬陣，則向量間的內積在新舊座標系不變

$\vec{x} \cdot \vec{y} = x_i y_i = x_i{}' y_i{}' = \vec{x}' \cdot \vec{y}'$

$$\begin{aligned} x_i y_i &= x_j{}' S_{ji} y_k{}' S_{ki} \\ &= x_j{}' S_{ji} (S^T)_{ik} y_k{}' \\ &= x_j{}' (SS^T)_{jk} y_k{}' \\ &= x_j{}' \delta_{jk} y_k{}' = x_j{}' y_j{}' \end{aligned}$$

＊例題 2 的 S 就是一個正交矩陣

例題 3 求 $\sin\left\{\begin{pmatrix} \frac{3}{2} & 2 \\ 2 & -\frac{3}{2} \end{pmatrix}\pi\right\}$

$$A = \begin{pmatrix} \frac{3}{2} & 2 \\ 2 & -\frac{3}{2} \end{pmatrix}$$

$$\begin{vmatrix} \frac{3}{2}-\lambda & 2 \\ 2 & -\frac{3}{2}-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - \frac{9}{4} - 4 = 0 \quad \lambda = \pm\frac{5}{2}$$

(a) $\lambda_1 = \frac{5}{2} \quad \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\vec{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

(b) $\lambda_2 = -\frac{5}{2} \quad \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\vec{v}_2 = \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad \vec{v}_1 \cdot \vec{v}_2 = 0$$

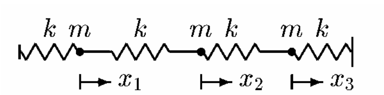
(c) $\mathrm{S}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \Rightarrow \mathrm{S} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

$\mathrm{S}^T = \mathrm{S}^T$

$$\mathrm{S}\sin(\mathrm{A}\pi)\mathrm{S}^{-1} = \begin{pmatrix} \sin(\frac{5\pi}{2}) & 0 \\ 0 & \sin(-\frac{5\pi}{2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sin(\mathrm{A}\pi) = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

習題：



求簡正模式及其頻率

習題：推廣到 N 個粒子

矩陣對角化的其他應用

$$f(\mathrm{A})=\sum_{n=0}^{\infty}a_n\mathrm{A}^n$$

$$\mathrm{S}f(\mathrm{A})\mathrm{S}^{-1}=\sum_{n=0}^{\infty}a_n\mathrm{S}\mathrm{A}^n\mathrm{S}^{-1}=\sum_{n=0}^{\infty}a_n(\mathrm{SAS}^{-1})^n$$

若 $\mathrm{SAS}^{-1}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_n\end{pmatrix}$

$$\Rightarrow \mathrm{S}f(\mathrm{A})\mathrm{S}^{-1}=\begin{pmatrix}f(\lambda_1) & 0\\ 0 & f(\lambda_n)\end{pmatrix}$$

$$f(\mathrm{A})=\mathrm{S}^{-1}\begin{pmatrix}f(\lambda_1) & 0\\ 0 & f(\lambda_n)\end{pmatrix}\mathrm{S}$$

n 維的特殊正交變換 s0(n)，當 $n=2\Rightarrow s0(2)$ 平面上的旋轉

$$\mathrm{S}\cdot\mathrm{S}^T=\mathrm{I}\Rightarrow\begin{pmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{pmatrix}\begin{pmatrix}s_{11} & s_{21}\\ s_{12} & s_{22}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

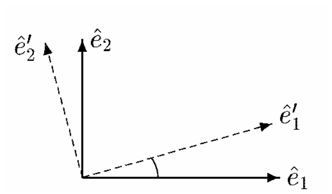
$$\begin{pmatrix}s_{11}^2+s_{12}^2 & s_{11}s_{21}+s_{12}\\ s_{11}s_{21}+s_{12}s_{22} & s_{21}^2+s_{22}^2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\Rightarrow\begin{cases}s_{11}=\cos\theta_1,\ s_{12}=\sin\theta_1\\ s_{21}=\cos\theta_2,\ s_{22}=\sin\theta_2\end{cases}$$

$$s_{11}s_{21}+s_{12}s_{22}=0\quad\Rightarrow\cos(\theta_1-\theta_2)=0$$

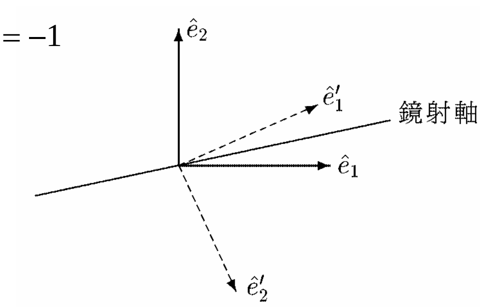
$$\theta_1=\theta,\ \theta_2=\theta\pm\frac{\pi}{2}$$

(a) $\theta_2=\pi+\dfrac{\theta}{2}:\mathrm{S}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, $\det(\mathrm{S})=1$



(b) $\theta_2=\theta-\dfrac{\pi}{2}:\mathrm{S}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$, $\det(\mathrm{S})=-1$

$$\underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{\text{鏡射}}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$



例題 4. 橢圓：$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

雙曲線：$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=\pm 1$

斜橢圓(或雙曲線)：$cx^2 + 2dxy + ey^2 = 1$

$$(x\,,\,y)\underbrace{\begin{pmatrix} c & d \\ d & e \end{pmatrix}}_{\text{A}}\begin{pmatrix} x \\ y \end{pmatrix} = 1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \text{S}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$(x\,,\,y) = (x'\,,\,y')(\text{S}^{-1})^T = (x'\,,\,y')\text{S} \quad (\text{若 } \text{SS}^T = \text{I})$$

$$(x'\,,\,y')(\underset{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}{\underset{\downarrow}{\text{SAS}^{-1}}})\begin{pmatrix} x' \\ y' \end{pmatrix} = 1$$

例：$x^2 + \frac{3}{2}xy + 3y^2 = 1$

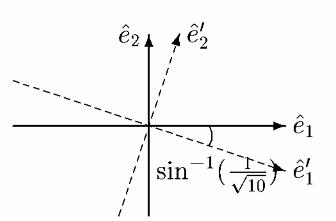
$$\text{A} = \begin{pmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & 3 \end{pmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & \frac{3}{4} \\ \frac{3}{4} & 3-\lambda \end{vmatrix} = 0$$

$\lambda = \frac{3}{4}$ 或 $\frac{13}{4}$

$$\text{(a)}\ x_1 = \frac{3}{4}\begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{9}{4} \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \overline{v_1} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$\text{(b)}\ \lambda_2 = \frac{13}{4}\begin{pmatrix} -\frac{9}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \overline{v_2} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}$$

$$\text{S}^{-1} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \Rightarrow \text{S} = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} \vec{e}_1{}' \\ \vec{e}_2{}' \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$

習題：求$\cos\frac{\pi}{3}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

習題：求$\int_{-\infty}^{\infty} dx \cdot dy \cdot dz \cdot \exp\{-7(x^2 + y^2 + z^2) + 6xy + 8yz\}$