(d)**定義** $\vec{R}=(x\ ,y\ ,z)\ \vec{A_1}=(a_1\ ,b_1\ ,c_1),\ \vec{A_2}=(a_2\ ,b_2\ ,c_2),\ \vec{A_3}=(a_3\ ,b_3\ ,c_3)$

$$\begin{cases}\vec{R}\cdot\vec{A_1}=d_1\\ \vec{R}\cdot\vec{A_2}=d_2\\ \vec{R}\cdot\vec{A_3}=d_3\end{cases}$$

練習：證明 $\vec{B_1}=\dfrac{\vec{A_2}\times\vec{A_3}}{(\vec{A_1}\times\vec{A_2})\cdot\vec{A_3}},\ \vec{B_2}=\dfrac{\vec{A_3}\times\vec{A_1}}{(\vec{A_1}\times\vec{A_2})\cdot\vec{A_3}},\ \vec{B_3}=\dfrac{\vec{A_1}\times\vec{A_2}}{(\vec{A_1}\times\vec{A_2})\cdot\vec{A_3}}$

$$\vec{R}=d_1\vec{B_1}+d_2\vec{B_2}+d_3\vec{B_3}$$

(e)**如何推廣到** n **維？**

※ $\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=\sum_{ij=1}^{2}\in ij(\vec{A_1})i\ (\vec{A_2})j$

※ $\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=\sum_{i\ ,\ j\ ,\ k=1}^{3}\in ijk(\vec{A_1})i(\vec{A_2})\ j(\vec{A_3})k$

n **維列**

$$\begin{array}{c|cccc|}1 & (\vec{A_1})_1, & (\vec{A_1})_2, & \cdots, & (\vec{A_1})_n\\ 2 & (\vec{A_2})_1, & (\vec{A_2})_2, & \cdots, & (\vec{A_2})_n\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ n & (\vec{A_n})_1, & (\vec{A_n})_2, & , & (\vec{A_n})_n\end{array}$$

n **階行列式：** $\sum_{i_1,\ i_2\cdots i_n=1}^{n}\in i_1,\ i_2\cdots i_n(\vec{A_1})i_1\ (\vec{A_2})i_2\cdots(\vec{A_n})i_n$

$\in 1,\ 2,\ 3,\ \cdots,\ n=1$

$$\begin{cases}\in 2,\ 1,\ 3,\ \cdots,\ n=-1\ \Rightarrow \text{奇數次交換}\ \Rightarrow -1\\ \in 2,\ 3,\ 1,\ 4,\ \cdots,\ n=1\ \Rightarrow \text{偶數次交換}\ \Rightarrow +1\end{cases}\qquad \text{跳過}\ (\vec{A_n})$$

$$(\vec{B_p})i_p=\frac{\sum_{i_1,\ i_2\cdots i_{p-1},\ i_{p+1},\ \cdots i_n=1}^{n}\in i_1,\ i_2\cdots i_n(\vec{A_1})_{i_2}(\vec{A_2})_{i_2}\cdots\cdots(\vec{A_n})_{i_n}}{\sum_{i_1,\ i_2\cdots i_n=1}^{n}\in i_1,\ i_2\cdots i_n(\vec{A_1})_{i_1}(\vec{A_2})_{i_2}\cdots(\vec{A_n})i_n}$$

$$\vec{A_p}:\vec{B_q}=d_{pq}$$

定義 $\Delta(\vec{A_1},\ \vec{A_2},\ \cdots,\ \vec{A_n})$ **為** $\vec{A_1},\ \vec{A_2}\cdots,\ \vec{A_n}$ **所構成的行列式**

$$(\overrightarrow{B_1})_1 = \frac{\displaystyle\sum_{i_2,\ i_3,\ i_4\,(2,\ 3,\ 4)} \in_{1\ i_2 i_3 i_4} (\overrightarrow{A_2})_{i_2} (\overrightarrow{A_3})_{i_3} (\overrightarrow{A_4})_{i_4}}{\Delta(\overrightarrow{A_1},\ \overrightarrow{A_2},\ \overrightarrow{A_3},\ \overrightarrow{A_4})}$$

$$(\overrightarrow{B_2})_1 = \frac{\displaystyle\sum_{i_1,\ i_3,\ i_4=(2,\ 3,\ 4)} \in_{i_1,\ 1,\ i_3,\ i_4} (\overrightarrow{A_1})_{i_1} (\overrightarrow{A_3})_{i_3} (\overrightarrow{A_4})_{i_4}}{\Delta(\overrightarrow{A_1},\ \overrightarrow{A_2},\ \overrightarrow{A_3},\ \overrightarrow{A_4})}$$

$$(\overrightarrow{B_1})_2 = \frac{\displaystyle\sum_{i_2,\ i_3,\ i_4=(1,\ 3,\ 4)} \in_{2\ i_2 i_3 i_4} (\overrightarrow{A_2})_{i_2} (\overrightarrow{A_3})_{i_3} (\overrightarrow{A_4})_{i_4}}{\Delta(\overrightarrow{A_1},\ \overrightarrow{A_2},\ \overrightarrow{A_3},\ \overrightarrow{A_4})}$$

$$(\overrightarrow{B_1})_2 = -\begin{vmatrix} (\overrightarrow{A_1})_1 & (\overrightarrow{A_1})_2 & (\overrightarrow{A_1})_3 & (\overrightarrow{A_1})_4 \\ (\overrightarrow{A_2})_1 & (\overrightarrow{A_2})_2 & (\overrightarrow{A_2})_3 & (\overrightarrow{A_2})_4 \\ (\overrightarrow{A_3})_1 & (\overrightarrow{A_3})_2 & (\overrightarrow{A_3})_3 & (\overrightarrow{A_3})_4 \\ (\overrightarrow{A_4})_1 & (\overrightarrow{A_4})_2 & (\overrightarrow{A_4})_3 & (\overrightarrow{A_4})_4 \end{vmatrix} \Big/ \Delta(\overrightarrow{A_1},\ \overrightarrow{A_2},\ \overrightarrow{A_3},\ \overrightarrow{A_4})$$

餘因子：$(-1)^{p+q}$ **刪掉** p **列，** q **行的行列式。**

2. **線性方程組的矩陣表示：**

(a)**二元方程組：**

$$\begin{cases} (\overrightarrow{A_i})_j \equiv a_{ij} \\ x \equiv x_1,\ \ y \equiv x_2 \end{cases} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(b)**三元方程組：**

$$x \equiv x_1,\ \ y \equiv x_2,\ \ z = x_3 \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

(c)n **元方程組：**

$$\begin{pmatrix} a_{11}, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22}, & \cdots, & a_{2n} \\ \vdots & & & \\ a_{n1}, & a_{n2}, & \cdots, & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

左手邊：$\displaystyle\sum_{j=1}^{n} a_{ij} x_j$ ， **右手邊** $di,\ \ i=1,\ 2,\ \cdots,\ n$

3. **矩陣：**

$$A:\ \begin{array}{c} \text{列} \\ (row) \end{array} \begin{array}{c} 1 \\ \downarrow \\ m \end{array} \overset{1 \xrightarrow{\text{行}\ (cdumn)} n}{\begin{pmatrix} A_{11}, & A_{12}, & \cdots, & A_{1n} \\ A_{21}, & A_{22}, & \cdots, & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1}, & A_{m2}, & \cdots, & A_{mn} \end{pmatrix}}$$

$A: m\times n$ **的短陣，** $m=n$ **方陣**

(a)**矩陣的係數積：**

$$CA=\begin{pmatrix} CA_{11}, & CA_{12}, & \cdots, & CA_{1n} \\ CA_{21}, & CA_{22}, & \cdots, & CA_{2n} \\ \vdots & \vdots & & \vdots \\ CA_{m1}, & CA_{m2}, & \cdots, & CA_{mn} \end{pmatrix}$$

(b)**矩陣相加：**A，B **同為** $m\times n$ **的矩陣**

$$A+B=\begin{pmatrix} A_{11}+B_{11}, & A_{12}+B_{12}, & \cdots, & A_{1n}+B_{1n} \\ A_{21}+B_{21}, & A_{22}+B_{22}, & \cdots, & A_{2n}+B_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1}+B_{m1}, & A_{m2}+B_{m2}, & \cdots, & A_{mn}+B_{mn} \end{pmatrix}$$

※**線性組合** $cA+dB$

(c)**矩陣的乘法：**A **為** $l\times m$ **的矩陣，** B **為** $m\times n$ **的矩陣則可定義** $C=AB$ **且** C **為** $l\times n$ **的矩陣** $C_{ij}=\sum_{k=1}^{m} A_{ik}B_{kj}$

例：
$$\begin{pmatrix} 1, & 3, & 0 \\ 2, & -1, & -2 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ -3 & 0 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 5+(-9)+0, & 2+0+0 \\ 10+3+(-2), & 4+0+2 \end{pmatrix}=\begin{pmatrix} -4 & ** \\ 11 & ** \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ -3 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1, & 3, & 0 \\ 2, & -1, & -2 \end{pmatrix}=\begin{pmatrix} 5+4, & 15+(-2), & 0+(-4) \\ -3+0, & -9+0, & 0+0 \\ 1+(-2), & 3+1, & 0+2 \end{pmatrix}$$
$$=\begin{pmatrix} 9, & 13, & ** \\ -3, & -9, & ** \\ -1, & 4, & ** \end{pmatrix}$$

※ $(AB)C=A(BC)$

習題：$A=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$, $B=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$

求 A^2, A^3, A^4, B^2, B^3, B^4

4. **一次方程式的解與反矩陣**：

(a)**矩陣的列運算**：

i. **將矩陣中某一列的每一個元都乘上一個不為零的數。**

ii. **將矩陣中的某一列加到另外一列。**

iii. **將矩陣中的某兩列位置互調。**

(b)**高斯消去法**：

$$\begin{pmatrix}1&2&3\\2&4&5\\3&7&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}8\\11\\9\end{pmatrix}$$

第一列乘 (-2) **加到第二列**：$\left(\begin{array}{ccc:c}1&2&3&8\\0&0&-1&-5\\3&7&-2&9\end{array}\right)$

第一列乘 (-3) **加到第三列**：$\left(\begin{array}{ccc:c}1&2&3&8\\0&0&-1&-5\\0&1&-11&-15\end{array}\right)$

第三列乘 (-2) **加到第一列**：$\left(\begin{array}{ccc:c}1&0&25&38\\0&0&-1&-5\\0&1&-11&-15\end{array}\right)$

第二列乘 (25) **加到第一列；乘** (-11) **加到第三列**：$\left(\begin{array}{ccc:c}1&0&0&-87\\0&0&-1&-5\\0&1&0&40\end{array}\right)$

第二列乘 (-1) **；交換二，三列**：$\left(\begin{array}{ccc:c}1&0&0&-87\\0&1&0&40\\0&0&1&5\end{array}\right)$

(c)**單位矩陣**：$\mathrm{I\!I}A=A$, $A\mathrm{I\!I}=A$ (**任意矩陣**)

$$
\mathrm{II}=\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
$$

(d)**反矩陣：** A^{-1}, $A^{-1}A=\mathrm{II}=A\cdot A^{-1}$

※**若** $AB=C$

$$A^{-1}AB=A^{-1}C$$

$$\Rightarrow B=A^{-1}C$$

※**反矩陣存在的條件** $Det(A)\neq 0$

(e)**用高斯消去法求反矩陣。**

習題： $A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 7 & -2 \end{pmatrix}$, **求** A^{-1}

(f) $(A^{-1})_{ij}=\underbrace{Cofactor}_{\text{餘因子}}(A)_{ji}/Det(A)$

$(-1)^{i+j}\times$**刪掉** j **列** i **行後剩下的行列式值**

習題： $B=\begin{pmatrix} 1 & 3 & -1 \\ 2 & -4 & 5 \\ 4 & 7 & -3 \end{pmatrix}$ **求** B^{-1} (**檢查** $B^{-1}B$)

5. **行列式：**

(a)**行列式的列運算：**

i. **將某一列乘上非零的數並加到另外一列(行列式值不變)**

ii. **從某一列提出公因子** C(**行列式值變成原來的** $\frac{1}{C}$)

iii. **行列互換(行列式值變號)**

$$
\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}
$$

$$
=(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}=(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}
$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 0 & 1\end{vmatrix} = (b-a)(c-a)(c-b)$$

習題：求 $\begin{vmatrix}1 & a & a^2 & a^3\\ 1 & b & b^2 & b^3\\ 1 & c & c^2 & c^3\\ 1 & d & d^2 & d^3\end{vmatrix}$ (Vander monde **行列式**)

(b)**轉置矩陣：** $(A^T)_{ij} = (A)_{ji}$ **行列式值不變**

(c)**行列式的降階：**

可對某一列式某一行降階

$$\sum_{j=1}^{n} A_{ij} \times \text{Cofactor }(A)_{ij}$$

例： $\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} = 1\times\begin{vmatrix}b & b^2\\ c & c^2\end{vmatrix} + (-1)\begin{vmatrix}a & a^2\\ c & c^2\end{vmatrix} + (1)\begin{vmatrix}a & a^2\\ b & b^2\end{vmatrix}$

$$= (bc^2 - b^2c) - (ac^2 - a^2c) + (ab^2 - a^2b)$$

習題： $\begin{vmatrix}a & 1 & 0 & \cdots & \cdots & \cdots & 0\\ 1 & a & 1 & 0 & \cdots & \cdots & 0\\ 0 & 1 & a & \ddots & \ddots & & \vdots\\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \vdots & & \ddots & \ddots & \ddots & 1\\ 0 & 0 & \cdots & \cdots & 0 & 1 & a\end{vmatrix}$

6. **基底：**

(a)**線性獨立：**

$\vec{V}_1,\ \vec{V}_2,\ \cdots,\ \vec{V}_n$ **是一組向量**

若 $\sum_{i=1}^{n} a_1\vec{V}_i = 0$ **唯一的解是** $a_1 = a_2 = \cdots = a_n = 0$

則 $\vec{V}_1,\ \vec{V}_2,\ \cdots\vec{V}_n$ **稱為線獨立**(linear dependent)

反之稱為線相依(linear dependent)

例：$\vec{V_1}=\hat{e}_1+2\hat{e}_2$, $\vec{V_2}=3\hat{e}_1+6\hat{e}_2$，則$\vec{V_1}$, $\vec{V_2}$ 線性相依

$\vec{V_3}=\hat{e}_1-3\hat{e}_2$, $\vec{V_1}$, $\vec{V_3}$ 線性獨立

i. 一個向量時，只有當向量本身是零向量才是線性相依

ii. 兩個向量時，若互相平行，則為線相依。

iii. 三個向量時，若共平面，則為線相依。

iv. 一般而言，若$\vec{V_c}=\sum_{j=1}^{m}V_{ij}\cdot\hat{e}_j$

則V_{ij} 可構成$n\times m$的矩陣，用高斯消去法簡化矩陣 V，若 V 乘下 n 列不

為零$(rank\ n)$則$\vec{V_1}$, ⋯, $\vec{V_n}$ 為線性獨立。否則$(rank<n)$則為線性相依

例 $\vec{V_1}=\hat{e}_1+2\hat{e}_2-\hat{e}_3$

$\vec{V_2}=2\hat{e}_1+\hat{e}_3+3\hat{e}_4$

$\vec{V_3}=-\hat{e}_1+2\hat{e}_2-2\hat{e}_3-3\hat{e}_4$

$$V=\begin{pmatrix}1&2&-1&0\\2&0&1&3\\-1&2&-2&-3\end{pmatrix}\rightarrow\begin{pmatrix}1&2&-1&0\\0&-4&3&3\\0&4&-3&-3\end{pmatrix}$$

$$\rightarrow\begin{pmatrix}1&2&-1&0\\0&-4&3&3\\0&0&0&0\end{pmatrix}rankn\ 2。$$

V. 在 n 維空間最多只有 n 個互相線性獨立的向量

證明：$\sum_{i=1}^{n+1}a_i\vec{V_i}=0$

$$\Rightarrow\begin{pmatrix}V_{11}&V_{12}&\cdots&V_{1n}\\V_{21}&V_{22}&\cdots&V_{2n}\\\vdots&&&\\V_{n+1,\ 1}&V_{n+1,\ 2}&\cdots&V_{n+1,\ n}\end{pmatrix}\begin{pmatrix}a_1\\a_2\\\vdots\\a_{n+1}\end{pmatrix}=0$$

(b)基底：

$\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$構成一組基底的條件。

空間任何一個向是$\vec{X}$ 可以用唯一的一種展開。

i. $\{\vec{e_1}, \cdots, \vec{e_n}\}$ **是線性獨立。**

反言正法。

ii. n **維空間需要** n **個向量才能構成一組基底。**

iii. **若** $\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$ **為一組基底**

$$\vec{x} = \sum_{i=1}^{n} x_i \vec{e_i} = (x_1, \cdots, x_n)\begin{pmatrix} \vec{e_1} \\ \vdots \\ \vec{e_n} \end{pmatrix}$$

(c)**線性(座標)變換：**

$\{\vec{e_1}, \vec{e_2} \cdots, \vec{e_n}\}, \{\vec{e_1}', \vec{e_2}', \cdots, \vec{e_n}'\}$

是新舊兩組基底：

且 $\vec{e_i}' = \sum_{j=1}^{n} S_{ij} \hat{e}_j$

若 $\vec{x} = \sum_{j=1}^{n} x_j \widehat{e_j} = \sum_{i=1}^{n} x'_i \widehat{e_i}'$

$\Rightarrow \sum_{j=1}^{n} x_j \widehat{e_j} = \sum_{i=1}^{n} x_i' \sum_{j=1}^{n} S_{ij} \widehat{e_j} = \sum_{j=1}^{n} \{\sum_{i=1}^{n} x_i' S_{ij}\} \widehat{e_j}$

$\Rightarrow x_j = \sum_{j=1}^{n} x_i' S_{ij}$

即 $\begin{pmatrix} \vec{e_1}' \\ \vec{e_2}' \\ \vdots \\ \vec{e_n}' \end{pmatrix} = S \begin{pmatrix} \vec{e_1} \\ \vec{e_2} \\ \vdots \\ \vec{e_n} \end{pmatrix}$

$(x_1, x_2, \cdots, x_n) = (x_1', x_2', \cdots x_n')S$

※ $\begin{pmatrix} \vec{e_1}' \\ \vec{e_2}' \\ \vdots \\ \vec{e_n}' \end{pmatrix} = S \begin{pmatrix} \vec{e_1} \\ \vec{e_2} \\ \vdots \\ \vec{e_n} \end{pmatrix}; \begin{pmatrix} \vec{e_1}'' \\ \vec{e_2}'' \\ \vdots \\ \vec{e_n}'' \end{pmatrix} = S_2 \begin{pmatrix} \vec{e_1}' \\ \vec{e_2}' \\ \vdots \\ \vec{e_n}' \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \vec{e_1}'' \\ \vec{e_2}'' \\ \vdots \\ \vec{e_n}'' \end{pmatrix} = S_2 S_1 \begin{pmatrix} \vec{e_1}' \\ \vec{e_2}' \\ \vdots \\ \vec{e_n} \end{pmatrix}$

(d)**線性關係(運算子)：** $\vec{y} = A(\vec{x})$

在 $\{\vec{e_1}, \cdots, \vec{e_n}\}$ **基底下** $y_j \vec{e_j} = A(x_j \vec{e_j})$

$$y_j \vec{e_j} = x_i A(\vec{e_i})$$
$$y_j \vec{e_j} = x_i A_{ij} \vec{e_j}$$
$$\Rightarrow y_j = x_i A_{ij}$$

在 $\{\vec{e_1}', \cdots \vec{e_n}'\}$ **基底下** $y'_j \vec{e_j}' = A(x'_i \vec{e'_i})$

$\Rightarrow y'_j = x'_i A'_{ij}$

$(y_1, \cdots, y_n)S = (x_1, \cdots, x_n)S\ A$

$(y_1, \cdots y_n) = (x_1 \cdots, x_n)\ S\ A\ S^{-1}$

$\Rightarrow A' = S\ A\ S^{-1}$(**相似變換**)

$\det(S) \neq 0$

※ $A(\vec{e_i}) = A_{ij} \vec{e_j}$