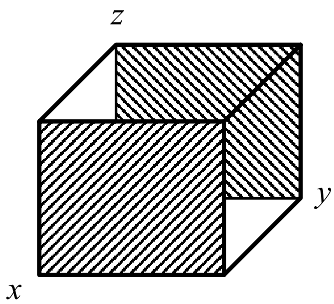
Gauss定理與Stokes定理(續)

●Gauss定理(另一種講法)

考慮中心點為$\vec{r}=(x,y,z)$的小長方體(邊長為$\Delta x,\Delta y,\Delta z$)，流速場為$\vec{v}(\vec{r})$。流過前面的流量為



$$
\begin{aligned}
\Delta\phi_{\text{前}} &\simeq \vec{v}\left(x+\frac{\Delta x}{2},y,z\right)\cdot\hat{n}\Delta A \\
&= v_x\left(x+\frac{\Delta x}{2},y,z\right)\Delta y\Delta z \\
&\simeq \left(v_x(\vec{r})+\frac{\partial v_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z
\end{aligned}
$$

流過後面的量為

$$
\begin{aligned}
\Delta\phi_{\text{後}} &\simeq \vec{v}\left(x-\frac{\Delta x}{2},y,z\right)\cdot\hat{n}\Delta A \\
&= -v_x\left(x-\frac{\Delta x}{2},y,z\right)\Delta y\Delta z \\
&\simeq -\left(v_x(\vec{r})-\frac{\partial v_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z
\end{aligned}
$$

$$
\therefore \Delta\phi_{\text{前}}+\Delta\phi_{\text{後}}\simeq\frac{\partial v_x}{\partial x}\Delta x\Delta y\Delta z \tag{1}
$$

同理，流過左右兩面的流量為

$$
\Delta\phi_{\text{左}}+\Delta\phi_{\text{右}}\simeq\frac{\partial v_y}{\partial y}\Delta x\Delta y\Delta z \tag{2}
$$

流過上下兩面的流量為

$$
\Delta\phi_{\text{上}}+\Delta\phi_{\text{下}}\simeq\frac{\partial v_z}{\partial z}\Delta x\Delta y\Delta z \tag{3}
$$

∴流過小長方體的淨流量為

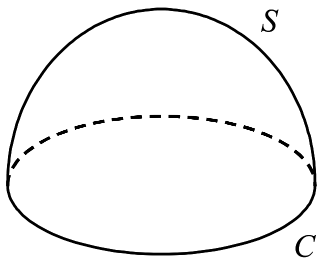
$$
\begin{aligned}
\Delta\phi &= (1)+(2)+(3) \\
&= \nabla\cdot\vec{v}\Delta x\Delta y\Delta z
\end{aligned}
$$

一曲面S包圍立體T時，可將T切割為無數的小長方體，則通過S的淨流量

$$\begin{aligned}\phi &= \iint\limits_S \vec{v}\cdot d\vec{A}\\ &= \lim_{\Delta x\Delta y\Delta z\to 0}\sum_{\text{小方塊}}\nabla\cdot\vec{v}\Delta x\Delta y\Delta z\\ &= \iiint\limits_T \nabla\cdot\vec{v}dxdydz\end{aligned}$$

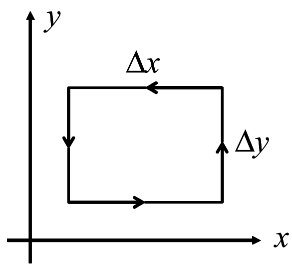
●Stokes定理

S為一平滑曲面，其邊界為C，則環流量



$$\oint\limits_C \vec{v}\cdot d\vec{r} = \iint\limits_S \nabla\times\vec{v}\cdot d\vec{A}$$

說明：只考慮較簡單的情況。S為x–y平面上的小長方塊，中心為$(x,y,0)$，則



$$\oint\limits_C \vec{v}\cdot d\vec{r} = \oint\limits_C v_x dx + \oint\limits_C v_y dy$$

$$\begin{aligned}\oint\limits_C v_x dx &\simeq v_x\left(x, y-\frac{\Delta y}{2}, 0\right)\Delta x - v_x\left(x, y+\frac{\Delta y}{2}, 0\right)\Delta x\\ &\simeq \left(v_x - \frac{\partial v_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(v_x + \frac{\partial v_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x\\ &= -\frac{\partial v_x}{\partial y}\Delta y\Delta x \qquad (4)\end{aligned}$$

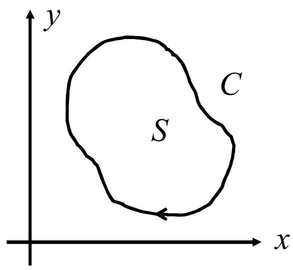
同理

$$\begin{aligned}\oint\limits_C v_x dx &\simeq v_y\left(x+\frac{\Delta x}{2}, y, 0\right)\Delta y - v_y\left(x-\frac{\Delta x}{2}, y, 0\right)\Delta y\\ &\simeq \frac{\partial v_y}{\partial x}\Delta x\Delta y \qquad (5)\end{aligned}$$

$$\begin{aligned}&\therefore \oint\limits_C \vec{v}\cdot d\vec{r} = (4)+(5)\\ &= \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\Delta x\Delta y\\ &= \nabla\times\vec{v}\cdot\hat{z}\Delta x\Delta y \qquad (6)\end{aligned}$$

當S為x-y平面上的任意區域時候，可將它切割為無數小方塊，則

$$\begin{aligned}\oint_C \vec{v}\cdot d\vec{r} &= \lim_{\Delta x\Delta y\Delta z\to 0}\sum_{\text{小方塊}} \nabla\times\vec{v}\cdot\hat{z}\Delta x\Delta y \\ &= \iint_S \nabla\times\vec{v}\cdot d\vec{A}\end{aligned}$$



Note: (6)式中，若小方塊法向量$\hat{n}$不在z方向，則$\oint_C \vec{v}\cdot d\vec{r} = \nabla\times\vec{v}\cdot\hat{n}dA$仍成立。一般曲面$S$切割成無數小方塊後仍可得到Stokes定理。

●Maxwell方程式

積分形式(MKS制)

1.
$$\iint_S \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

Q為曲面S內的總電荷。

2.
$$\iint_S \vec{B}\cdot d\vec{A} = 0$$

3. 安培定律

$$\oint_C \vec{B}\cdot d\vec{r} = \mu_0 I + \overbrace{\varepsilon_0\mu_0\frac{\partial\phi_E}{\partial t}}^{\text{位移電流}}$$

I為穿過迴路C的電流，ϕ_E為穿過迴路C的電通量。

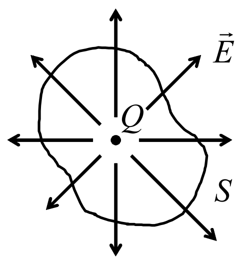
4. 法拉第定律

$$\oint_C \vec{E}\cdot d\vec{r} = -\frac{\partial\phi}{\partial t}$$

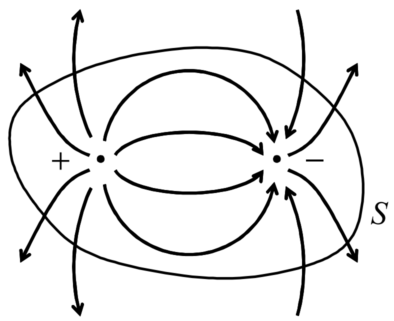
ϕ為穿過迴路C的磁通量。

說明

1.



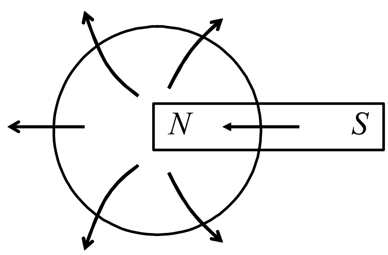
扭曲S不會改變電通量。



電通量為零。

2. 任何封閉曲面的磁通量為零。

3.



不存在磁單極。

微分形式

1.

$$\iint\limits_S \vec{E}\cdot d\vec{A} \overset{\text{Gauss定理}}{=} \iiint\limits_T \nabla\cdot\vec{E}d\tau = \frac{1}{\varepsilon_0}\iiint\limits_T \rho\left(\vec{r}\right)d\tau$$

T的體積$\to 0$時可得

$$\nabla\cdot\vec{E} = \frac{\rho\left(\vec{r}\right)}{\varepsilon_0}$$

2. 同理

$$\nabla \cdot \vec{B} = 0$$

3.

$$\oint\limits_C \vec{B} \cdot d\vec{r} \overset{\text{Stokes定理}}{=} \iint\limits_S \nabla \times \vec{B} \cdot d\vec{A} = \mu_0 \iint\limits_S \vec{J} \cdot d\vec{A} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \iint\limits_S \vec{E} \cdot d\vec{A}$$

S的面積$\rightarrow 0$時可得

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

4.

$$\begin{aligned} \oint\limits_C \vec{E} \cdot d\vec{r} &= \iint\limits_S \nabla \times \vec{E} \cdot d\vec{A} = -\frac{\partial}{\partial t} \iint\limits_S \vec{B} \cdot d\vec{A} \\ \therefore \quad \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}$$

自由空間裡的電磁場$\left(\rho = 0, \vec{J} = 0\right)$

$$\begin{aligned} \nabla \times (4) \quad \rightarrow \quad \nabla \times \nabla \times \vec{E} &= -\frac{\partial}{\partial t} \nabla \times \vec{B} \\ &\overset{(3)}{=} -\varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \end{aligned}$$

$$\because \quad \nabla \times \nabla \times \vec{E} = \nabla \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$\therefore \quad \nabla^2 \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \text{(電場波動方程)}$$