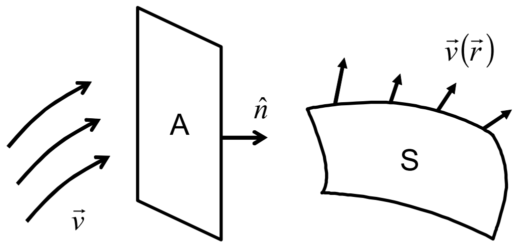
Gauss定理

●流量(flux)



均勻速度場$\vec{v}$，流過面積$A\hat{n} \equiv \vec{A}$（有向面積）的流量ϕ定義為

$$\begin{aligned}\phi &= \vec{v}\cdot\hat{n}A \\ &= v_{\perp}A\end{aligned}$$

單位$[\phi]$ =m^3/sec（每秒鐘流過A的量）
Note: 若$\hat{n}\to-\hat{n}$，則流量ϕ變號
若流速$\vec{v}\left(\vec{r}\right)$與$\vec{r}$有關，則通過$dA$的流量

$$d\phi = \vec{v}\left(\vec{r}\right)\cdot\hat{n}\left(\vec{r}\right)dA$$

通過曲面S的總流量為

$$\phi = \iint\limits_{S}\vec{v}\cdot\hat{n}dA$$

若S為$z=f\left(x,y\right)$，則$\vec{n}=\left(-\dfrac{\partial f}{\partial x},-\dfrac{\partial f}{\partial y},+1\right)$

$$\begin{aligned}\hat{n} &= \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}}\left(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},+1\right) \\ dA &= \sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}dxdy\end{aligned}$$

$$\therefore\ \ \phi = \iint\limits_{\Omega}\left(-v_x\frac{\partial f}{\partial x}-v_y\frac{\partial f}{\partial y}+v_z\right)dxdy$$

（Ω為S投影在x-y平面的區域）

Note: $\hat{n}$與$+z$軸的夾角為γ，則$\cos\gamma = \dfrac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}}$，$\therefore dA = \sec\gamma dxdy$

Example 1 速度場$\vec{v}(x,y,z)=x\hat{i}+y\hat{j}+z\hat{k}$，曲面$S$為$z=1-(x^2+y^2), z\geq 0$，計算流量$\iint\limits_S \vec{v}\cdot\hat{n}dA$，($\hat{n}$指向上)

Solution:

$$f(x,y)=1-\left(x^2+y^2\right)$$

$$\frac{\partial f}{\partial x}=-2x,\quad \frac{\partial f}{\partial y}=-2y$$

$$\begin{aligned}\therefore\quad \iint\limits_S \vec{v}\cdot\hat{n}dA &= \iint\limits_\Omega \left(-v_1\frac{\partial f}{\partial x}-v_2\frac{\partial f}{\partial y}+v_3\right)dxdy\\ &= \iint\limits_\Omega \left(1+x^2+y^2\right)dxdy=\cdots=\frac{3}{2}\pi\end{aligned}$$

●散度(divergence)與旋度(curl)

向量場$\vec{v}(x,y,z)$的散度定義為

$$\mathrm{div}\vec{v}=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

旋度定義為

$$\mathrm{curl}\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}+\cdots$$

令$\nabla\equiv\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$，則也可寫

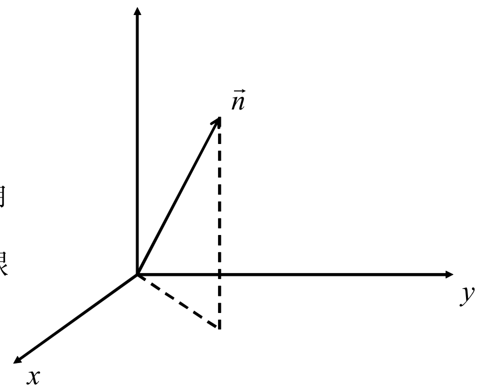
$$\begin{aligned}\mathrm{div}\vec{v} &= \nabla\cdot\vec{v}\\ \mathrm{curl}\vec{v} &= \nabla\times\vec{v}\end{aligned}$$

●Gauss定理(散度定理)

$$\iint\limits_S \vec{v}\cdot\hat{n}dA=\iiint\limits_T \nabla\cdot\vec{v}dxdydz$$

其中S為一平滑的封閉曲面，S所圍出的立體為T，法線$\hat{n}$指向外

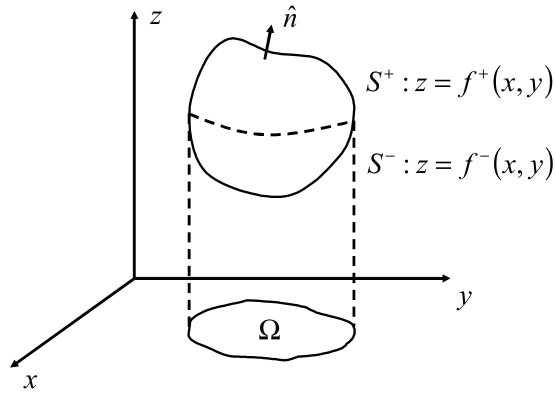
pf:



$$\hat{n} = (\cos\alpha_1, \cos\alpha_2, \cos\alpha_3)$$
$$\alpha_1, \alpha_2, \alpha_3\text{分別為}\hat{n}\text{與}x, y, z\text{軸的夾角}$$

為了簡化，假設平行於座標軸的任何直線與S相交都不超過 2 點，則可以證明

$$\iint_S v_1 \cos\alpha_1 dA = \iiint_T \frac{\partial v_1}{\partial x} dxdydz$$
$$\iint_S v_2 \cos\alpha_2 dA = \iiint_T \frac{\partial v_2}{\partial y} dxdydz$$
$$\iint_S v_3 \cos\alpha_3 dA = \iiint_T \frac{\partial v_3}{\partial z} dxdydz$$



以下只證第三式
立體T為所有滿足

$$f^-(x,y) \le z \le f^+(x,y) \quad (x,y) \in \Omega$$

的點(x,y,z)的集合

$$\begin{aligned}\iiint_T \frac{\partial v_3}{\partial z} dxdydz &= \iint_\Omega \left(\int_{f^-}^{f^+} \frac{\partial v_3}{\partial z} dz \right) dxdy \\ &= \iint_\Omega \left[v_3\left(x,y,f^+\right) - v_3\left(x,y,f^-\right) \right] dxdy \qquad (1)\end{aligned}$$

$$v_3 dxdy = v_3 \cos\alpha_3 dA$$
$$\therefore (1) = \iint_{S^+} v_3 \cos\alpha_3 dA + \iint_{S^-} v_3 \cos\alpha_3 dA = \iint_S v_3 \cos\alpha_3 dA$$

Note: 若T為$\vec{r}_0$點旁很小的區域，則$\vec{v}$流出S的流量

$$\phi = \iint_S \vec{v}\cdot\hat{n} dA = \iiint_T \nabla\cdot\vec{v} dxdydz \simeq \nabla\cdot\vec{v}\left(\vec{r}_0\right) \times T\text{的體積}V_T$$

$$\therefore \nabla\cdot\vec{v}\left(\vec{r}\right) = \lim_{V_T\to 0} \frac{\iint_S \vec{v}\cdot\hat{n} dA}{V_T}$$

$\nabla\cdot\vec{v} > 0$的點稱為源點(Source)
$\nabla\cdot\vec{v} < 0$的點稱為匯點(Sink)
$\nabla\cdot\vec{v} = 0$的點沒有淨流量