# 梯度與極值

## 1 連鎖律

- 單變數：

$$\frac{d}{dt}f\left(g\left(t\right)\right)=\frac{df}{dg}\frac{dg\left(t\right)}{dt}$$

- 多變數（單參數）：

$$\frac{d}{dt}f\left(\vec{r}\left(t\right)\right)\equiv\lim_{\Delta t\to 0}\frac{f\left(\vec{r}\left(t+\Delta t\right)\right)-f\left(\vec{r}\left(t\right)\right)}{\Delta t}=?$$

假設$f\left(\vec{r}\right)$可微分，則

$$f\left(\vec{r}+\Delta\vec{r}\right)=f\left(\vec{r}\right)+\nabla f\cdot\Delta\vec{r}+O\left(\vec{h}\right)$$

$$\therefore\frac{d}{dt}f\left(\vec{r}\left(t\right)\right)=\lim_{\Delta t\to 0}\nabla f\cdot\frac{\Delta\vec{r}}{\Delta t}=\nabla f\cdot\vec{r}'\left(t\right)$$

或寫為

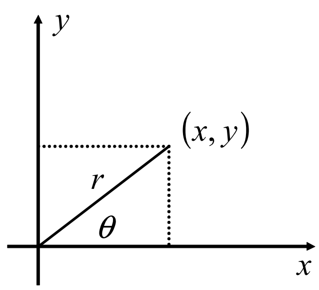
$$\frac{d}{dt}f\left(\vec{r}\left(t\right)\right)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$

- 多變數（多參數）：

$f\left(x,y\right)$, 若$x=x\left(s,t\right), y=y\left(s,t\right)$

$$\text{則}\ \frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

可類推到更多變數或參數



Exercise 1

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \text{函數為} f(x,y)$$

證明

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

pf: $s, t \to r, \theta$

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y} \end{aligned}$$

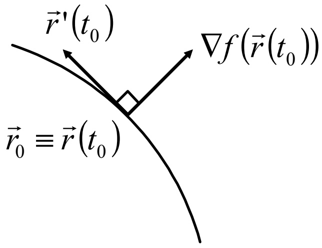
代入上式可証得

# 2 切線與切面

- 平面曲線方程式：$y = f(x)$ 或 $f(x,y) = c$ （$c$為常數）e.g. $x^2 + y^2 = c^2$

也可寫為參數式$\vec{r}(t) = (x(t), y(y))$

- 對$\forall t, f(x,y)$都是常數$c$



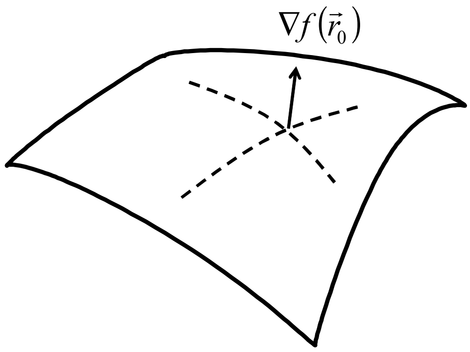

$$\therefore \frac{df(\vec{r}(t))}{dt} = \nabla f \cdot \underbrace{\vec{r}'(t)}_{\text{切線}} = 0$$

$\nabla f \perp$切線，$\therefore$為法線方向

$\therefore$通過$\vec{r}_0$點的切線方程式為

$$(\vec{r}(t) - \vec{r}_0) \cdot \nabla f(\vec{r}_0) = 0 \qquad \text{（法線式）}$$

- 曲面方程式：$z = f(x,y)$或$f(x,y,z) = c$



令$\vec{r}(t)=(x(t),y(t),z(t))$為曲面上通過$\vec{r}_0$點的某一曲線
對$\forall t, f(x,y,z)$都是常數$c$

$$\therefore \frac{df}{dt}=\nabla f\cdot\vec{r}'(t)=0$$

上式對任意通過$\vec{r}_0$的曲線都適用（$\vec{r}'(t_0)$為切線）
$\therefore \nabla f(\vec{r}_0)$為曲面在$\vec{r}_0$點的法線向量
→通過$\vec{r}_0$點的切面方程式為

$$(\vec{r}(t)-\vec{r}_0)\cdot\nabla f(\vec{r}_0)=0.$$

Example 2 求曲面

$$z=\frac{x^2}{2a}+\frac{y^2}{2b}$$

在$(x_0,y_0,z_0)$點的切面方程式

Sol'n:

$$f(x,y,z)\equiv\frac{x^2}{2a}+\frac{y^2}{2b}-z=0$$

$$\nabla f=\left(\frac{x}{a},\frac{y}{b},-1\right)$$

$$\to(x-x_0)\frac{x_0}{a}+(y-y_0)\frac{y_0}{b}-(z-z_0)=0$$

## 3 極大與極小

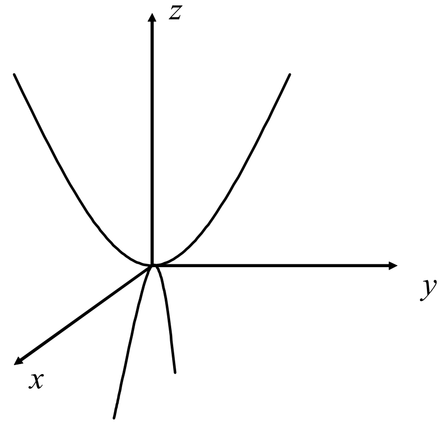
若$f$在$\vec{r}_0$可微分且在$\vec{r}_0$有一區域極值，則$\nabla f(\vec{r}_0)=\vec{0}$（反之不然）

pf: 考慮$z=f(x,y)$（3 變數同理可証）
由於$f(\vec{r}_0)$為極值點，沿$x,y$方向曲線的變化率應有

$$\left.\frac{\partial f(x,y_0)}{\partial x}\right|_{x=x_0}=0,\ \left.\frac{\partial f(x_0,y)}{\partial y}\right|_{y=y_0}=0,\ \therefore\nabla f(\vec{r}_0)=\vec{0}$$

Note: 若$\nabla f(\vec{r}_0)=\vec{0}$，則$\vec{r}_0$稱為靜止點（stationary point）. 它可能為極值點，也可能為鞍點（saddle point）

鞍點：




# 4 二變數Taylor展開式

$f(x,y)$對$(a,b)$點作Taylor展開
　引入參數$t$, 令

$$\begin{cases} x = a + ut \\ y = b + vt \end{cases}$$ （$u,v$視為常數，沿某一方向展開）

然後展開單變數函數$g(t) \equiv f(a+ut, b+vt)$

$$\begin{aligned} g(0) &= f(a,b) \\ g(1) &= f(a+u, b+v) \end{aligned}$$

$$g(1) = g(0) + g'(0) + \frac{1}{2!}g''(0) + ... \quad (1)$$

其中

$$\begin{aligned} g'(t) &= \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} \quad \text{（見１.連鎖律）} \\ &= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) f(a+ut, b+vt) \end{aligned}$$

$$g'(0) = u\frac{\partial f(a,b)}{\partial x} + v\frac{\partial f(a,b)}{\partial y}$$

$$\begin{aligned} g''(t) &= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) g'(t) \\ &= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^2 f(a+ut, b+vt) \end{aligned}$$

依此類推後，代入(1)式可得

$$
\begin{aligned}
f(a+u,b+v) &= f(a,b)+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)f(a,b) \\
&\quad +\frac{1}{2!}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)^2 f(a,b)+...
\end{aligned}
$$

或

$$
\begin{aligned}
f(x,y) &= f(a,b)+\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a,b) \\
&\quad +\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)+...
\end{aligned}
$$

# 5 極值的二階偏導數測試

單變數：$f'(x_0)=0$

若$f''(x_0) > 0$，則$x_0$為極小值點
若$f''(x_0) < 0$，則$x_0$為極大值點
若$f''(x_0) = 0$，則需進一步分析

二變數：$\nabla f(\vec{r}_0)=\vec{0}$ ，然後?

利用Taylor展式，在$(a,b)$附近

$$
\begin{aligned}
f(x,y) &\simeq f(a,b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) \\
&= f(a,b)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x-a)^2+\frac{\partial^2 f}{\partial x\partial y}(x-a)(y-b)+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(y-b)^2
\end{aligned}
$$

後三項形如

$$
\begin{aligned}
Ax^2+2Bxy+Cy^2 &\equiv h(x,y) \\
&= \frac{1}{A}\left(A^2x^2+2ABxy+ACy^2\right) \\
&= \frac{1}{A}\left[(Ax+By)^2+\left(AC-B^2\right)y^2\right]
\end{aligned}
$$

1. 若$A>0$，且$\overbrace{\begin{vmatrix} A & B \\ B & C \end{vmatrix}}^{\text{Hesse行列式}}>0$，則$h(x,y)>0,\forall x,y$（鄰近的）
2. 若$A<0$，且$\begin{vmatrix} A & B \\ B & C \end{vmatrix}>0$，則$h(x,y)<0,\forall x,y$（鄰近的）

3. 若$\begin{vmatrix} A & B \\ B & C \end{vmatrix} < 0$，則$(0,0)$為鞍點（證明略）

4. 若$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = 0$，則需進一步判定（e. g.，$f(x,y) = y^2 - x^3$）

令

$$A = \frac{\partial^2 f(a,b)}{\partial x^2}, B = \frac{\partial^2 f(a,b)}{\partial x \partial y}, C = \frac{\partial^2 f(a,b)}{\partial y^2}$$

則由以上方式可判定$f(a,b)$的極值性質.

hw 1 假設由$f(x,y) = 0$可得出$y(x)$，且$y(x)$為一可微分函數. 證明若$\frac{\partial f}{\partial y} \neq 0$，則

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$$

hw 2 證明由三個座標平面及曲面$xyz = a^3$的任一個切平面所形成的4面體皆有相同的體積. 此體積是多少？