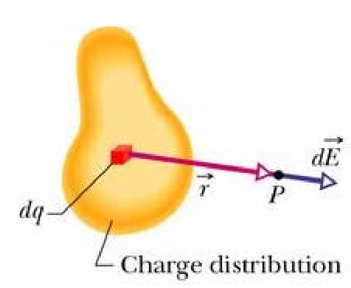
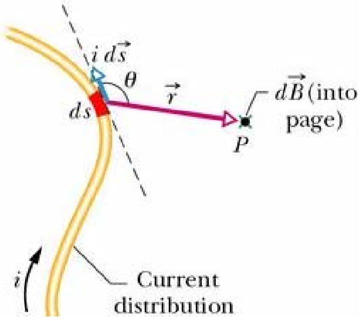
VII. Magnetic Fields due to Currents (電流)

1. Magnetic field (B) due to currents vs. electric field due to charges





$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}$$

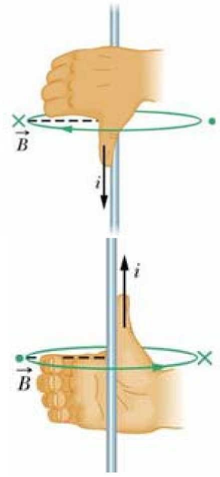
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^3}\vec{r} \quad \text{(Coulomb law)}$$

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{s}\times\vec{r}}{r^3} \quad \text{(Biot − Savart law)}$$

The permeability constant $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T\cdot m/A}$

2. B due to a current in a long straight line



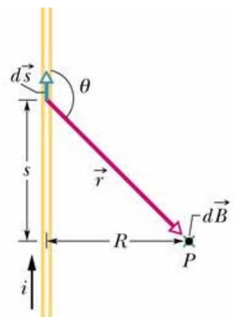
Right-Hand Rules

$$B = 2\int_0^{\infty} dB = \frac{\mu_0 i}{2\pi}\int_0^{\infty}\frac{\sin\theta\, ds}{r^2}$$

$$r = \sqrt{s^2 + R^2}$$

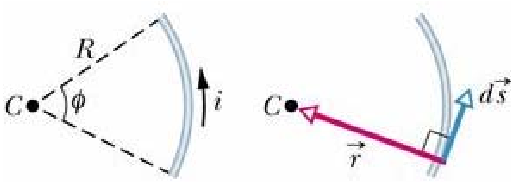
$$\sin\theta = \sin(\pi - \theta) = \frac{R}{\sqrt{s^2+R^2}}$$

$$B = \frac{\mu_0 i}{2\pi}\int_0^{\infty}\frac{R\,ds}{(s^2+R^2)^{3/2}} = \frac{\mu_0 i}{2\pi R}\left[\frac{s}{(s^2+R^2)^{1/2}}\right]_0^{\infty} = \frac{\mu_0 i}{2\pi R}$$



So $$B = \frac{\mu_0 i}{4\pi R} \quad \text{(semi − infinite straight wire)}$$

3. B due to a current in a circular arc



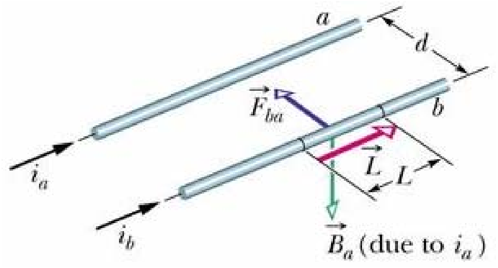
$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 90^\circ}{R^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{R^2}$$

$$B = \int dB = \int_0^{\phi}\frac{\mu_0}{4\pi}\frac{iR\,d\phi}{R^2} = \frac{\mu_0 i}{4\pi R}\int_0^{\phi} d\phi.$$

$$B = \frac{\mu_0 i\phi}{4\pi R} \quad \text{(at center of circular arc).}$$

So $$B = \frac{\mu_0 i(2\pi)}{4\pi R} = \frac{\mu_0 i}{2R} \quad \text{(at center of full circle).}$$

4. Force between two parallel currents



The magnitude of $\vec{B}_a$ at every point of wire b is $B_a = \frac{\mu_0 i_a}{2\pi d}$.
$\vec{F}_{ba}$ on a length L of wire b due to the external magnetic field $\vec{B}_a$ is

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a,$$

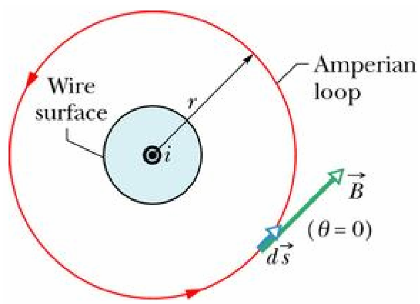
So $$F_{ba} = i_b L B_a \sin 90° = \frac{\mu_0 L i_a i_b}{2\pi d}$$

Note: ▸Parallel currents attract, and antiparallel currents repel.

5. Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} \quad \text{(Ampere's law)}$$

Example 1: **B** *outside of a current in a straight line*

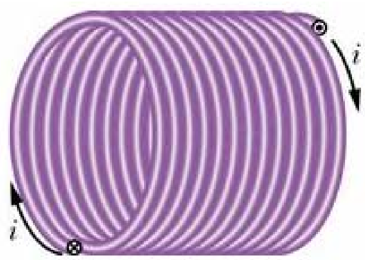


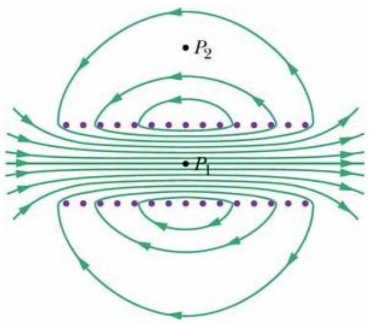
$$\oint \vec{B} \cdot d\vec{s} = \oint B\cos\theta\, ds = B \oint ds = B(2\pi r)$$

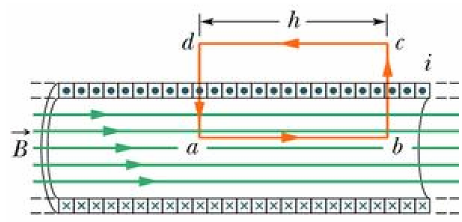
$$B(2\pi r) = \mu_0 i$$

So $$B = \frac{\mu_0 i}{2\pi r}.$$ ✓

6. Solenoids and Toroids







Solenoid

apply Ampere's law,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

So $$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \underbrace{\int_b^c \vec{B} \cdot d\vec{s}}_{0} + \underbrace{\int_c^d \vec{B} \cdot d\vec{s}}_{0} + \underbrace{\int_d^a \vec{B} \cdot d\vec{s}}_{0}$$

And thus

$$i_{\text{enc}} = i(nh).$$

$$Bh = \mu_0 inh$$

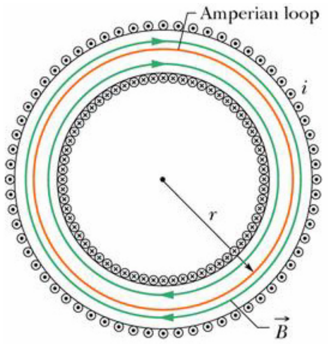
$$B = \mu_0 in \quad \text{(ideal solenoid).}$$

where n is the number of turns per unit length.

Toroid

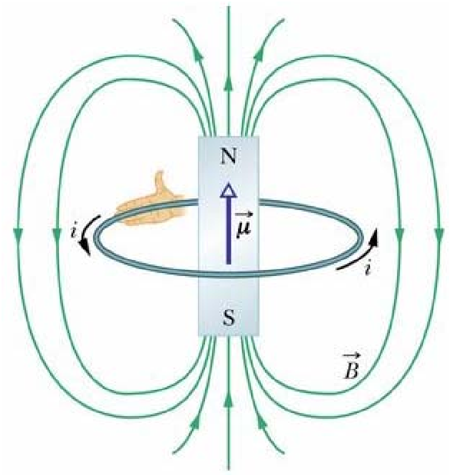
$$(B)(2\pi r) = \mu_0 iN,$$

$$B = \frac{\mu_0 iN}{2\pi}\frac{1}{r} \quad \text{(toroid).}$$



where N is the total number of turns.

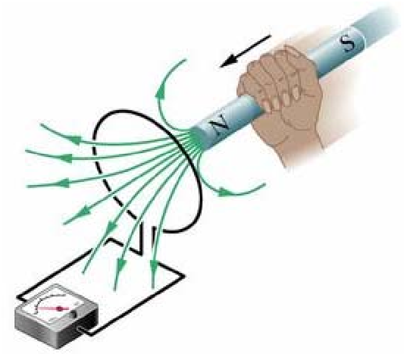
7. A current-carrying coil as a magnetic dipole



A current loop produces a magnetic field like that of a bar magnet and thus has associated north and south poles. The magnetic dipole moment $\vec{\mu}$ of the loop, given by a curled – straight right-hand rule, points from the south pole to the north pole, in the direction of the field $\vec{B}$ within the loop.

VIII. Magnetic Inductions (磁感)

1. Faraday's law of induction



An *emf* is induced in the loop when the number of magnetic field lines that pass through the loop is changing.

The current produced in the loop is called an **induced current;** the work done per unit charge to produce that current (to move the conduction electrons that constitute the current) is called an **induced emf;** and the process of producing the current and emf is called **induction.**

Magnetic flux: $\Phi_B = \int \vec{B} \cdot d\vec{A}$ (magnetic flux through area A).

Unit: $1 \text{ weber} = 1 \text{ Wb} = 1 \text{ T} \cdot \text{m}^2$

▸The magnitude of the emf $\mathscr{E}$ induced in a conducting loop is equal to the rate at which the magnetic flux Φ_B through that loop changes with time.

$$\mathscr{E} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)},$$

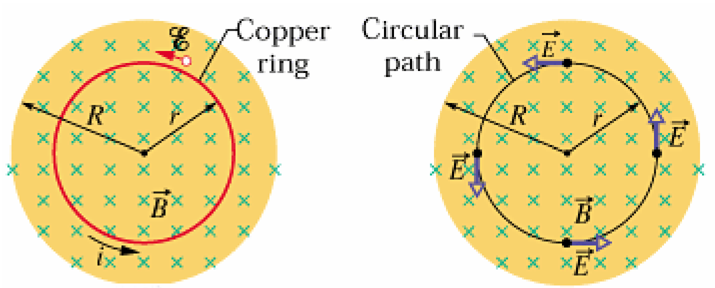
$$\mathscr{E} = -N\frac{d\Phi_B}{dt} \quad \text{(coil of } N \text{ turns)}.$$

2. Lenz's law

Soon after Faraday proposed his law of induction, Lenz devised a rule for determining the *direction* of an induced current in a loop.

▸An induced current has a direction such that the magnetic field due to *the current* opposes the change in the magnetic flux that induces the current.

3. Induced electric field – a restatement of Faraday's law



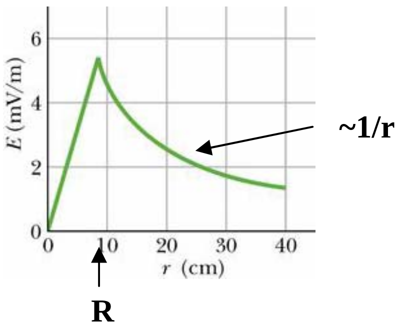
$$\mathscr{E} = \oint \vec{E} \cdot d\vec{s}.$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)}$$

If there is a current in the Cu ring, an electric field must be present along the ring; an *electric field* is needed to do the work of moving the conduction electrons. A changing magnetic field produces an electric field.

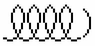
So

Exercise: Based on the 2nd figure above, show that



4. Inductors and Inductance

A **capacitor** (symbol ) can be used to produce a desired electric field.

While an **inductor** (symbol ) can be used to produce a desired magnetic field.

Define "**Inductance *L***":

$$L = \frac{N\Phi_B}{i}$$

, where Φ_B is the magnetic flux, i is the current, and N is the number of turns.

Unit of ***L***: 1 henry = 1H =1 T m^2/A

Exercise: Inductance of a solenoid is

$$L = \frac{N\Phi_B}{i} = \frac{(nl)(BA)}{i} = \frac{(nl)(\mu_0 in)(A)}{i} = \mu_0 n^2 lA.$$

where l is the total length of the solenoid.

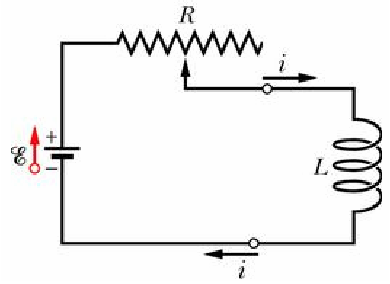
5. Self induction (自感)

If two coils—which we can now call inductors—are near each other, a current i in one coil produces a magnetic flux Φ_B through the second coil. We have seen that if we change this flux by changing the current, an induced emf appears in the second coil according to Faraday's law. An induced emf appears in the first coil as well.

However,

➤An induced emf $\mathscr{E}_L$ appears in any coil in which the current is changing.

This process is called **"self-induction",** and the emf that appears is called a **self-induced emf.**



From

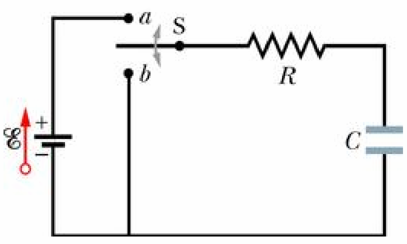
$$N\Phi_B = Li.$$

$$\mathscr{E}_L = -\frac{d(N\Phi_B)}{dt}.$$

So $\mathscr{E}_L = -L\frac{di}{dt}$ (self − induced emf)

6. *RL* and *RC* Circuits

First, consider an *RC* circuit for charging process (switch to point *a*)



Loop rule clockwise (starting from point *a*):

$$\mathscr{E} - iR - \frac{q}{C} = 0$$

(note potential difference $V_C (= q/C)$ across the capacitor.)

Because $i = \frac{dq}{dt}$ and thus $R\frac{dq}{dt} + \frac{q}{C} = \mathscr{E}$ (charging equation).

So $q = C\mathscr{E}\left(1 - e^{-t/RC}\right)$ (charging a capacitor). and $i = \frac{dq}{dt} = \left(\frac{\mathscr{E}}{R}\right)e^{-t/RC}$ (charging a capacitor).

Also $V_C = \frac{q}{C} = \mathscr{E}\left(1 - e^{-t/RC}\right)$ (charging a capacitor).

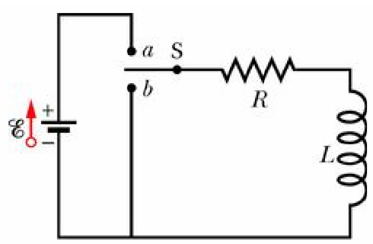
If, then switch to point *b* for a discharging process:

$$R\frac{dq}{dt} + \frac{q}{C} = 0 \quad \text{(discharging equation).}$$

Thus $q = q_0 e^{-t/RC}$ (discharging a capacitor),

and $i = \frac{dq}{dt} = -\left(\frac{q_0}{RC}\right)e^{-t/RC}$ (discharging a capacitor).

We next consider an *RL* circuit (switch to point *a*)



Since $-iR - L\frac{di}{dt} + \mathscr{E} = 0$ **Eq. (A)**

Then $i = \frac{\mathscr{E}}{R}\left(1 - e^{-Rt/L}\right)$, (rise of current).

When switch to point *b*:

$$L\frac{di}{dt} + iR = 0.$$

And then $i = \frac{\mathscr{E}}{R} e^{-t/\tau_L} = i_0 e^{-t/\tau_L}$ (decay of current). Here $\tau_L = \frac{L}{R}$ (time constant).

7. Energy stored in a magnetic field

From **Eq. (A)** in previous page, we obtain

$$\mathscr{E} i = Li\frac{di}{dt} + i^2 R,$$

The l.h.s. can be written as $(\mathscr{E}\ dq)/dt$ and it corresponds to the rate of energy gained when the charge is passing through the battery (that is, **work done by the battery on the charge**). This gained energy is "consumed" by two processes described in the r.h.s. The first term *Li di/dt* represents the rate of **magnetic energy** (U_B)stored in the solenoid, and the second term i^2R represents the rate of **thermal energy** produced by the resister.

Thus, $\frac{dU_B}{dt} = Li\frac{di}{dt}$. and $\int_0^{U_B} dU_B = \int_0^i Li\, di$, which results $U_B = \frac{1}{2}Li^2$ (magnetic energy)

which represents the total energy stored by an inductor L carrying a current i. Note the similarity in form between this expression and the expression for the energy stored by a capacitor with capacitance C and charge q; namely,

$$U_E = \frac{q^2}{2C}.$$

(The variable i^2 corresponds to q^2, and the constant L corresponds to $1/C$.)