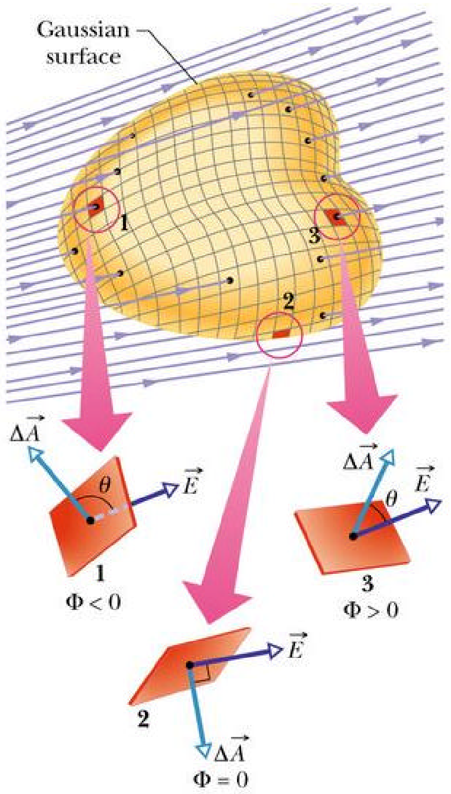
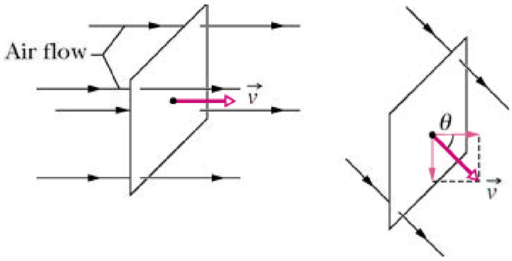
III. Gauss' Law (高斯定律)

Alternative look of Coulomb's law

1. Flux of electric field (電通量)



Example:



Flux of airstream (volume flow rate)

$$\Phi = vA\cos\theta = \vec{v}\cdot\vec{A}$$

Imagine: when $q = 0,\ \Phi = 0$**.**

So, flux of E is

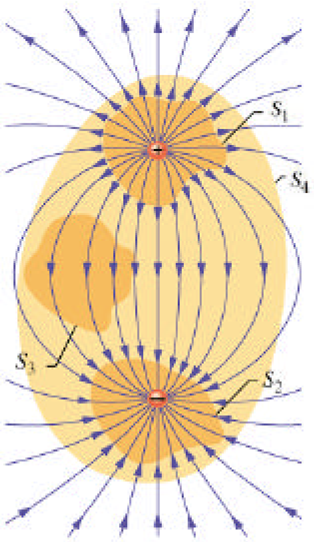
$$\Phi = \Sigma\vec{E}\cdot\Delta\vec{A}$$

$$\Phi = \oint \vec{E}\cdot d\vec{A} \quad \text{(electric flux through a Gaussian surface)}$$

➤The electric flux Φ through a Gaussian surface is proportional to the net number of electric field lines passing through that surface.

2. Gauss' Law (高斯定律)

$$\epsilon_0\Phi = \epsilon_0\oint \vec{E}\cdot d\vec{A} = q_{enc} \quad \text{(Gauss' law)}$$



Example:

$\Phi_1 > 0$

$\Phi_2 < 0$

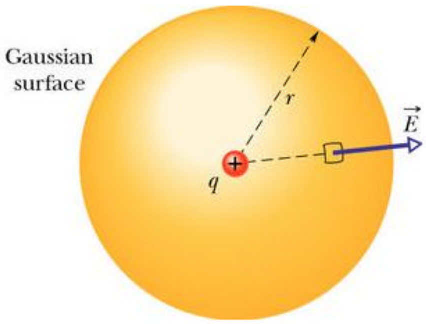
$\Phi_3 = 0$

$\Phi_4 = 0$

Gauss' Law can take advantage of special symmetry of the condition , through proper choices of the Gaussian surface.

3. Important Examples

a) Point charge (spherical symmetry)



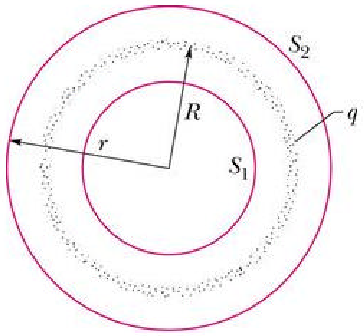
$$\epsilon_0 E \oint dA = \epsilon_0 E\left(4\pi r^2\right) = q$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

which is just the result of Coulomb's Law.

So, Gauss's law is equivalent to Coulomb's law.

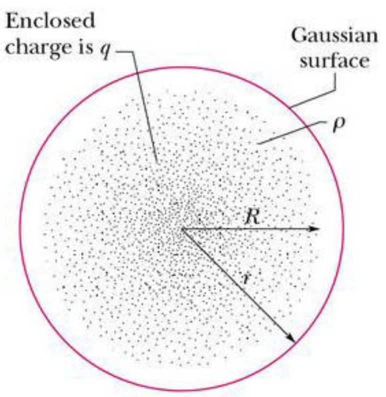
b) Charged spherical shell (spherical symmetry)



$$E = 0 \quad \text{(spherical shell, field at } r < R\text{)},$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad \text{(spherical shell, field at } r \geq R\text{)}$$

c) Charged sphere (spherical symmetry)



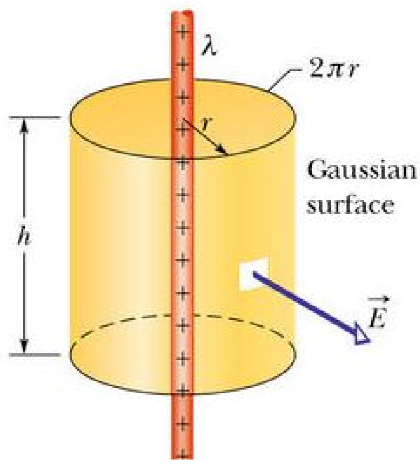
$r \geq R$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

$r < R$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q'}{r^2} = \left(\frac{q}{4\pi\epsilon_0 R^3}\right)r$$

d) Charged line (cylindrical symmetry)

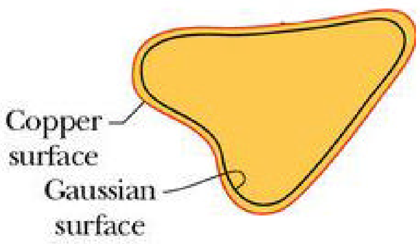


$$\epsilon_0 \Phi = q_{\text{enc}},$$

$$\epsilon_0 E(2\pi r h) = \lambda h,$$ λ is the line charge density

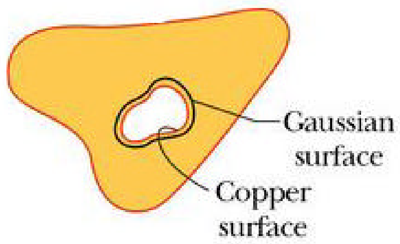
$$E = \frac{\lambda}{2\pi\epsilon_0 r} \quad \text{(line of charge)}$$

4. Isolated charged conductors (和外界隔離的帶電導體)



Conductor: charge can move around

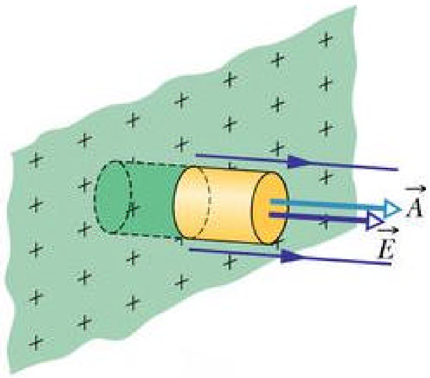
When the system is in equilibrium, inside **E** = 0, and there will be *no charge* in bulk (all the charges are on the surface).



If there is a cavity (凹洞) inside of the conductor, there will be *no charge* on the wall (surface) of the cavity.

So, all the excess charge will go to the surface of a conductor and **E** exists only outside of the surface.

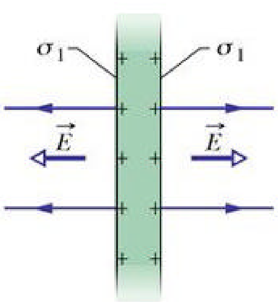
Now, consider a small piece of a outer surface:



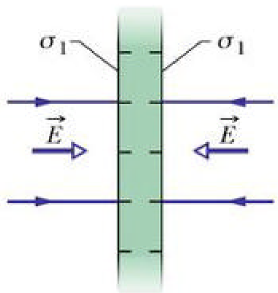
$\epsilon_0 EA = \sigma A,$ s is the surface charge density

$$E = \frac{\sigma}{\epsilon_0}$$

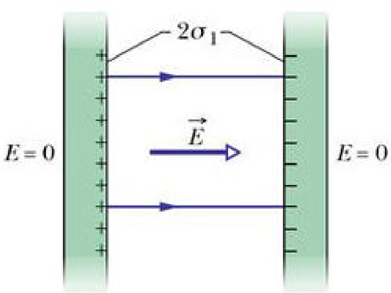
Example: conducting plates



or



$E = \sigma_1/\varepsilon_0$



The electric field between the plates is

$$E = \frac{2\sigma_1}{\epsilon_0} = \frac{\sigma}{\epsilon_0}.$$

IV. Electric Potential (電位)

1. Introduction: electric potential energy and electric potential（電位能與電位）

- Newton's law for *gravitational* force and Coulomb's law for *electrostatic* force are mathematically identical.

- Similar to gravitation that electrostatic force is a *conservative* force, so one can define an *electric potential energy*

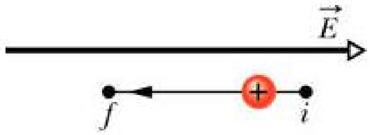
$$\Delta U = U_f - U_i = -W$$

where W is the work done by the electrostatic force on the particles.

- For convenience, we usually take $U_i = 0$ (for example, imagine that charges are initially infinitely separated), then U_f º $U = -W_\infty$, with W_∞ the work done by electrostatic force between the charged particles during the move in from infinity.

Question:

In the figure, a proton moves from point *i* to point *f* in a uniform electric field directed as shown. (a) Does the electric field do positive or negative work on the proton? (b) Does the electric potential energy of the proton increase or decrease?

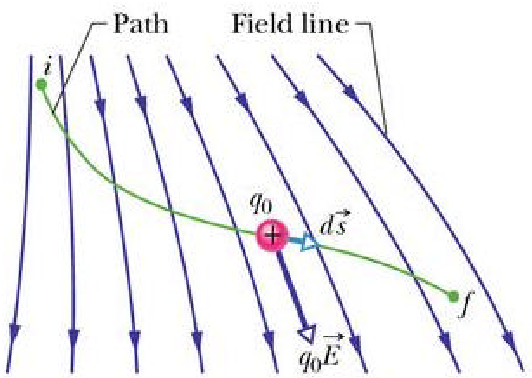


Answer: (a) negative (b) increases (because that kinetic energy decreases).

- *Electric potential* is defined as the electric potential energy per unit charge at a point under an electric field:

$$V = \frac{U}{q} \quad \text{or} \quad \Delta V = V_f - V_i = \frac{U_f}{q} - \frac{U_i}{q} = \frac{\Delta U}{q}$$

2. Electric potential and electric field



Work W done by the electric field on the charge q_0 is

$$dW = \vec{F} \cdot d\vec{s} = q_0 \vec{E} \cdot d\vec{s}$$

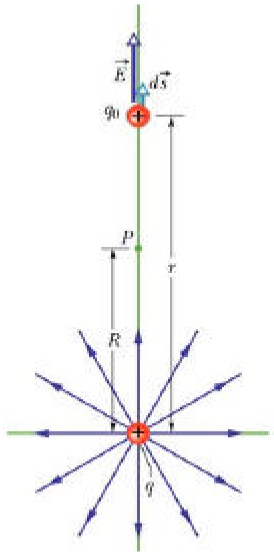
$$W = q_0 \int_i^f \vec{E} \cdot d\vec{s}$$

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$$

If we set $Vi = 0$, then the electric potential V at point f is

$$V = -\int_i^f \vec{E} \cdot d\vec{s}$$

3. Potential due to a point charge



$$\vec{E}\cdot\vec{ds} = E\cos\theta\, ds$$ For the present case, *q* = 0 and *ds* = *dr* .

$$V_f - V_i = -\int_R^{\infty} E\,dr$$

If we set $V_f = 0$ (at ∞) and $V_i = V$(at R), then

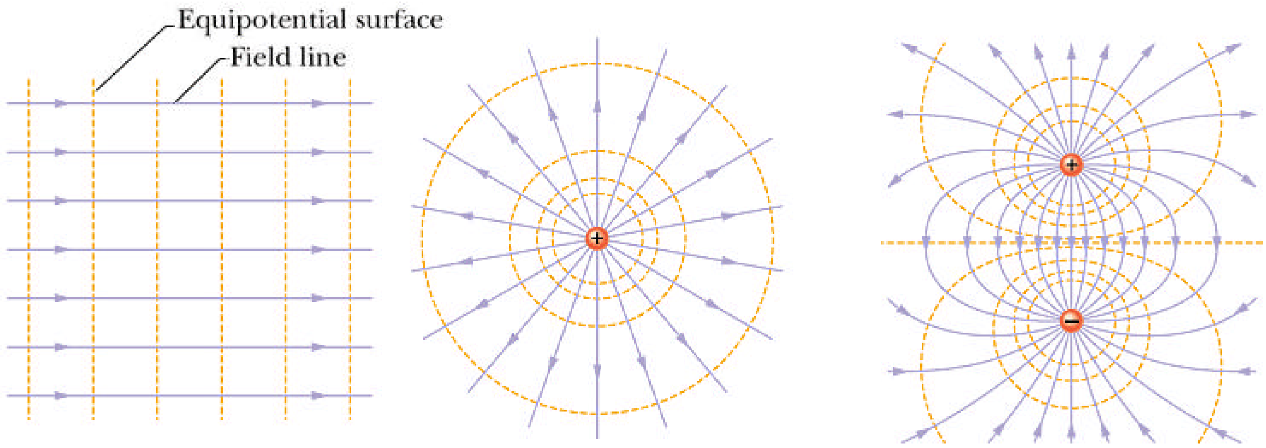
$$0 - V = -\frac{q}{4\pi\epsilon_0}\int_R^{\infty}\frac{1}{r^2}dr = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_R^{\infty} = -\frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

So $$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}.$$

Potential due to more than one charges is then given by

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} \quad (n \text{ point charges})$$

4. Equipotential surfaces (等位面)

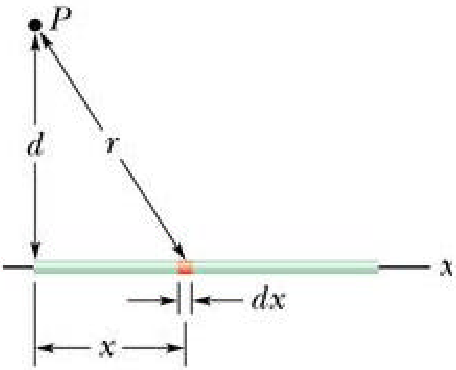


5. Potential due to a continuous charge distribution

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} \quad (\text{positive or negative } dq)$$ and then

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$$ scalar integration

Example: a line of charge



$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dx}{\left(x^2+d^2\right)^{1/2}}$$

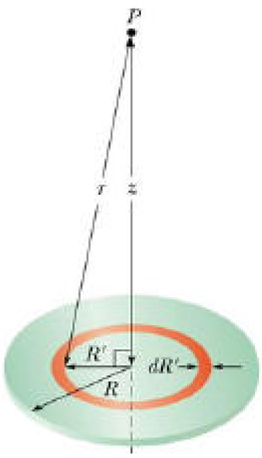
$$V = \int dV = \int_0^L \frac{1}{4\pi\epsilon_0}\frac{\lambda}{\left(x^2+d^2\right)^{1/2}}dx$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{dx}{\left(x^2+d^2\right)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(x+\left(x^2+d^2\right)^{1/2}\right)\right]_0^L$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(L+\left(L^2+d^2\right)^{1/2}\right) - \ln d\right]$$

Example: a charged disk



$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{\sigma\left(2\pi R'\right)\left(dR'\right)}{\sqrt{z^2+R'^2}}$$

$$V = \int dV = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{R'\,dR'}{\sqrt{z^2+R'^2}} = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - z\right)$$

6. Electric potential energy of a system of point charges

The potential set up by q_1 at a point where q_2 is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r}$$

So the potential energy of this two-charge system is

$$U = W = q_2 V = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$$