1 月 17 號　數學作業解答

習題 1. $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$

會不會 $= 1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots - 2\,(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\cdots) = 0$

解：不會！

① $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

此級數也收斂到 ln2

② 而一個級數要拆成奇數一國，偶數一國的運算必需符合“均勻收斂”的性質才可以，否則運算不可更序。

③ 而 ln2 不是一個均勻收斂級數，故運算不可更序！

習題 2. 求(1) $\cos h^2 x - \sin h^2 x =$？　(2) $1 - \tan h^2 x =$？

解：(1) $\cos hx = \frac{e^x + e^{-x}}{2}$

$\sin hx = \frac{e^x - e^{-x}}{2}$

$$\cos h^2 x - \sin h^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1$$

(2) $1 - \tan h^2 x = 1 - \frac{\sin h^2 x}{\cos h^2 x}$

$$= \frac{\cos h^2 x - \sin h^2 x}{\cos h^2 x} = \frac{1}{\cos h^2 x} = \sec h^2 x$$

習題 3. (1) $(\sin hx) = (\frac{e^x - e^{-x}}{2})^1 = \frac{e^x + e^{-x}}{2} = \cos hx$

(2) $(\cos hx) = (\frac{e^x + e^{-x}}{2})^1 = \frac{e^x - e^{-x}}{2} = \sin hx$

(3) $(\tan hx) = (\frac{\sin hx}{\cos hx})^1 = \frac{\cos h^2 x - \sin h^2 x}{\cos h^2 x} = \frac{1}{\cos h^2 x} = \sec h^2 x$

習題 4. 已知 $x > 0$，求 $x^3 + \frac{1}{x^3}$ 之極值

解：(1) $x>0$，故 x^3，$\dfrac{1}{x^3}>0$

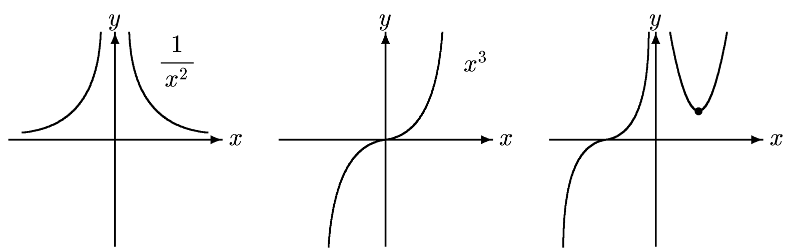
由算術平均 $\geq$ 幾何平均

$$\Rightarrow \frac{\frac{x^3}{2}+\frac{x^3}{2}+\frac{1}{3x^2}+\frac{1}{3x^2}+\frac{1}{3x^2}}{5} \geq \sqrt[5]{\frac{x^3}{2}\cdot\frac{x^3}{2}\cdot\frac{1}{3x^2}\cdot\frac{1}{3x^2}\cdot\frac{1}{3x^2}}$$

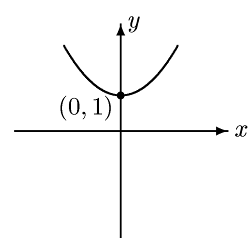
$$\Rightarrow x^3+\frac{1}{x^2} \geq 5\sqrt[5]{\frac{1}{108}}$$

當 $\dfrac{x^3}{2}=\dfrac{1}{3x^2}$，即 $x=\sqrt[5]{\dfrac{2}{3}}$ 時等號成立

(2) $x<0$時



習題 5.求 coshx 和 tanhx 的反函數



解：① $\because\ y=\cos hx=\dfrac{e^x+e^{-x}}{2}$ $\therefore\ y\geq 0$

$\Rightarrow 2y=e^x+e^{-x}$

$\Rightarrow 2y\,e^x=e^{2x}+1$ 令 $e^x=A$

$\Rightarrow A^2-2y\,A+1=0$ $\Rightarrow A=y\pm\sqrt{y^2-1}\geq 0$

$\therefore e^x=y\pm\sqrt{y^2-1}$ $\therefore x=\ln(y\pm\sqrt{y^2-1})$

(又 $\because \ln(y-\sqrt{y^2-1})=\ln\left[(y-\sqrt{y^2-1})\left(\dfrac{y+\sqrt{y^2-1}}{y+\sqrt{y^2-1}}\right)\right]$

$$=\ln\left(\frac{1}{y+\sqrt{y^2-1}}\right)=-\ln(y+\sqrt{y^2-1})$$

$\therefore x=\pm\ln(y+\sqrt{y^2-1})$ *or* $\ln(y\pm\sqrt{y^2-1})$ 多值函數

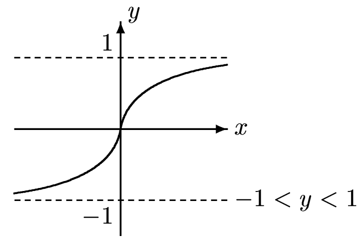
② $y = \tan hx = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

$\therefore y = \dfrac{A^2 - 1}{A^2 + 1} \quad \Rightarrow (A^2 + 1)y = A^2 - 1$

$\Rightarrow A^2(y - 1) = -(1 + y)$

$\Rightarrow A^2 = \dfrac{1 + y}{1 - y} > 0 \quad \Rightarrow A^2 = e^{2x} = \dfrac{1 + y}{1 - y}$

$\Rightarrow 2x = \ln(\dfrac{1 + y}{1 - y}) \quad \Rightarrow x = \dfrac{1}{2}\ln(\dfrac{1 + y}{1 - y})$



習題 6. ① 求 $\sin h^{-1}x$ 的微分

② 求 $\tan h^{-1}x$ 的微分

解：

①

$y = \tan h^{-1}x \quad \Rightarrow x = \tan hy \quad \dfrac{dx}{dy} = \dfrac{d\ \tan\ hy}{dy} = \sec h^2 y$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{\sec h^2 y} = \dfrac{1}{1 - \tan h^2 y} = \dfrac{1}{1 - x^2}$

$(\because 1 - \tan h^2 y = \sec h^2 y)$

$\because y = \sin h^{-1}x \quad \Rightarrow x = \sin hy \quad \dfrac{dx}{dy} = \dfrac{d\ \sin\ hy}{dy} = \cos hy$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{\cos hy} = \dfrac{1}{\sqrt{1 + \sin h^2 y}} = \dfrac{1}{\sqrt{1 + x^2}}$

$(\because \cos h^2 y = 1 + \sin h^2 y)$

②

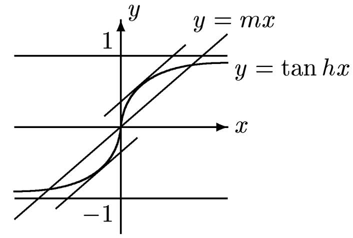
$y = \tan h^{-1}x \quad \Rightarrow x = \tan hy \quad \dfrac{dx}{dy} = \dfrac{d\ \tan\ hy}{dy} = \sec h^2 y$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{\sec h^2 y} = \dfrac{1}{1 - \tan h^2 y} = \dfrac{1}{1 - x^2}$

$(\because 1 - \tan h^2 y = \sec h^2 y)$

習題 7. tanhx=mx ,若 m 大於等於 m_0 則此方程式只有 0 一根,求 m_0

解：∴ 當m夠大時 $\begin{cases} y = mx \\ y = \tan hx \end{cases}$ 的交點僅於$x=0$處



$y = \tan hx$

$\therefore \frac{dy}{dx} = \frac{d \tan hx}{dx} = \sec h^2 x = \frac{1}{\cos h^2 x}$

$(\therefore \frac{dy}{dx} \downarrow as\ |x| \uparrow)$

∴ $y = \tan hx$ 有最大斜率的切線在 $x = 0$ 處

∴ 當 $y = mx$　$m = 1$ 時與 $y = \tan hx$ 恰切於一點