Mathematics for physicists

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From SHM to ellipse and Planck's quantization condition

1.

$$x = A \cdot \cos(\omega \cdot t + \delta) = A \cdot \cos\theta$$

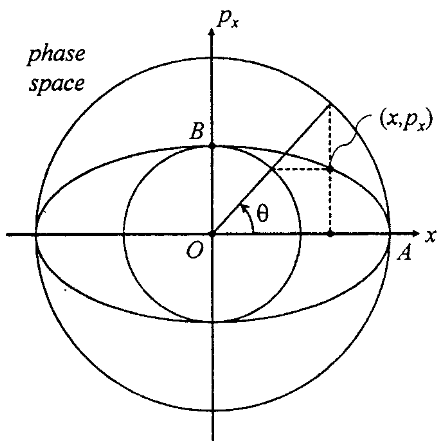
$$mv_x = p_x = -m\omega \cdot A \cdot \sin(\omega \cdot t + \delta) = B \cdot \sin\theta \quad , \quad B \equiv m\omega A$$

$$\left(\frac{x}{A}\right)^2 + \left(\frac{p_x}{m\omega A}\right)^2 = 1$$

area enclosed by the ellipse

$$\text{Area} = \pi \cdot A \cdot m\omega A = \int p_x dx \quad \text{for a cycle}$$

As the time increase, the point corresponding to the state of the motion (x, p) moves along the ellipse clockwise in the phase space.



2. Calculation of the area enclosed by the ellipse with equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

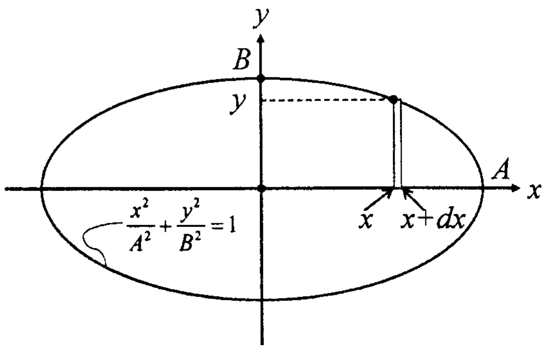
$\text{Area} = 4\int_0^y y \cdot dx$

$y = B\sin\theta$

$x = A\cos\theta \rightarrow dx = A\sin\theta \cdot d\theta$

$x : 0 \rightarrow A$

$\therefore \theta : \frac{\pi}{2} \rightarrow 0$



$$\text{Area} = -4\int_{\pi/2}^{0} B\sin\theta \cdot A\sin\theta \cdot d\theta$$
$$= -4AB\int_{\frac{\pi}{2}}^{0} \sin^2\theta \cdot d\theta$$
$$= -4AB\cdot\frac{1}{2}\int_{\pi/2}^{0}(1-\cos 2\theta)\,d\theta = -2AB\cdot\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{2}}^{0} = \pi\cdot AB$$

3. Total mechanical energy:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2m}(m\omega A)^2 = \frac{1}{2}kA^2$$

$$\therefore \omega^2 = \frac{k}{m} \qquad \omega = 2\pi\nu \qquad , \nu : \text{frequency}$$

$$\text{Area} = \oint p_x dx = \pi m\omega A^2 = \frac{E\cdot 2\pi}{\omega} = \frac{E}{\nu}$$

(1) area $=$ any value ≥ 0 (Newtonian mechanics $E =$ any value ≥ 0)

(2) area $= n\cdot h$, $n = 0, 1, 2, \cdots$, $h =$ Planck const. (Planck's quantum theory $E = nh\nu$

(3) area $= \left(n + \frac{1}{2}\right)h$ (Heisenberg's quantum theory $E = \left(n+\frac{1}{2}\right)h\nu$)

Heisenberg's result is exact.

4. area of an ellipse with equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

$$\text{Area} = \int dA = \int y(x)\,dx = \int \frac{B}{A}\left(A^2 - x^2\right)^{\frac{1}{2}} dx$$

integration by change of the variable:

$$x = A\cos\theta\ ,\ dx = -A\sin\theta d\theta\ ;\ x = 0 \rightarrow \theta = \frac{\pi}{2}\ ,\ x = a \rightarrow \theta = 0$$

$$\left(A^2 - x^2\right)^{\frac{1}{2}} = A\sin\theta$$

$$\text{Area} = -4\int_{\frac{\pi}{2}}^{0} B\sin\theta\cdot A\sin\theta d\theta = -4AB\int_{\frac{\pi}{2}}^{0}\sin^2\theta d\theta = 2AB\int_{0}^{\frac{\pi}{2}}(1-\cos 2\theta)\,d\theta$$
$$= 2AB\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{0}^{\frac{\pi}{2}} = \pi AB$$

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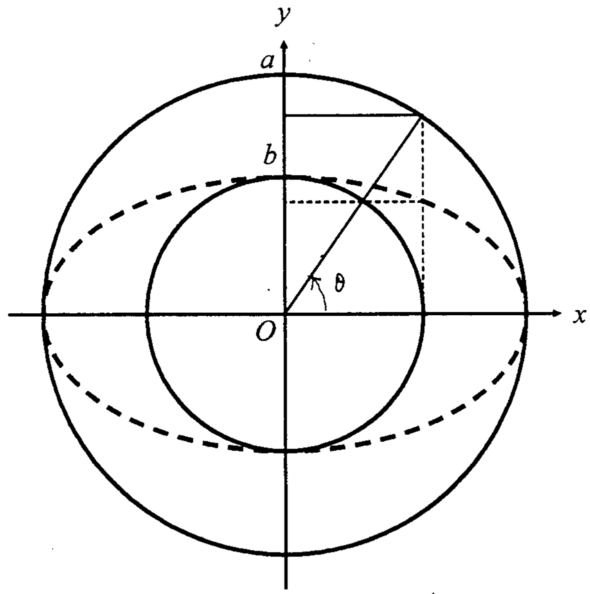
More on ellipse-related math.

ellipse $x = a \cdot \sin\theta$, $y = b \cdot \cos\theta$



1. Arc length

$$ds = \left((dx)^2 + (dy)^2\right)^{\frac{1}{2}} = \left(a^2 \cdot \cos^2\theta + b^2 \cdot \sin^2\theta\right)^{\frac{1}{2}} \cdot d\theta$$

$$\because \cos^2\theta = 1 - \sin^2\theta$$

$$\therefore ds = a \cdot \left(1 - k^2 \cdot \sin^2\theta\right)^{\frac{1}{2}} d\theta \quad , \quad k \equiv \left(a^2 - b^2\right)^{\frac{1}{2}} / a \rightarrow \text{eccentricity}$$

$0 \leq k \leq 1$

$k = 0 \rightarrow$ circle

$0 < k < 1 \rightarrow$ ellipse

$k = 1 \rightarrow$ parabola ($\Longrightarrow a \gg b$)

$k > 1 \rightarrow$ hyperbola

arc length from $\theta = 0$ to $\theta = \phi$

$$s = \int ds = a \int_0^{\phi} \left(1 - k^2 \cdot \sin^2\theta\right)^{\frac{1}{2}} d\theta = a \cdot E(k, \phi) \cdot$$

$E(k,\phi)$: elliptic integral of the 2nd kind

binomial theorem

$$\left(1 - k^2 \cdot \sin^2\theta\right)^{\frac{1}{2}} = 1 - \frac{1}{2}k^2 \sin^2\theta - \frac{1}{8}k^4 \cdot \sin^4\theta - \cdots \qquad (\text{for } k < 1 \text{ i.e. } b \neq 0)$$

$$\therefore E(k,\phi) = \phi - \frac{1}{2}k^2 \int_0^{\phi} \sin^2\theta d\theta - \frac{k^4}{2\cdot 4}\int_0^{\infty} \sin^4\theta d\theta - \cdots$$
$$- \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n)} k^{2n} \int \sin^{2n}\theta d\theta - \cdots$$

2. Simple pendulum

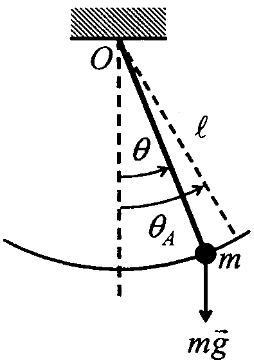
$v(\theta_A) = 0$, θ_A : amplitude of the oscillation

$$ds = \ell \cdot d\theta \ , \quad \frac{ds}{dt} = \ell \cdot \frac{d\theta}{dt} = v$$

conservation of energy:

$$\frac{1}{2}mv^2(\theta) = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}m\ell^2 \cdot \left(\frac{d\theta}{dt}\right)^2 = mg\ell \cdot (\cos\theta - \cos\theta_A)$$

$= $ work done by gravity from θ_A to θ



$$\frac{d\theta}{dt} = \left[2g\ell(\cos\theta - \cos\theta_A)\right]^{\frac{1}{2}}$$

$\therefore \int_0^T dt = \ell \cdot \int d\theta / \left(2g\ell(\cos\theta - \cos\theta_A)\right)^{\frac{1}{2}} = \frac{T}{4} \implies$ period

$\therefore$ the period of the pendulum deps on the amplitude θ_A

$\because \cos\theta = 1 - 2\sin^2(\theta/2)$, $\quad \cos\theta_A = 1 - 2\sin^2(\theta_A/2)$

$T = 2\sqrt{\frac{\ell}{g}} \int_0^{\theta_A} d\theta / \left(k^2 - \sin^2(\theta/2)\right)^{\frac{1}{2}}$, $\quad k = \sin(\theta_A/2)$

change of variable:

$$\text{set} \quad \sin\frac{\theta}{2} = k \cdot \sin\phi \qquad \therefore \theta : 0 \to \theta_A \ , \ \phi : 0 \to \frac{\pi}{2}$$

$$d\theta = \frac{2k \cdot \cos\phi d\phi}{\cos\frac{\theta}{2}} = \frac{2\left(k^2 - \sin^2(\theta/2)\right)^{\frac{1}{2}} d\theta}{\left(1 - k^2\sin^2\phi\right)^{\frac{1}{2}}}$$

$$\rightarrow T = 4\left(\frac{\ell}{g}\right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\left(1-k^2\sin^2\phi\right)^{\frac{1}{2}}} = 4\sqrt{\frac{\ell}{g}} \cdot F\left(k, \frac{\pi}{2}\right) \rightarrow$$ elliptic integral of the 1st kind

by expansion

$$\left(1-k^2\sin^2\phi\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}k^2\sin^2\phi + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\sin^4\phi + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}k^{2n}\cdot\sin^{2n}\phi + \cdots$$

$$\therefore F(k,\eta) = \eta + \frac{1}{2}k^2\int_0^{\eta}\sin^2\phi d\phi + \frac{1\cdot 3}{2\cdot 4}k^4\int_0^{\eta}\sin^4\phi d\phi + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}k^{2n}\int_0^{\eta}\sin^{2n}\phi d\phi$$

$$\therefore T = 4\sqrt{\frac{\ell}{g}}\left[\frac{\pi}{2} + \frac{1}{2}k^2\int_0^{\pi/2}\sin^2\phi d\phi + \frac{1\cdot 3}{2\cdot 4}k^4\int_0^{\pi/2}\sin^4\phi d\phi + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6\int_0^{\pi/2}\sin^6\phi d\phi + \cdots\right]$$

$$= 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \underbrace{\left(\frac{1}{2}\right)^2 k^2}_{\approx\frac{1}{16}\theta_A^2} + \underbrace{\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4}_{\approx\frac{11}{3072}\theta_A^4} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right]$$