Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

(Dated: April 15, 2006)

Application of Calculus

1. Newton's law of motion in time domain

$$\vec{p}(t) = m \cdot \vec{v}(t) \qquad \text{(momentum)}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \qquad \text{(velocity)}$$

$$t \to \vec{p}(t) \;;\; t + dt \to \vec{p}(t + dt) = \vec{p}(t) + d\vec{p}(t)$$

$$d\vec{p}(t) = \vec{F}(t)\, dt \qquad \text{(Newton's law of motion)}$$

$$\vec{F}(t) \to \text{Force}$$

in one-dimension case

$$dp_x(t) = F_x(t)\, dt \quad \text{or} \quad \frac{dp_x(t)}{dt} = F_x(t) \implies m\frac{dv_x(t)}{dt} = m\frac{d^2x(t)}{dt^2} = F_x(t)$$

$$t \to \vec{r}(t) \;;\; t + dt \to \vec{r}(t + dt) = \vec{r}(t) + d\vec{r}(t)$$

$d\vec{r}(t) = ?$

$$d\vec{r}(t) = \frac{d\vec{r}(t)}{dt} \cdot dt = \vec{v}(t) \cdot dt$$

in one-dim. case

$$dx(t) = \frac{dx(t)}{dt} \cdot dt = v_x(t) \cdot dt$$

$$\frac{d\vec{p}(t)}{dt} = m\frac{d\vec{v}(t)}{dt} = m\vec{a}(t) \to m\vec{a} = \vec{F}(t) \qquad \vec{a} : \text{acceleration}$$

in general

$$\vec{F} = \vec{F}(\vec{r}, \vec{v}) = \vec{F}(\vec{r}(t), \vec{v}(t), \cdots)$$

(a) $\vec{F} = -k \cdot \vec{r}$ Hooke's force

one-dim. case $F_x = -k \cdot x$

$$\rightarrow m\frac{d^2x(t)}{dt^2} = -k \cdot x(t)$$

$$\rightarrow x(t) = A \cdot \cos(2\pi f \cdot t + \delta) \qquad \text{(SHM)}$$

$$(2\pi f)^2 = \frac{k}{m} \because \frac{d^2\cos(\omega t + \delta)}{dt^2} = -\omega^2 \cos(\omega t + \delta)$$

$$\therefore f = \frac{1}{2\pi} \cdot \left(\frac{k}{m}\right)^{1/2} \quad , \quad x(0) = A \cdot \cos\delta \quad , \quad v(0) = -2\pi f \cdot A \sin\delta$$

$$\therefore \forall \; x(0), v(0) \implies A, \delta \qquad (1)$$

$\therefore$ A and δ are two constants of integration and determined by the initial conditions.

(b) $\vec{F} = -\gamma \cdot \vec{v}$

one-dim. case $F_x = -\gamma \cdot v_x$

$$m\frac{dv_x(t)}{dt} = -\gamma \cdot v_x(t)$$

$$\therefore v_x(t) = B \cdot e^{-\frac{\gamma}{m} \cdot t} \quad , \quad B = v_x(0) \qquad \because \frac{de^{\beta \cdot t}}{dt} = \beta \cdot e^{\beta \cdot t}$$

$$\frac{dx(t)}{dt} = v_x(0) \cdot e^{-\gamma \cdot \frac{t}{m}}$$

$$x(t) = C - \frac{m \cdot v_x(0)}{\gamma} e^{-\gamma \cdot \frac{t}{m}} \quad , \quad x(0) = C - \frac{m \cdot v_x(0)}{\gamma}$$

Note: B and C are two constants of integration and determined by $v_x(0)$ and $x(0)$.

$$\therefore x(t) = x(0) + \frac{m \cdot v_x(0)}{\gamma}\left(1 - e^{-\gamma \cdot \frac{t}{m}}\right)$$

$$x(\infty) = x(0) + \frac{m \cdot v_x(0)}{\gamma}$$

2. Newton's equation of motion in space domain

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

one-dim. case $F = m\dfrac{dv}{dt}$

if $F = F(x)$

then

$$F(x) = m\frac{dv(t)}{dt} = m\frac{dv(x)}{dx} \cdot \frac{dx}{dt} = m \cdot v(x) \cdot \frac{dv(x)}{dx}$$
$$= \frac{d\left(\frac{1}{2}mv^2(x)\right)}{dx}$$

$$\therefore \frac{1}{2}mv^2(x) - \frac{1}{2}mv^2(x_0) = \int_{x_0}^{x} F(x) \cdot dx = W \qquad \text{(work-energy theorem)}$$

when $x = x_0 \to v = v_0 \equiv v(x_0)$

$$\frac{1}{2}mv^2(x) = K \qquad \text{(kinetic energy)}$$

$$\int_{x_0}^{x} F(x) \cdot dx = W \qquad \text{(work done by } F \text{ from } x_0 \text{ to } x\text{)}$$

$$\to v = v(x) = \frac{dx}{dt} \therefore dt = \frac{dx}{v(x)} \to t = t(x)$$

(a) Hooke's force $F = -k \cdot x$ if $t = 0$, $x = x_0$, $v = v_0$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} (-k \cdot x)\, dx = -\frac{1}{2}k \cdot x^2 + \frac{1}{2}k \cdot x_0^2$$

or

$$\frac{1}{2}mv^2 + \frac{1}{2}k \cdot x^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}k \cdot x_0^2 = E = \text{total mechanical energy}$$

Note $\frac{1}{2}k \cdot x^2 \to$ elastic potential energy

$$\therefore v = \left(\frac{2}{m}\left(E - \frac{1}{2}k \cdot x^2\right)\right)^{\frac{1}{2}} = \frac{dx}{dt}$$
$$\therefore dt = \frac{dx}{\left(\frac{2}{m}\left(E - \frac{1}{2}k \cdot x^2\right)\right)^{\frac{1}{2}}} \implies t - 0 = \int_{x_0}^{x} \frac{dx}{\left(\frac{2}{m}\left(E - \frac{1}{2}k \cdot x^2\right)\right)^{\frac{1}{2}}}$$

change of variable: set $x^2 = \frac{2E}{k} \cdot \cos^2\theta$ i.e. $x = \left(\frac{2E}{k}\right)^{1/2}\cos\theta$, $x = x_0 \implies \theta = \theta_0$

then

$$dx = \left(\frac{2E}{k}\right)^{\frac{1}{2}} (-\sin\theta)\, d\theta$$

$$\left(\frac{2}{m}\left(E - \frac{1}{2}k \cdot x^2\right)\right)^{\frac{1}{2}} = \left(\frac{2E}{m}\right)^{\frac{1}{2}} \sin\theta$$

$$\therefore t - 0 = \int_{\theta_0}^{\theta} \left(\frac{m}{k}\right)^{\frac{1}{2}} (-d\theta) = -\left(\frac{m}{k}\right)^{\frac{1}{2}} (\theta - \theta_0)$$

$$\left[x_0 = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \cos\theta_0 \ , \quad t = 0 \implies \theta = \theta_0 \quad \text{(initial phase angle)} \right]$$

$$\therefore \theta - \theta_0 = -\left(\frac{k}{m}\right)^{\frac{1}{2}} \cdot t = -\omega t$$

$$\text{set } \omega \equiv \left(\frac{k}{m}\right)^{\frac{1}{2}} = 2\pi\nu = \text{angular frequency}$$

$$\therefore x = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \cos\theta = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \cos(\omega t - \theta_0)$$

Note: $\omega \cdot T = 2\pi$, $T = \frac{2\pi}{\omega} = \frac{1}{\nu}$; $T \to$ period, $\nu \to$ frequency

Note: when $x \to$ max i.e. $x = A =$ Amplitude, then $v = 0$

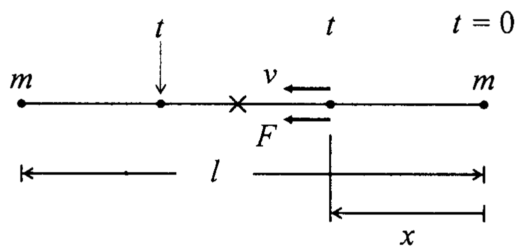
$$\therefore E = \frac{1}{2}k \cdot A^2 \qquad A = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \to x = A \cdot \cos(\omega t - \theta_0)$$

$$\therefore x = A \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t - \theta_0\right)$$

or

$$t = \left(\frac{m}{k}\right)^{\frac{1}{2}} \left(\cos^{-1}\frac{x_0}{A} - \cos^{-1}\frac{x}{A}\right)$$

(b) Gravitational force



gravitational force acting on the particle at time t :

$$F = \frac{Gm^2}{(\ell - 2x)^2} = F(x)$$

Find the time required for the two initially rest particles to collide.

at time t , $v = \dfrac{dx}{dt}$, $v(0) = 0$

$$\frac{1}{2}mv^2 - 0 = \int_0^x F(x) \cdot dx = \int_0^x \frac{Gm^2}{(\ell - 2x)^2} \cdot dx$$

$$\therefore \frac{1}{2}mv^2 = \frac{1}{2}Gm^2 \cdot \left.\frac{1}{\ell - 2x}\right|_0^x = \frac{Gm^2}{2}\left(\frac{1}{\ell - 2x} - \frac{1}{\ell}\right) = Gm^2 \cdot \frac{x}{\ell(\ell - 2x)}$$

$$\therefore v(x) = \left(\frac{2Gm}{\ell} \cdot \frac{x}{\ell - 2x}\right)^{\frac{1}{2}} = \frac{dx}{dt}$$

$$\therefore t_c - 0 = \int_0^{t_c} dt = \int_0^{\frac{\ell}{2}} \left(\frac{\ell}{2Gm} \cdot \frac{\ell - 2x}{x}\right)^{\frac{1}{2}} \cdot dx$$

change of variable: set $x = \frac{\ell}{2}\sin^2\theta$ $\quad \therefore \quad \begin{array}{l} x = 0 \rightarrow \theta = 0 \rightarrow t = 0 \\ x = \frac{\ell}{2} \rightarrow \theta = \frac{\pi}{2} \rightarrow t = t_c \end{array}$

$$dx = \ell \cdot \sin\theta \cdot \cos\theta \cdot d\theta$$

$$\left(\frac{\ell - 2x}{x}\right)^{\frac{1}{2}} = \left(\frac{\ell \cdot \cos^2\theta}{\frac{\ell}{2}\sin^2\theta}\right)^{\frac{1}{2}} = \sqrt{2} \cdot \frac{\cos\theta}{\sin\theta}$$

$$\therefore t_c = \int_0^{\frac{\pi}{2}} \left(\frac{\ell}{Gm}\right)^{\frac{1}{2}} \cdot \ell\cos^2\theta d\theta = \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$
$$= \frac{\pi}{4} \cdot \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}}$$

$$\because \int_0^{\pi/2} \cos^2\theta d\theta = \frac{1}{2}\int_0^{\pi/2} (\cos 2\theta + 1)\, d\theta = \frac{1}{2}\left.\left(\frac{1}{2}\sin 2\theta + \theta\right)\right|_0^{\pi/2} = \frac{\pi}{4}$$

Between $t = 0$ and $t = t_c$

$$t = t(x) = \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}} \left(\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right) \quad \text{, where } \theta = \sin^{-1}\left(\frac{2x}{\ell}\right)^{\frac{1}{2}}$$

$$\sin 2\theta = 2 \cdot \left(\frac{2x}{\ell}\right)^{\frac{1}{2}} \cdot \left(1 - \frac{2x}{\ell}\right)^{\frac{1}{2}}$$

$$\therefore t(x) = \left(\frac{\ell^3}{Gm}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left[\left(\frac{2x}{\ell}\right)^{\frac{1}{2}}\left(1 - \frac{2x}{\ell}\right)^{\frac{1}{2}} + \sin^{-1}\left(\frac{2x}{\ell}\right)^{\frac{1}{2}}\right]$$

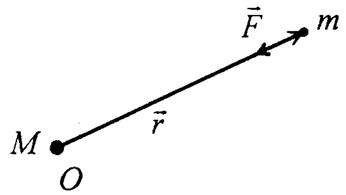
Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

(Dated: May 13, 2006)

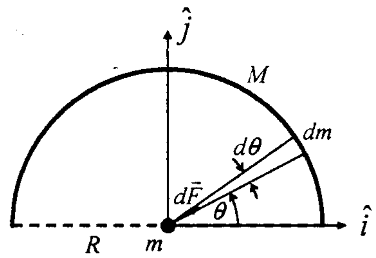
Application of Calculus on Mechanics

1. Universal gravity



$$\vec{F} = -\frac{GMm\hat{r}}{r^2} = -\frac{GMm\vec{r}}{r^3}$$

Example: Total force on m at the center by a uniform semicircular wire with mass M



$$\theta \text{ to } \theta + d\theta \implies dM = \frac{M}{\pi R} \cdot R \cdot d\theta = \frac{M}{\pi} d\theta$$

$$\implies d\vec{F} = dF \cos\theta \cdot \hat{\imath} + dF \sin\theta \cdot \hat{\jmath}$$

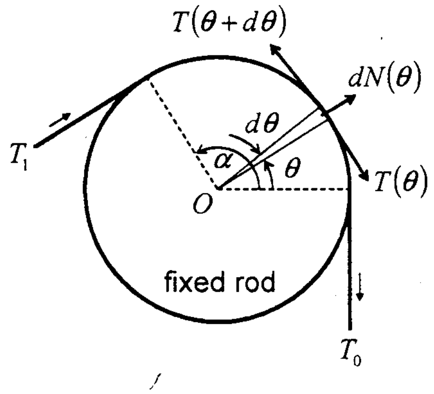
$$dF = \frac{GmdM}{R^2} = G\frac{mM}{\pi R^2} d\theta$$

$$\therefore \vec{F} = \int d\vec{F} = \hat{\imath}\frac{GMm}{\pi R^2} \underbrace{\int_0^\pi \cos\theta \cdot d\theta}_{=\sin\theta|_0^\pi = 0} + \hat{\jmath}\frac{GMm}{\pi R^2} \underbrace{\int_0^\pi \sin\theta \cdot d\theta}_{=-\cos\theta|_0^\pi = 2}$$

$$\vec{F} = \hat{\jmath}\frac{2GMm}{\pi R^2}$$

2. Friction force and Tension: $f_\mu = \mu N$, f_μ : sliding friction force, N : normal force, μ :coefficient of sliding friction

Example: Lift a weight by a string tossed over a fixed circular rod



$\theta \to T(\theta) \qquad \theta + d\theta \to T(\theta + d\theta) = T(\theta) + dT(\theta)$

Normal component:

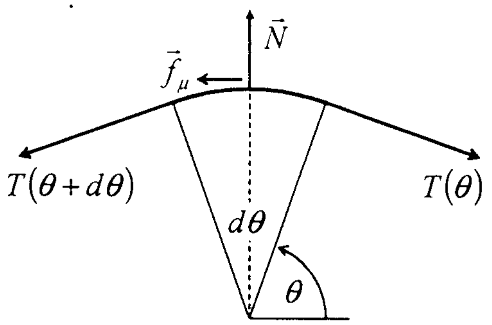
$(T + dT) \sin\frac{d\theta}{2} + T \sin\frac{d\theta}{2} = dN$ Normal force

Tangent component:

$(T + dT) \cos\frac{d\theta}{2} - T \cos\frac{d\theta}{2} = -\mu dN$ Friction force

$\because d\theta \ll 1 \therefore \sin\frac{d\theta}{2} \to \frac{d\theta}{2}; \ \cos\frac{d\theta}{2} \to 1$

$\therefore dT \cdot d\theta \ll T \cdot d\theta$ the $dT \cdot d\theta$ term is neglected

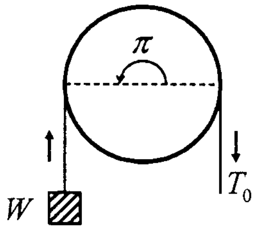


$\therefore T \cdot d\theta \approx dN \quad$ and $\quad dT \approx -\mu dN$

$\dfrac{dT}{T \cdot d\theta} \approx -\mu_k \quad$ or $\quad \dfrac{dT}{d\theta} = -\mu \cdot T$

$\therefore T = A \cdot e^{-\mu \cdot \theta} \qquad T(0) = T_0 = A$

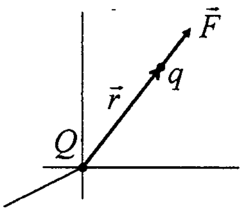
$T(\alpha) = T_1 = T_0 e^{-\mu \cdot \alpha}$



$T_0 = W e^{\mu \pi} \qquad \therefore T_0 > W$

Application of Calculus in Electrostatics

3. Coulomb's force

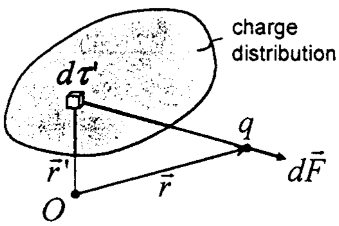


force on point charge q by Q at origin

$$\vec{F} = k \cdot \frac{Q \cdot q \cdot \hat{r}}{r^2}$$

$\vec{r} = r \cdot \hat{r} \qquad \therefore \hat{r} = \vec{r}/r$

$$\vec{F} = k \frac{Q \cdot q \cdot \vec{r}}{r^3}$$

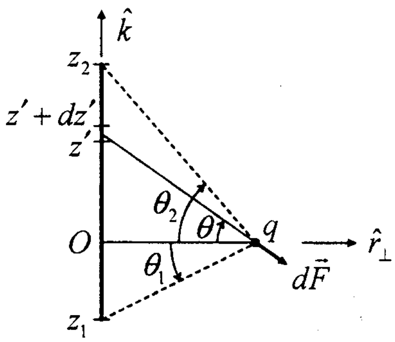


$dQ = \rho(\vec{r}')\, d\tau'$, $\rho(\vec{r}')$: charge density, $d\tau'$: space element

$\vec{F} = \int d\vec{F}$

$$d\vec{F} = k \cdot \frac{\rho(\vec{r}')\, d\tau' \cdot q \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Example: Total Coulomb force on q by a uniformly charged line:



$\rho_L = \dfrac{dQ}{dz}$ linear charge density, $dQ = \rho_L \cdot dz'$

$\vec{F} = \int d\vec{F}$

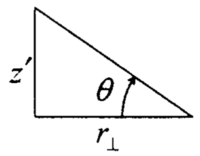
$d\vec{F} = dF \cos\theta \cdot \hat{r}_\perp + dF \sin\theta \left(-\hat{k}\right)$

$\therefore \vec{F} = F_{r_\perp} \cdot \hat{r}_\perp + F_k \cdot \hat{k}$

$$dF = k \cdot \frac{\rho_L dz' \cdot q}{z'^2 + r_\perp^2}$$

$$F_{r_\perp} = \int dF \cdot \cos\theta = \int k \cdot \frac{\rho_L dz' \cdot q}{z'^2 + r_\perp^2} \cdot \cos\theta$$

change of variable from z' to θ :



$$z' = r_\perp \cdot \tan\theta$$

$$\therefore z'^2 + r_\perp^2 = r_\perp^2 \cdot \sec^2\theta$$

$$dz' = r_\perp \cdot \sec^2\theta d\theta \qquad \left(\because \frac{d\tan\theta}{d\theta} = \sec^2\theta\right)$$

$$\therefore F_{r_\perp} = k\cdot\rho_L\cdot q\frac{r_\perp}{r_\perp^2}\int\frac{\sec^2 d\theta}{\sec^2\theta}\cdot\cos\theta = \frac{k\rho_L q}{r_\perp}\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \frac{k\rho_L q}{r_\perp}(\sin\theta_2 - \sin\theta_1)$$

Similarly:

$$F_z = -\frac{k\rho_L q}{r_\perp}\int_{\theta_1}^{\theta_2}\sin\theta d\theta = \frac{k\rho_L q}{r_\perp}(\cos\theta_2 - \cos\theta_1) = k\rho_L q\cdot\left(\frac{1}{(z_2^2 + r_\perp^2)^{1/2}} - \frac{1}{(z_1^2 + r_\perp^2)^{1/2}}\right)$$

(1) If $\theta_1 \to -\frac{\pi}{2}$, $\theta_2 \to +\frac{\pi}{2}$ $\Longrightarrow$ ∞-long line charge:

$$F_{r_\perp} = \frac{2\pi\rho_L q}{r_\perp}\ , \quad F_z = 0$$

(2) If $\theta_1 \to 0$, $\theta_2 \to +\frac{\pi}{2}$ $\Longrightarrow$ semi-∞-long line charge:

$$F_{r_\perp} = \frac{k\rho_L q}{r_\perp}\ , \quad F_z = -\frac{k\rho_L q}{r_\perp}$$