Mathematics for physicists

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Logarithm function

$$y = e^x \qquad x_1 = \ln y_1\ ,\ \ x_2 = \ln y_2\ ,\ \ e^{x_1+x_2} = e^{x_1}\cdot e^{x_2} = y_1\cdot y_2$$

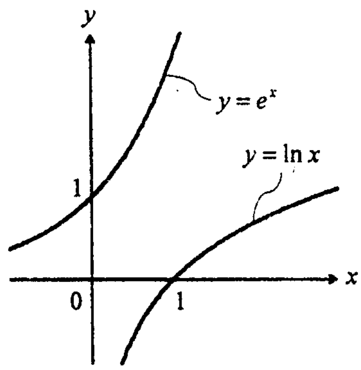
$$x = \ln y \qquad x_1 + x_2 = \ln(y_1\cdot y_2) = \ln y_1 + \ln y_2\ ,\ \ x_1 - x_2 = \ln\left(\frac{y_1}{y_2}\right) = \ln y_1 - \ln y_2$$

$$\frac{dy}{dx} = e^x$$

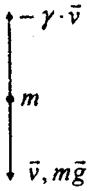
$$\therefore dx = \frac{dy}{e^x} = \frac{dy}{y}$$

$$\therefore \frac{d\ln y}{dy} = \frac{1}{y}$$

$$\text{or} \quad \int\frac{dy}{y} = \int d\ln y = \ln y + const.$$



Application: falling with drag



$$m\frac{d\vec{v}}{dt} = m\vec{g} - \gamma\cdot\vec{v} \qquad \rightarrow \vec{v}(t) = v(t)\cdot\downarrow\ ,\ \ \gamma : \text{damping coefficient}$$

$$u(v) = g - \frac{\gamma}{m}v\ ,\ \ du = -\frac{\gamma}{m}dv$$

$$\frac{dv}{g-\frac{\gamma}{m}v} = dt \qquad \therefore \frac{dv}{g-\frac{\gamma}{m}v} = \left(-\frac{m}{\gamma}\right)\frac{du}{u} = \left(-\frac{m}{\gamma}\right)d\ln u = dt$$

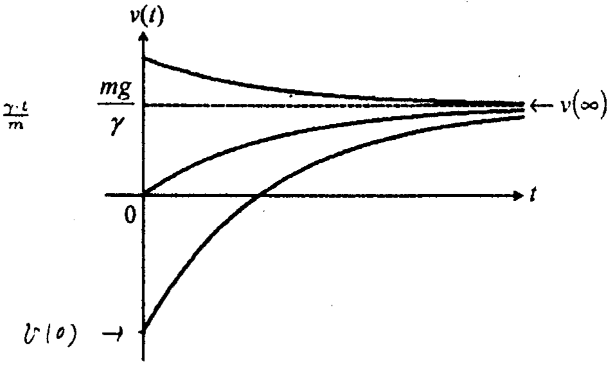
$$\therefore \int_{v_0}^{v} d\ln\left(g - \frac{\gamma}{m}v\right) = \int_0^t -\frac{\gamma}{m}dt \quad , \qquad v_0 = v(0) : \text{initial velocity}$$

$$\ln\left|g-\frac{\gamma}{m}v\right|\Big|_{v_0}^{v} = \ln\left|g-\frac{\gamma}{m}v\right| - \ln\left|g-\frac{\gamma}{m}v_0\right| = \ln\left|\frac{g-\frac{\gamma}{m}v}{g-\frac{\gamma}{m}v_0}\right|$$

$$\therefore \ln\left|\frac{g-\frac{\gamma}{m}v}{g-\frac{\gamma}{m}v_0}\right| = -\frac{\gamma\cdot t}{m} \quad\Longrightarrow\quad g - \frac{\gamma}{m}v(t) = \left(g-\frac{\gamma}{m}v_0\right)\cdot e^{-\frac{\gamma\cdot t}{m}}$$

$$\therefore v(t) = v(\infty) - (v(\infty) - v(0))\, e^{-\frac{\gamma \cdot t}{m}}$$

$$v(\infty) = \frac{mg}{\gamma} \quad \text{terminal velocity}$$



Other approach to solve the differential equation: $m\frac{dv(t)}{dt} = mg - \gamma v$

particular solution $v_p(t) = const. = \frac{mg}{\gamma} \qquad \therefore v(t) = v_{\text{complementary}}(t) + v_p(t)$

$\rightarrow m\frac{dv_{\text{comp}}(t)}{dt} = -\gamma \cdot v_{\text{comp}}(t) \qquad \therefore v_{\text{comp}}(t) = A \cdot e^{-\frac{\gamma \cdot t}{m}}$

$\therefore v(t) = \frac{mg}{\gamma} + A \cdot e^{-\frac{\gamma \cdot t}{m}}$

$\because v(0) = v_0 \qquad \therefore A = v_0 - \frac{mg}{\gamma} = v(0) - v(\infty) \quad , \quad v(\infty) = \frac{mg}{\gamma} \qquad (\because e^{-\infty} = 0)$

$\rightarrow v(t) = v(\infty) + (v(0) - v(\infty))\, e^{-\gamma \cdot t/m}$

$x(t) = ?$

$x(t) - x(0) = \int \frac{dx(t)}{dt} dt = \int_0^t v(t)\, dt = \int_0^t v(\infty)\, dt + A \int_0^t e^{-\gamma \cdot t/m} dt$

$x(t) = x(0) + \frac{mg}{\gamma} \cdot t - \frac{m}{\gamma} \cdot A \cdot \left(e^{-\frac{\gamma \cdot t}{m}} - 1\right)$

Complex numbers

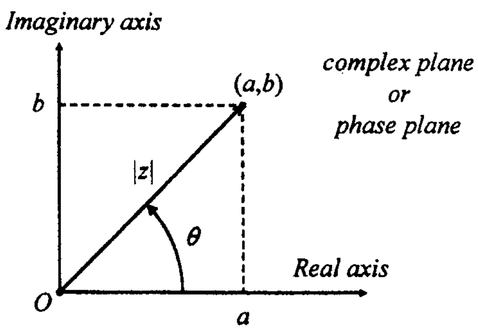
$z = a + i \cdot b$

$a \rightarrow$ real part, $b \rightarrow$ imaginary part

$a = |z| \cdot \cos\theta, \; b = |z| \cdot \sin\theta$

$|z| = (a^2 + b^2)^{\frac{1}{2}}, \quad \tan\theta = \frac{b}{a}$

$\rightarrow z = |z|(\cos\theta + i \cdot \sin\theta)$



a)

$$z_1 = |z_1| (\cos\theta_1 + i \cdot \sin\theta_1) \overset{\curvearrowright \text{by Euler relation}}{=} |z_1| \cdot e^{i\theta_1} \qquad \text{polar form of phasor}$$

$$z_2 = |z_2| \cdot e^{i\theta_2}$$

$$\therefore z_1 \cdot z_2 = |z_1| |z_2| \cdot e^{i(\theta_1+\theta_2)}$$

$$z^n = |z|^n e^{in\theta} = |z|^n \cdot (\cos n\theta + i \cdot \sin n\theta) \longrightarrow \text{de Moivre formula}$$

$$\because \cos\theta = \cos(\theta \pm 2k\pi)$$

$$z^{1/n} = \sqrt[n]{z} = |z|^{1/n} \left(\cos\left(\frac{\theta \pm 2k\pi}{n}\right) + i \cdot \sin\left(\frac{\theta \pm 2k\pi}{n}\right)\right) \quad , \quad k = 0, 1, 2, \cdots, n-1$$

$$z^{1/2} = |z|^{1/2} \cdot \left(\cos\frac{\theta}{2} + i \cdot \sin\frac{\theta}{2}\right) \quad \text{and} \quad |z|^{1/2} \cdot \left(\cos\left(\frac{\theta}{2} + \pi\right) + i \cdot \sin\left(\frac{\theta}{2} + \pi\right)\right)$$

$$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i \cdot \sin\frac{2k\pi}{n}$$

$$\sqrt[3]{1} = 1 \ , \ \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \ , \ \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 1 \ , \ \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \ , \ \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

b)

$$\forall \ f(t) = A \cdot e^{\pm i\omega t}$$

$$\longrightarrow \frac{d^2 f(t)}{dt^2} = -\omega^2 \cdot f(t)$$

$$\text{SHM} \longrightarrow \frac{d^2 x(t)}{dt^2} = -\kappa \cdot x(t) \longrightarrow x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

$$= B_1 \cos\omega t + B_2 \sin\omega t$$

$$= C \cdot \cos(\omega t + \delta)$$

$A_1, A_2; B_1, B_2; C, \delta$ are all constants of integration and determined by the initial conditions, e.g., $x(0)$ and $dx/dt|_{t=0} = v(0)$

c)

$$a \cdot x^2 + b \cdot x + c = 0$$

$$\longrightarrow x_{\frac{1}{2}} = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a} \qquad \text{roots of the equation}$$

$x_{\frac{1}{2}}$ are real if $b^2 \geq 4ac$

if $b^2 < 4ac \rightarrow x_{\frac{1}{2}} = -\frac{b}{2a} \pm i\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)^{\frac{1}{2}}$ complex conjugate

Applications: damped oscillation

$$m\frac{d^2\vec{v}}{dt^2} = \sum \vec{F} = \underbrace{-\kappa\cdot\vec{r}}_{\text{Hook's force}} + \underbrace{-\gamma\cdot\vec{v}}_{\text{damping force}}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

one-dimensional case

$$m\frac{d^2x(t)}{dt^2} + \gamma\cdot\frac{dx(t)}{dt} + \kappa\cdot x(t) = 0$$

$$\rightarrow x(t) = \ ?$$

set $x(t) = Ae^{\beta t}$ $\therefore$ $\frac{d^2x(t)}{dt^2}, \frac{dx(t)}{dt}, x(t)$ are linear dependent, i.e., have the same function form.

$$\rightarrow \left(m\beta^2 + \gamma\cdot\beta + \kappa\right)\cdot A\cdot e^{\beta\cdot t} = 0$$

$$\therefore m\beta^2 + \gamma\cdot\beta + \kappa = 0 \qquad \therefore \beta_{\frac{1}{2}} = -\frac{\gamma}{2m} \pm \left(\frac{\gamma^2}{4m^2} - \frac{\kappa}{m}\right)^{\frac{1}{2}}$$

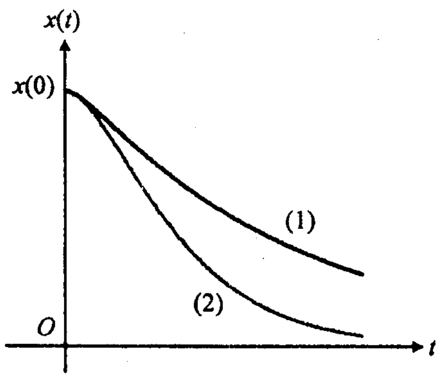
$$\therefore x(t) = A_1\cdot e^{\beta_1\cdot t} + A_2\cdot e^{\beta_2\cdot t}$$

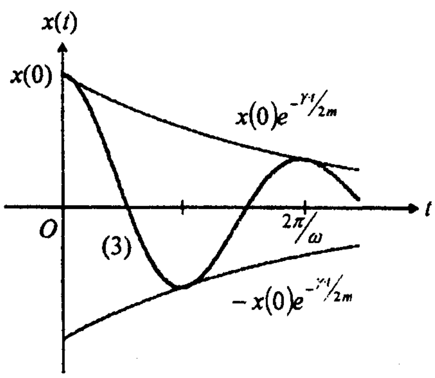
(1) $\gamma^2 > 4m\kappa$ overdamped

$\beta_{\frac{1}{2}}$ real number and < 0; A_1, A_2 determined by $x(0), v(0)$

(2) $\gamma^2 = 4m\kappa$ critically damped

$$x(t) = \left(\underbrace{A_3}_{x(0)} + \underbrace{A_4\cdot t}_{\text{by other method}}\right)e^{-\beta\cdot t}, \quad \beta = -\frac{\gamma}{2m}$$





(3) $\gamma^2 < 4m\kappa$ underdamped

$$\beta_{\frac{1}{2}} = -\frac{\gamma}{2m} \pm i \cdot \underbrace{\left(\frac{\kappa}{m} - \frac{\gamma^2}{4m^2}\right)^{\frac{1}{2}}}_{\omega}$$

$$x(t) = e^{-\frac{\gamma \cdot t}{2m}} \cdot \left(A_5 e^{i\omega t} + A_6 e^{-i\omega t}\right) = e^{-\frac{\gamma \cdot t}{2m}} \cdot A\cos(\omega t + \delta)$$

Oscillation with exponentially decaying amplitude.

(4) if $\gamma \to 0$, $\omega \to \omega_0 = \sqrt{\frac{\kappa}{m}}$, $x(t) \to A\cos(\omega_0 \cdot t + \delta) \to SHM.$