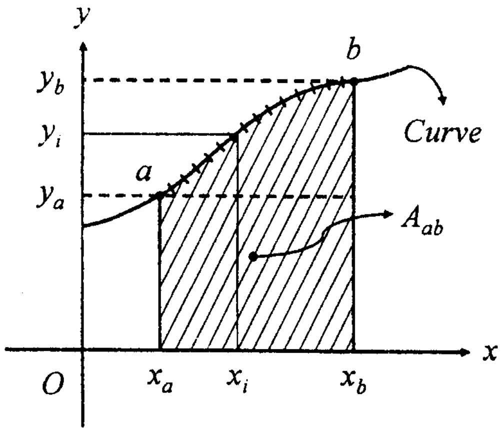
Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

(Dated: December 24, 2005)

Integration &. Differentiation



$\overline{ab} = \left[(y_b - y_a)^2 + (x_b - x_a)^2\right]^{\frac{1}{2}}$

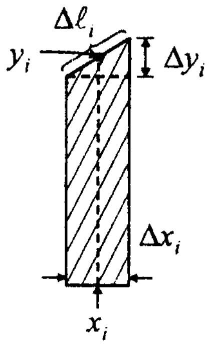
$\hookrightarrow$ length of line section between a & b

$\overset{\frown}{ab} =$? path length along $y(x)$ between a & b

length element at (x_i, y_i)

$$\Delta \ell_i \approx \left[(\Delta x)_i^2 + (\Delta y)_i^2\right]^{\frac{1}{2}} \approx \left[1 + \left(\frac{\Delta y}{\Delta x}\right)_i^2\right]^{\frac{1}{2}} \cdot \Delta x_i$$

differential length at (x_i, y_i)



$$d\ell_i = \lim_{\Delta x_i \to 0} \left[1 + \left(\frac{\Delta y}{\Delta x}\right)_i^2\right]^{\frac{1}{2}} \cdot \Delta x_i$$

$$d\ell(x) = \left[1 + \left(\frac{dy(x)}{dx}\right)^2\right]^{\frac{1}{2}} \cdot dx$$

$$x_1 = x_a \quad , \quad x_N = x_b$$

$$\overset{\frown}{ab} = \lim_{\Delta x_i \to 0} \sum_{i=1}^{N} \Delta \ell_i = \lim_{\Delta x_i \to 0} \sum_{i=1}^{N} \left[1 + \left(\frac{\Delta y}{\Delta x}\right)_i^2\right]^{\frac{1}{2}} \cdot \Delta x_i \equiv \int_{x=x_a}^{x=x_b} d\ell(x)$$

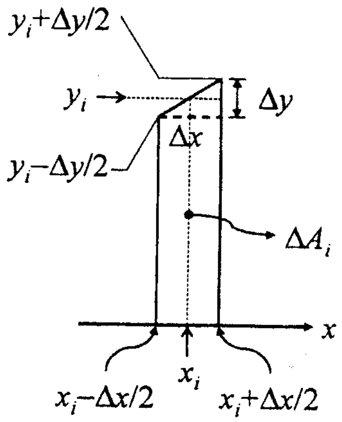
$$= \int_{x_a}^{x_b} \left[1 + \left(\frac{dy(x)}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

$$\frac{dy(x)}{dx} \equiv \lim_{\Delta x_i \to 0} \left(\frac{\Delta y}{\Delta x}\right)_i \to \text{slope of the curve } y(x) \text{ at } x = x_i$$

$$y_b - y_a = y(x_b) - y(x_a) = \ ?$$

$$\Delta y_i \approx \frac{\Delta y_i}{\Delta x_i} \cdot \Delta x_i \implies dy(x) = \frac{dy(x)}{dx} \cdot dx$$

$$\implies y_b - y_a = \lim_{\Delta y_i \to 0} \sum_{i=1}^{N} \Delta y_i \equiv \int_{y_a}^{y_b} dy = \int_{x_a}^{x_b} \frac{dy(x)}{dx} dx$$



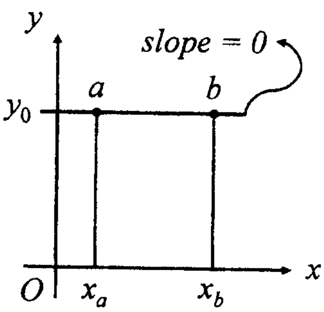
A_{ab}: area between $y(x)$ and x-axis from x_a to x_b

$$A_{ab} = \lim_{\Delta A_i \to 0} \sum \Delta A_i = \lim_{\Delta x_i \to 0} \sum_{i=1}^{N} y_i \cdot \Delta x_i = \int_{x_a}^{x_b} y(x) \cdot dx = \ ?$$

if $y(x) = \dfrac{dg(x)}{dx}$ then

$$\int_{x_a}^{x_b} y(x)\, dx = \int_{x_a}^{x_b} \frac{dg(x)}{dx} \cdot dx = \int_{x_a}^{x_b} dg(x) = g(x_b) - g(x_a)$$

Example 1 *Constant function*



$y = y_0 = const$

$y_b - y_a = \int dy = \int \dfrac{dy}{dx} \cdot dx$

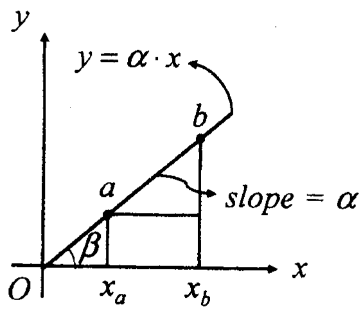
$dy/dx = 0 \qquad \therefore y_b - y_a = 0$

$\overline{ab} = x_b - x_a$

$\overset{\frown}{ab} = \int_{x_a}^{x_b} dx = x_b - x_a$

$$A_{ab} = \int_{x_a}^{x_b} y_0 \cdot dx = y_0 \cdot \int_{x_a}^{x_b} dx = y_0 \cdot (x_b - x_a)$$

Example 2 *Linear function*



$\because \dfrac{dy(x)}{dx} = \tan\beta = \underline{\underline{\alpha}} \longrightarrow$ *slope of the line*

$\therefore \overset{\frown}{ab} = \int_{x_a}^{x_b} (1+\alpha^2)^{\frac{1}{2}} \cdot dx = (1+\alpha^2)^{\frac{1}{2}} \cdot (x_b - x_a)$

$\overset{\frown}{ab} = \overline{ab}$

$\overline{ab} \cdot \cos\beta = x_b - x_a$

$(1+\alpha^2)^{\frac{1}{2}} = (1+\tan^2\beta)^{\frac{1}{2}} = \sec\beta = \dfrac{1}{\cos\beta}$

$\therefore$ *The result is consistent with the trigonometry*

$$A_{ab} = \int_{x_a}^{x_b} y \cdot dx = \int_{x_a}^{x_b} \alpha \cdot x \cdot dx \qquad \because y = \alpha \cdot x = \frac{d\left(\frac{1}{2}\alpha \cdot x^2\right)}{dx} \leftarrow \textit{formula 1}$$

$$= \int d\left(\frac{1}{2}\alpha \cdot x^2\right) \qquad \hookrightarrow \textit{special case of } \frac{dx^n}{dx} = n \cdot x^{n-1}$$

$$= \frac{1}{2}\alpha x^2\Big|_{x_a}^{x_b} = \frac{1}{2}\alpha \cdot \left(x_b^2 - x_a^2\right)$$

$$= \frac{1}{2}\left(y_b + y_a\right) \cdot \left(x_b - x_a\right) \rightarrow \textit{consistent with the geometry}$$

check

$$\frac{d\left(\frac{1}{2}\alpha \cdot x^2\right)}{dx} = \lim_{\Delta x \to 0} \frac{\frac{1}{2}\alpha \cdot (x + \Delta x)^2 - \frac{1}{2}\alpha \cdot x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left(\frac{1}{2}\alpha \cdot 2 \cdot x + \frac{1}{2}\alpha \cdot (\Delta x)\right)$$

$$= \alpha \cdot x$$

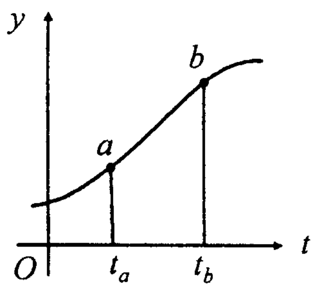
OR

$$\int_{x_a}^{x_b} \underbrace{\alpha \cdot x}_{y(x)} dx = \int d\underbrace{\left(\frac{1}{2}\alpha \cdot x^2\right)}_{g(x)} = \frac{1}{2}\alpha \cdot x^2\Big|_{x_a}^{x_b} = \underbrace{\frac{1}{2}\alpha x_b^2}_{g(x_b)} - \underbrace{\frac{1}{2}\alpha x_a^2}_{g(x_a)}$$

$$\left(\because \frac{dg(x)}{dx} = y(x) \therefore y(x) \cdot dx = dg(x) \implies \int_{x_a}^{x_b} y(x)\, dx = \int_{x_a}^{x_b} dg(x) = g(x_b) - g(x_a)\right)$$

One-dimensional motion:

if $y = y(t) \rightarrow$ one-dimensional motion along y-axis

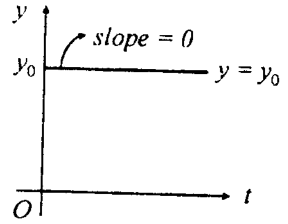


$y_b - y_a \rightarrow$ displacement

$t_b - t_a \rightarrow$ time duration

$\frac{dy(t)}{dt} \rightarrow$ velocity at time t

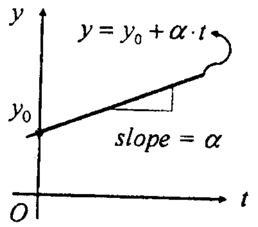
Case 1: Statics



$$\frac{dy}{dt} = 0 \qquad \text{static}$$

$$y = y_0 = const.$$

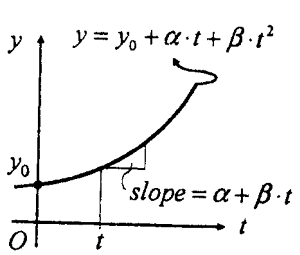
Case 2: Constant velocity motion



$$\alpha = \frac{dy(t)}{dt} = const. = v_y(t) \rightarrow y \text{ component of velocity}$$

$$a_y(t) = \frac{dv_y(t)}{dt} = 0 \rightarrow y \text{ component of acceleration}$$

Case 3: Constant acceleration motion

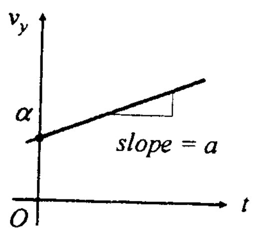


$$\frac{dy(t)}{dt} = \overbrace{\alpha}^{v_y(0)} + 2\beta \cdot t = v_y(t) \rightarrow y \text{ component of velocity}$$

$$\frac{dv_y(t)}{dt} = \underbrace{2\beta}_{a_y(0)} = a_y(t) \rightarrow y \text{ component of acceleration}$$

$$\rightarrow y(t) = y(0) + v_y(0) \cdot t + \frac{1}{2} a_y(0) \cdot t^2$$

Case 4: From given $v_y(t)$ to find $y(t)$



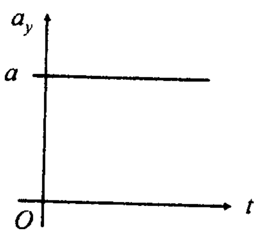
$$\because v_y(t) = \frac{dy(t)}{dt}$$

$$\therefore y(t) - y(t_0) = \int_{t_0}^{t} dy(t) = \int \frac{dy(t)}{dt} \cdot dt = \int_{t_0}^{t} v_y(t)\, dt$$

$$\int_0^t (\alpha + at)\, dt = \left(\alpha \cdot t + \tfrac{1}{2} a \cdot t^2\right)\Big|_0^t = \alpha \cdot t + \tfrac{1}{2} a \cdot t^2$$

$\therefore$ displacement = area between $v_y(t)$ and t-axis from t_a to t_b

Case 5: From given $a_y(t)$ to find $v_y(t)$ and $y(t)$



$$v_y(t) - v_y(0) = \int_0^t \frac{dv(t)}{dt} dt = \int_0^t a(t)\, dt = \int_0^t a \cdot dt = a \cdot t$$

$$\therefore v_y(t) = v_y(0) + \int_0^t a(t)\, dt = v_y(0) + a \cdot t$$

$$y(t) = y(0) + \int_0^t v_y(t)\, dt = y(0) + v_y(0) \cdot t + \tfrac{1}{2} \cdot a \cdot t^2$$

4.

$$y(x) = a \cdot \sin \beta x$$
$$\frac{dy(x)}{dx} = a \cdot \beta \cdot \cos \beta x$$

5. Series expansion of a function

$$\text{set} \quad y(x) = \sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 \cdot x + \cdots + a_n \cdot x^n + \cdots$$
$$y(0) = a_0$$
$$\frac{dy(x)}{dx} = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} = a_1 + 2 \cdot a_2 \cdot x + \cdots + n \cdot a_n \cdot x^{n-1} + \cdots$$
$$\therefore \left. \frac{dy(x)}{dx} \right|_{x=0} = a_1 = y'(0) \qquad y'(x) \equiv \frac{dy(x)}{dx}$$
$$\therefore \frac{d^n y(x)}{dx^n} \equiv \left(\frac{d}{dx} \right)^n \cdot y(x) = n! \cdot a_n + \frac{(n+1) \cdot n \cdot \cdots \cdot 2}{1} \cdot x^1 + \cdots$$
$$\therefore a_n = \frac{1}{n!} \cdot \left. \frac{d^n y(x)}{dx^n} \right|_{x=0} , \quad n! \equiv 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$
$$\rightarrow y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left. \frac{d^n y(x)}{dx^n} \right|_{x=0} \cdot x^n \qquad \text{Maclaurin series expansion}$$

6.

$$y(x) = \cos x = \sum_{n=0}^{\infty} a_n \cdot x^n$$
$$a_n = \ ?$$
$$a_n = \left. \frac{d^n \cos x}{dx^n} \right|_{x=0}$$
$$a_0 = 1 \ , \quad a_1 = 0 \ , \quad a_2 = -1 \ , \quad a_3 = 0 \ , \quad a_4 = a_0$$
$$\therefore \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$$

7.

$$y(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

try it!

Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

(Dated: January 7, 2006)

Differentiation

1.

$$
\begin{aligned}
y(x) &= a \cdot x^n \\
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{a \cdot (x+\Delta x)^n - a \cdot x^n}{\Delta x} \\
(x+\Delta x)^n &= x^n + n \cdot x^{n-1} \cdot \Delta x + \frac{n \cdot (n-1)}{1 \cdot 2} \cdot x^{n-2} \cdot (\Delta x)^2 + \cdots + (\Delta x)^n \\
\therefore \frac{dy(x)}{dx} &= a \cdot n \cdot x^{n-1} \quad , \quad (n \neq 0)
\end{aligned}
$$

2.

$$
\begin{aligned}
y(x) &= \sum_{i=0}^{n} a_i \cdot x^i = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n \\
\frac{dy(x)}{dx} &= a_1 + 2 \cdot a_2 \cdot x^2 + \cdots + n \cdot a_n \cdot x^{n-1} \\
&= \sum_{i=1}^{n} i \cdot a_i \cdot x^{i-1}
\end{aligned}
$$

3.

$$
\begin{aligned}
y(x) &= a \cdot \cos \beta x \\
\frac{dy(x)}{dx} &= \lim_{\Delta x \to 0} \frac{a \cdot \cos[\beta \cdot (x+\Delta x)] - a \cdot \cos \beta}{\Delta x} \\
\cos[\beta \cdot (x+\Delta x)] &= \cos \beta x \cdot \cos \beta \Delta x - \sin \beta x \cdot \sin \beta \Delta x \\
\frac{dy(x)}{dx} &= \lim_{\Delta x \to 0} \frac{a \cdot (\cos \beta \Delta x - 1)}{\Delta x} - \lim_{\Delta x \to 0} \sin \beta x \frac{\sin \beta \Delta x}{\Delta x}
\end{aligned}
$$

$$
\because \lim_{\Delta x \to 0} \frac{\sin \beta}{\Delta x} = \beta \qquad \text{OR} \qquad \sin \beta \Delta x \overset{\Delta x \to 0}{\longrightarrow} \beta \cdot \Delta x
$$

$$
\lim_{\Delta x \to 0} \frac{\cos \beta \Delta x - 1}{\Delta x} = 0 \qquad \text{OR} \qquad \cos \beta \Delta x \overset{\Delta x \to 0}{\longrightarrow} 1
$$

$$
\therefore \frac{dy(x)}{dx} = -a \cdot \beta \cdot \sin \beta x
$$

Integration

1. $y = x^n \rightarrow \dfrac{dy}{dx} = n \cdot x^{n-1}$

$$\text{set} \quad f(x) = x^n \quad \text{then} \quad \int_a^b f(x)\,dx = \int_a^b x^n \cdot dx = \frac{1}{n+1} \cdot x^{n+1}\Big|_a^b$$

$$= \frac{b^{n+1} - a^{n+1}}{n+1}$$

$$\because \frac{dx^{n+1}}{dx} = (n+1) \cdot x^n$$

2. $\int \cos \beta x dx = ?$

$$\because \frac{d \sin \beta x}{dx} = \beta \cdot \cos \beta x$$

$$\therefore \int \cos \beta x dx = \frac{1}{\beta} \int d(\sin \beta x) = \frac{1}{\beta} \sin \beta x + C \quad , \quad C = \text{constant.}$$

Mathematics for physicists

Lecturer: Prof. Ven-Chung Lee

(Dated: January 21, 2006)

What we can do with the power series expansion

1.

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n \quad , \quad a_n = \frac{f^{(n)}(0)}{n!} \qquad \text{(Maclaurin expansion)}$$

by the similar way

2.

$$f(x) = \sum_{n=0}^{\infty} b_n \cdot (x - x_0)^n \quad , \quad b_n = \frac{f^{(n)}(x_0)}{n!} \qquad \text{(Taylor expansion)}$$

3.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} \quad ; \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$$

4. If $f(x_0) = 0$, $g(x_0) = 0$ then $\lim_{x \to x_0} \frac{f(x_0)}{g(x_0)} = ?$

$$\because f(x) = f'(x_0) \cdot (x - x_0) + f''(x_0) \cdot \frac{(x - x_0)^2}{2!} + \cdots$$

$$g(x) = g'(x_0) \cdot (x - x_0) + g''(x_0) \cdot \frac{(x - x_0)^2}{2!} + \cdots$$

$\therefore$ if $f'(x_0)$ or $g'(x_0) \neq 0$

then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$

or if $f'(x_0)$, $g'(x_0) = 0$

then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f''(x_0)}{g''(x_0)}$ and so on

∴ L'hôpital's rule

Application:

$$\lim_{x \to 0} \frac{\sin \beta x}{x} = \lim_{x \to 0} \frac{\beta \cdot \cos \beta x}{1} = \beta$$

5. If $f = f(t)$ & $\dfrac{df}{dt} = \beta \cdot t$, then $f(t) = ?$

assume

$$f(t) = \sum_{n=0}^{\infty} a_n \cdot t^n = a_0 + a_1 \cdot t + a_2 \cdot t^2 + \cdots \qquad \text{(power series expansion)}$$

$$\rightarrow \frac{df}{dt} = a_1 + 2a_2 \cdot t + \cdots + na_n \cdot t^{n-1}$$
$$= \beta \cdot \left(a_0 + a_1 \cdot t + a_2 \cdot t^2 + \cdots\right)$$

$$\therefore a_1 = \beta \cdot a_0 \ , \quad a_2 = \beta \cdot \frac{1}{2} a_1 \ , \quad a_3 = \beta \cdot \frac{1}{3} \cdot a_2 \ , \quad \cdots$$

OR

$$a_n = \frac{\beta}{n} \cdot a_{n-1} = \beta^2 \cdot \frac{1}{n \cdot (n-1)} \cdot a_{n-2} = \cdots = \beta^n \cdot \frac{1}{n!} a_0$$

$$\therefore f(t) = \sum_{n=0}^{\infty} a_n \cdot t^n = a_0 \cdot \sum_n \frac{\beta^n}{n!} t^n = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} (\beta \cdot t)^n$$

define:

$$g(t) = e^{\beta \cdot t} = \sum_{n=0}^{\infty} \frac{(\beta \cdot t)^n}{n!} \rightarrow \text{why?}$$

$$\because g(t_1 + t_2) = g(t_1) \cdot g(t_2) \ \ \& \ g(0) = 1$$

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_n \frac{1}{n!} \left(x^n + nx^{n-1} \cdot y + \cdots + \tbinom{n}{m} x^{n-m} y^m + \cdots + y^n\right)$$
$$\tbinom{n}{m} = \frac{n \cdot (n-1) \cdots (n-m+1)}{m!}$$
$$e^x \cdot e^y = \left(\sum_n \frac{x^n}{n!}\right) \cdot \left(\sum_m \frac{y^m}{m!}\right)$$

check $x^{n-m} \cdot y^m$ term

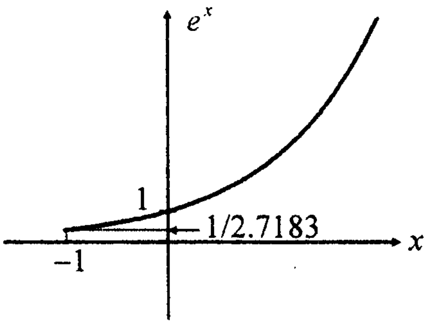
$$\rightarrow \frac{1}{(n-m)!} x^{n-m} \cdot y^m \cdot \frac{1}{m!} \Leftrightarrow \frac{n \cdot (n-1) \cdots (n-m+1)}{n! m!} = \frac{1}{m!\,(n-m)!}$$

Q.E.D.

then

$$f(t) = a_0 \cdot e^{\beta \cdot t}$$
$$f(0) = a_0 \qquad \therefore f(t) = f(0)\, e^{\beta \cdot t} \Leftrightarrow \frac{df}{dt} = \beta \cdot f(t)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \therefore e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.7183\cdots$$



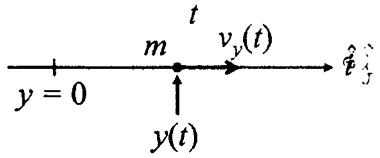
some rule
$$e^{x+y} = e^x \cdot e^y$$
$$e^{-y} = \frac{1}{e^y}$$

another definition: $e = \lim_{x\to 0} (1+x)^{\frac{1}{x}}$

6. One-dimensional motion with drag force

$$m\frac{d\vec{v}_y(t)}{dt} = -\gamma \cdot \vec{v}_y(t) = \vec{F}$$

γ : coefficient of drag



$$\therefore v_y(t) = \alpha \cdot e^{\beta \cdot t}$$

$$\Longrightarrow \beta = -\frac{\gamma}{m} \quad \text{and} \quad \alpha = v_y(0) \quad \text{initial velocity}$$

$$v_y(t) = v_y(0)\, e^{-\gamma \cdot t/m} = \frac{dy(t)}{dt}$$

$$\therefore y(t) = k + \eta \cdot e^{\zeta \cdot t} \rightarrow \frac{dy(t)}{dt} = \zeta \cdot \eta \cdot e^{\zeta \cdot t} = v_y(0)\, e^{-\gamma \cdot t/m}$$

$$\therefore \eta \cdot \zeta = v_y(0) \quad \text{and} \quad \zeta = -\frac{\gamma}{m}, \quad \eta = -\frac{m \cdot v_y(0)}{\gamma}$$

$$y(0) = k - \frac{m \cdot v_y(0)}{\gamma} \qquad \therefore k = y(0) + \frac{m \cdot v_y(0)}{\gamma}$$

$$\therefore y(t) = y(0) + \frac{m \cdot v_y(0)}{\gamma}\left(1 - e^{-\gamma \cdot t/m}\right) \xrightarrow[t\to\infty]{} y(0) + \frac{m \cdot v_y(0)}{\gamma}$$