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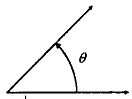
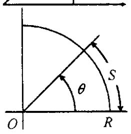
2007. 11. 10 高一數學講義 (取自 2005. 11. 12 者)

New Topics.

1. Trigonometric function

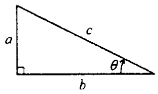


angle θ



$S = R \cdot \theta$ θ in radian

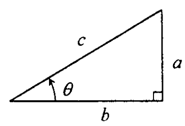
$\therefore 2\pi$ radian $= 360°$

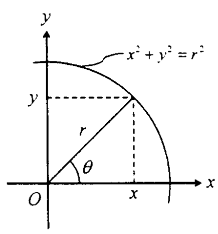


$a^2 + b^2 = c^2$ Pythagorean theorem

2. Definitions:

sine	$\sin\theta = \dfrac{a}{c} = \dfrac{y}{r}$
cosine	$\cos\theta = \dfrac{b}{c} = \dfrac{x}{r}$
tangent	$\tan\theta = \dfrac{a}{b} = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$
cotangen	$\cot\theta = \dfrac{b}{a} = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta} = \dfrac{x}{y}$
secant	$\sec\theta = \dfrac{c}{b} = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$
cosecant	$\csc\theta = \dfrac{c}{a} = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$





3. Identities:

(1). $\cos^2\theta + \sin^2\theta = 1$

$$\because \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{(a^2+b^2)}{c^2} = 1$$

$$\text{or} \quad \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{(x^2+y^2)}{r^2} = 1$$

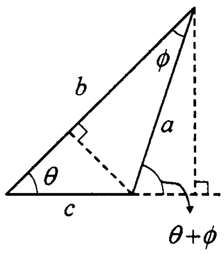
(2). $\tan^2 + 1 = \sec^2\theta$

$$\because \frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \left(\frac{1}{\cos\theta}\right)^2$$

(3). $\cot^2\theta + 1 = \csc^2\theta$

(4). $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

(5). $\cos(\theta \pm \phi) = \cos\theta \cdot \cos\phi \mp \sin\theta \cdot \sin\phi$



$$\because a \cdot \cos(\theta+\phi) = b \cdot \cos\theta - c$$

$$= (a \cdot \cos\phi + c \cdot \cos\theta) \cdot \cos\theta - c$$

$$= a \cdot \cos\theta \cdot \cos\phi - c \cdot \sin^2\theta \qquad \curvearrowleft \because \cos^2\theta - 1 = -\sin^2\theta$$

$$= a \cdot (\cos\theta \cdot \cos\phi - \sin\theta \cdot \sin\phi) \qquad \curvearrowleft \because c \cdot \sin\theta = a \cdot \sin\phi$$

(6). $\sin(\theta \pm \phi) = \sin\theta \cdot \cos\phi \pm \cos\theta \cdot \sin\phi$

$$\because a \cdot \sin(\theta+\phi) = b \cdot \sin\theta = (a \cdot \cos\phi + c \cdot \cos\theta) \cdot \sin\theta$$

$$= a \cdot \sin\theta \cdot \cos\phi + c \cdot \sin\theta \cdot \cos\theta$$

$$= a \cdot (\sin\theta \cdot \cos\phi + \cos\theta \cdot \sin\phi) \qquad \curvearrowleft \because c \cdot \sin\theta = a \cdot \sin\phi$$

or

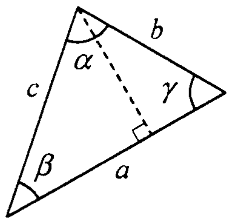
$$\because \sin(\theta+\phi) = \cos\left(\frac{\pi}{2} - (\theta+\phi)\right) = \cos\left(\frac{\pi}{2} - \theta\right) \cdot \cos\phi + \sin\left(\frac{\pi}{2} - \theta\right) \cdot \sin\phi$$

$$= \sin\theta \cdot \cos\phi + \cos\theta \cdot \sin\phi$$

(7). $a^2 = b^2 + c^2 - 2b \cdot c \cdot \cos\theta$

$$\because a^2 = (b \cdot \sin\theta)^2 + (b \cdot \cos\theta - c)^2 = b^2 \cdot \left(\sin^2\theta + \cos^2\theta\right) - 2 \cdot b \cdot c \cdot \cos\theta + c^2$$

(8). $a/\sin\alpha = b/\sin\beta = c/\sin\gamma$



$\because b\cdot\sin\gamma = c\cdot\sin\beta$

$\therefore \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$ etc.

(9). $\sin 2\theta = 2\sin\theta\cdot\cos\theta$

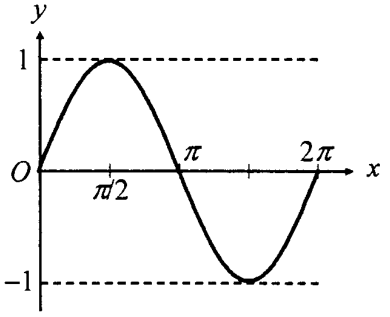
(10). $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\cdot\sin^2\theta = 2\cdot\cos^2\theta - 1$

(11). $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

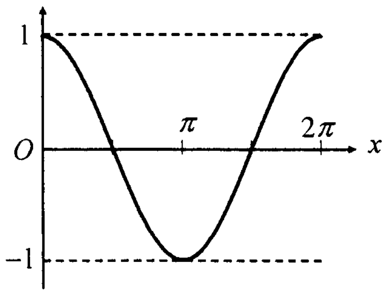
(12). $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

4. Graph

(1). $y = \sin x$



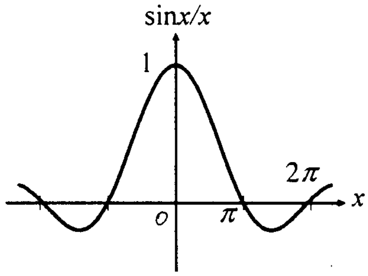
(2). $y = \cos x$



5. Limit

(1). $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(2). $\lim_{x \to 0} \frac{\sin 4x}{x} = 4$



For reference

power series expansion of $\sin\theta$ and $\cos\theta$; useful approximation for $\theta \ll 1$

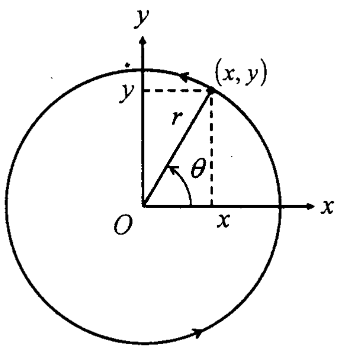
$$\sin\theta = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\theta^{2n+1}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

$$\cos\theta = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$$

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$$

6. Applications

(1). **uniform circular motion** coordinates: (x, y)



$$x = r \cdot \cos\theta$$
$$y = r \cdot \sin\theta$$
$$\theta = \omega \cdot t + \theta_0$$

$t \to$ time

$\omega = \frac{\Delta\theta}{\Delta t} \to$ angular velocity

$\theta_0 \to$ initial angle

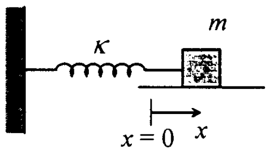
$$\therefore x(t) = r \cdot \cos(\omega t + \theta_0)$$

$$y(t) = r \cdot \sin(\omega t + \theta_0)$$

$x(0) = r \cdot \cos\theta_0$, $y(0) = r \cdot \sin\theta_0 \rightarrow$ initial position

projection on x-axis: $x(t) = r \cdot \cos(\omega t + \theta_0) \rightarrow$ simple harmonic motion-like

(2). Simple harmonic motion of the mass-spring system



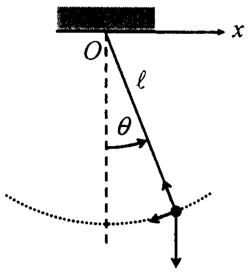
$x(t) = A \cdot \cos(\omega t + \theta_0)$

$\omega = (\kappa/m)^{\frac{1}{2}}$, k : force const. of the spring, m : mass

$A \rightarrow$ Amplitude, $\theta_0 \rightarrow$ initial phase angle

$\omega \rightarrow$ angular frequency

(3). Simple pendulum



$x = \ell \cdot \sin\theta$

$\because \theta \ll 1 \quad \sin\theta \approx \theta$

$\therefore x = \ell \cdot \theta$

$\theta = \theta(t) = \theta_A \cdot \cos(\omega t + \delta_0)$

$\therefore x = A \cdot \cos(\omega t + \delta_0)$, $A = \ell \cdot \theta_A \rightarrow$ amplitude

$x(t)$ and $\theta(t)$ are both simple harmonic functions of time.

$\omega = \left(\frac{g}{\ell}\right)^{\frac{1}{2}}$, δ_0 : initial phase angle , g : acceleration due to gravity

Vector

Lecturer: Prof. Ven-Chung Lee

(Dated: November 26, 2005)

1. Representation of vector $\vec{A}$ by basis vector $\hat{e}_i$

$$\vec{A} = \sum_{i=1}^{3} A_i \cdot \hat{e}_i \quad , \quad \hat{e}_i : \text{basis vector}, \ |\hat{e}_i| = 1$$

if $\hat{e}_1 = \hat{i}, \hat{e}_2 = \hat{j}, \hat{e}_3 = \hat{k}$ along the $x, y,$ and z axis respectively.

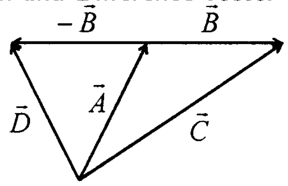
then

$$\vec{A} = A_x \cdot \hat{i} + A_y \cdot \hat{j} + A_z \cdot \hat{k}$$

A_x : projection or component of $\vec{A}$ on x-axis, etc.

$$\left|\vec{A}\right| \equiv A = \left(A_x^2 + A_y^2 + A_z^2\right)^{\frac{1}{2}} \quad \text{magnitude of the vector}$$

2. Sum and Difference vector

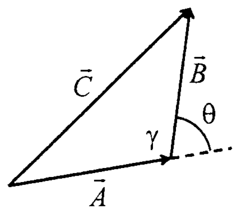


$$\vec{C} = \vec{A} + \vec{B} \ , \ \vec{D} = \vec{A} - \vec{B}$$

$$\vec{C} = \sum_{i=1}^{3} C_i \cdot \hat{e}_i = \sum A_i \cdot \hat{e}_i + \sum B_i \cdot \hat{e}_i = \sum_{i=1}^{3} (A_i + B_i) \cdot \hat{e}_i \qquad \therefore C_i = A_i + B_i$$

$$\vec{D} = \sum_i (A_i - B_i) \cdot \hat{e}_i \ , \qquad \therefore D_i = A_i - B_i$$

3. Scalar product



$$\vec{A} \cdot \vec{B} = A \cdot B \cdot \cos\theta$$

$$\left|\vec{A}\right| = \left(\vec{A} \cdot \vec{A}\right)^{\frac{1}{2}}$$

$$\left(\vec{A} + \vec{B}\right)^2 = \vec{C}^2 = C^2 = \left(\vec{A} + \vec{B}\right) \cdot \left(\vec{A} + \vec{B}\right) = A^2 + B^2 + 2\vec{A} \cdot \vec{B}$$

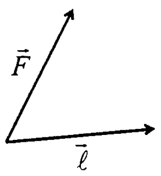
By cosine theorem $\quad A^2 + B^2 - 2A \cdot B \cdot \cos\gamma = C^2$

$\because \theta = \pi - \gamma \quad \therefore \vec{A} \cdot \vec{B} = -A \cdot B \cos\gamma = A \cdot B \cos\theta$

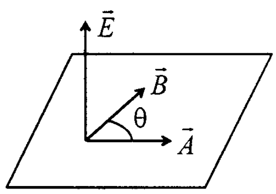
$$\begin{aligned}\vec{A} \cdot \vec{B} &= \left(\sum_i A_i \cdot \hat{e}_i\right) \cdot \left(\sum_j B_j \cdot \hat{e}_j\right) \\ &= \sum_i \sum_j A_i \cdot B_j \hat{e}_i \cdot \hat{e}_j \\ &= \sum_i A_i \cdot B_i \qquad (\because \hat{e}_i \cdot \hat{e}_j = 1 \text{ if } i = j \text{ , otherwise} = 0)\end{aligned}$$

Example: work W done by a force $\vec{F}$ along a displacement $\vec{\ell}$

$W = \vec{F} \cdot \vec{\ell}$



4. Vector product



$\vec{A} \times \vec{B} = \vec{E} \quad$ then $\quad E = A \cdot B \cdot \sin\theta$

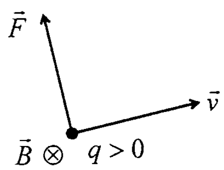
and $\vec{E} \perp \left(\text{coplane of } \vec{A} \,\&\, \vec{B}\right)$ as shown

$$\begin{aligned}\vec{E} &= \left(\sum_i A_i \cdot \hat{e}_i\right) \times \left(\sum_j B_j \cdot \hat{e}_j\right) = \sum_i \sum_j A_i \cdot B_j \hat{e}_i \times \hat{e}_j \\ &= (A_1 \cdot B_2 - A_2 \cdot B_1) \cdot \hat{e}_3 + (A_2 \cdot B_3 - A_3 \cdot B_2) \cdot \hat{e}_1 + (A_3 \cdot B_1 - A_1 \cdot B_3) \cdot \hat{e}_2\end{aligned}$$

$(\because \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \quad , \quad \hat{e}_2 \times \hat{e}_1 = -\hat{e}_3 \quad , \quad \hat{e}_1 \times \hat{e}_1 = 0 \quad , \quad \text{etc.})$

Example 1 *Force $\vec{F}$ on a q charged particle moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$*

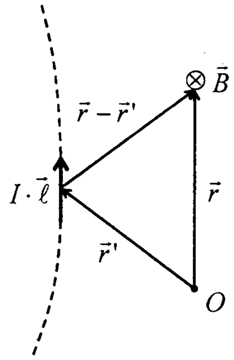
$\vec{F} = q\vec{v} \times \vec{B}$



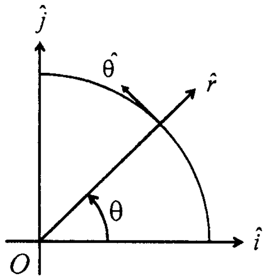
Example 2 *Magnetic field $\vec{B}$ at $\vec{r}$ generated by a current element $I \cdot \vec{\ell}$ at $\vec{r}'$*

$$\vec{B}(\vec{r}) = \frac{\mu I \cdot \vec{\ell} \times (\vec{r} - \vec{r}')}{4\pi (\vec{r} - \vec{r}')^3}$$

(Biot-Savart law)



5. Change of basis or rotation of the basis vectors



$\vec{A} = A_x \cdot \hat{i} + A_y \cdot \hat{j}$ basis $\hat{i}, \hat{j}$

$\vec{A} = A_r \cdot \hat{r} + A_\theta \cdot \hat{\theta}$ basis $\hat{r}, \hat{\theta}$

$\hat{r}$: radical vector

$\hat{\theta}$: tangent vector

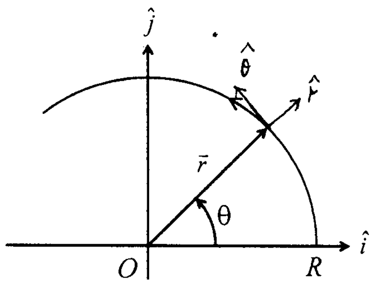
$\hat{r} = \cos\theta \cdot \hat{i} + \sin\theta \cdot \hat{j}$, $\hat{\theta} = -\sin\theta \cdot \hat{i} + \cos\theta \cdot \hat{j}$

$A_r = \vec{A} \cdot \hat{r} = A_x \cdot \cos\theta + A_y \cdot \sin\theta$, $A_\theta = \vec{A} \cdot \hat{\theta} = -A_x \cdot \sin\theta + A_y \cdot \cos\theta$

$\therefore$ Both of A_r and A_θ are linear combination of A_x , A_y

$A_r^2 + A_\theta^2 = A_x^2 + A_y^2$

Example: Uniform circular motion with radius R and angular frequency ω



position vector of the particle at time t

$\vec{r}(t) = R \cdot \hat{r}$ position vector

$\hat{r}(t) = \cos\theta(t) \cdot \hat{i} + \sin\theta(t) \cdot \hat{j}$

$\theta(t) = \omega \cdot t + \theta_0$

velocity $\vec{v} = R \cdot \omega \cdot \hat{\theta}$, centripetal acceleration $\vec{a} = -R \cdot \omega^2 \cdot \hat{r}$

$\hat{\theta}(t) = -\sin\theta(t) \cdot \hat{i} + \cos\theta(t) \cdot \hat{j}$