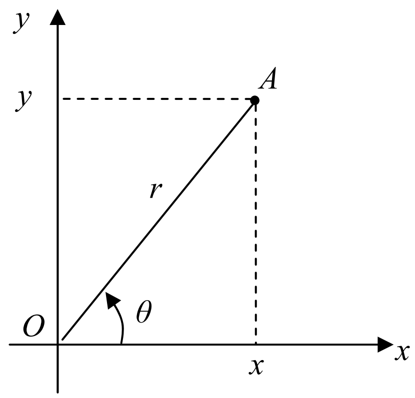
10.13 高一數學講義

1. Cartesian plane



Coordinates of the point A → (x , y) Cartesian coordinates

$\overline{OA} = (x^2 + y^2)^{\frac{1}{2}} = r$ → distance between A and the origin O

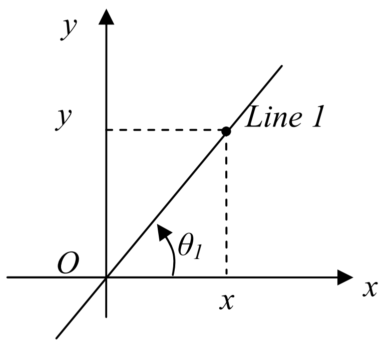
defi. $x \equiv r \cdot \cos\theta$, $y \equiv r \cdot \sin\theta$

$\therefore \cos^2\theta + \sin^2\theta = 1$

$\frac{\sin\theta}{\cos\theta} \equiv \tan\theta = \frac{y}{x}$ → slope of $\overline{OA}$ with respect to x axis

coordinates of A → (r, θ) polar coordinates

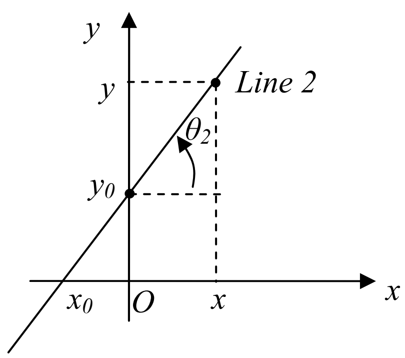
2. Equation of line



For any point on the line 1, its coordinates (x, y) satisfy the relation:

$\frac{y}{x} = \tan\theta_1 = m_1 = const.$

$\therefore y = x \cdot \tan\theta = m_1 \cdot x$



For any point on the line 2, its coordinates (x, y) satisfy the equation:

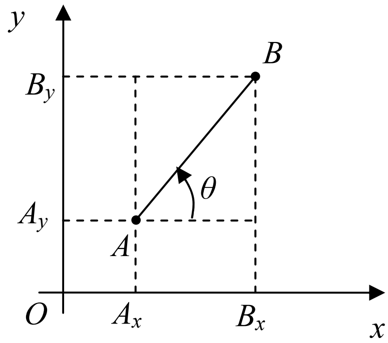
$y = y_0 + x \cdot \tan\theta_2$, y_0 →intercept of the line on y axis

or

$y = (x - x_0) \cdot \tan\theta_2$, x_0 →intercept of the line on x axis

$\therefore \tan\theta_2 = \frac{y_0}{x_0} = m_2$= slope of the line 2

3. Distance and euqtion of a line



$\overline{AB} = ((B_x - A_x)^2 + (B_y - A_y)^2)^{\frac{1}{2}}$

$A \rightarrow (A_x, A_y)$, $B \rightarrow (B_x, B_y)$

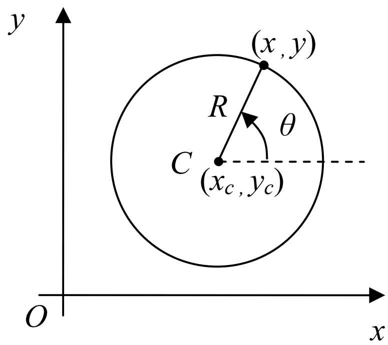
Slope of $\overline{AB} = \dfrac{B_y - A_y}{B_x - A_x} = m = \tan\theta =$ const.

For any point on the line which passes through the point A and B, its coordinates (x, y) satisfy the equation:

$$\frac{y - B_y}{x - B_x} = m = \frac{y - A_y}{x - A_x}$$

or $\begin{matrix} y - B_y = m \cdot (x - B_x) \\ y - A_y = m \cdot (x - A_x) \end{matrix}$, $(A_y - B_y) \cdot x + (B_x - A_x) \cdot y + (B_x A_y - A_x B_y) = 0$

4. Equation of a circle with (x_c, y_c) as its center

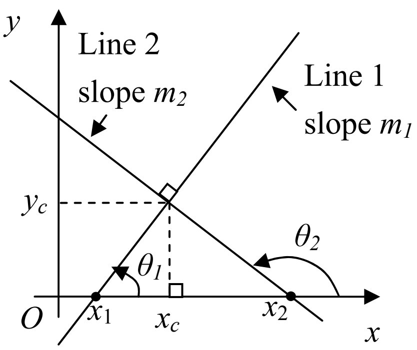


$(x - x_c)^2 + (y - y_c)^2 = R^2$, $R \rightarrow$ radius

or $\begin{matrix} x - x_c = R \cdot \cos\theta \\ y - y_c = R \cdot \sin\theta \end{matrix}$ and $\theta : 0 \rightarrow 2\pi$

$\therefore x = x_c + R \cdot \cos\theta$, $y = y_c + R \cdot \sin\theta$

5. Two lines which are perpendicular to each other



For line 1 $\dfrac{y_c}{x_c - x_1} = m_1$

For line 2 $\dfrac{y_c}{x_c - x_2} = m_2$

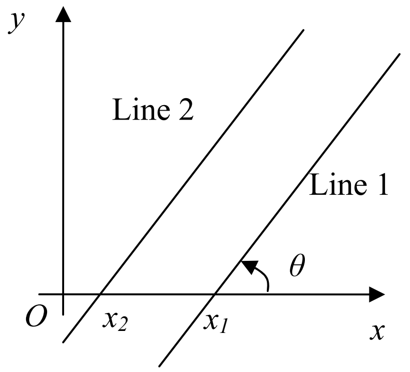
$\because \dfrac{x_2 - x_c}{y_c} = \dfrac{y_c}{x_c - x_1} = \tan\theta_1 = m_1$

$\therefore -\dfrac{1}{m_2} = m_1$ or $m_1 \cdot m_2 = -1$

note: $\because \theta_2 = \theta_1 + \frac{\pi}{2}$ $\quad \therefore \tan\theta_2 = m_2 = \tan(\theta_1 + \frac{\pi}{2}) = -\frac{1}{m_1} = -\frac{1}{\tan\theta_1}$

defi. $\cot\theta \equiv \frac{1}{\tan\theta}$ $\qquad \therefore \tan(\theta + \frac{\pi}{2}) = -\cot\theta$

6. Line which are parallel to each other have the same slope but different intercepts

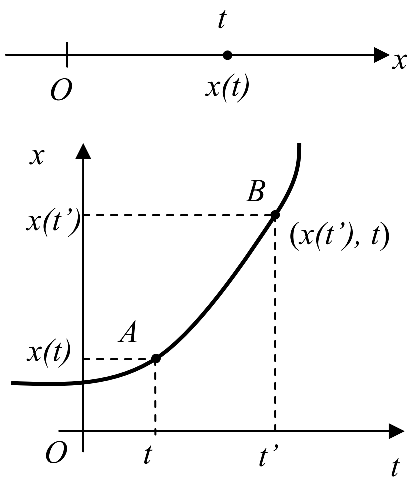


Line 1 $\quad \frac{y}{x - x_1} = \tan\theta = m$

Line 2 $\quad \frac{y}{x - x_2} = \tan\theta = m$

$\therefore y = m \cdot x - m \cdot x_i \quad , \quad i = 1,2$

7. Space-time coordinates for a mass point, linear motion



At time t, the coordinate of the mass point is $x(t)$

$A \rightarrow (x(t), t)$, $\quad B \rightarrow (x(t'), t')$, $\quad t' = t + \Delta t$

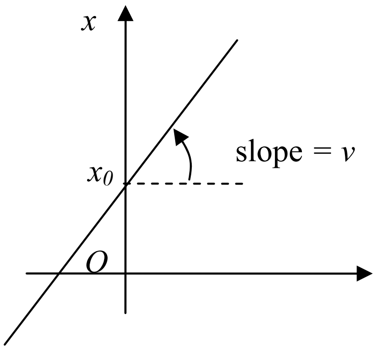
The distance between the points A and B

$$\Delta x = x(t') - x(t)$$

The slope of $\overline{AB} = \frac{x(t') - x(t)}{t' - t} \equiv \frac{\Delta x}{\Delta t}$

What is the physical meaning of this slope?

8. Linear motion with constant velocity



$x(t) = x_0 + v \cdot t$, v = constant, set $x(0) = x_0$

$\frac{\Delta x(t)}{\Delta t} \equiv \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{v\Delta t}{\Delta t} = v$ independent of Δt

$\therefore$ Slope of the line in space-time coordinates is the average velocity.

9. Linear motion with constant acceleration

$$x(t) = x_0 + v_0 \cdot t + \frac{1}{2} a \cdot t_2 \quad \rightarrow \text{position}$$

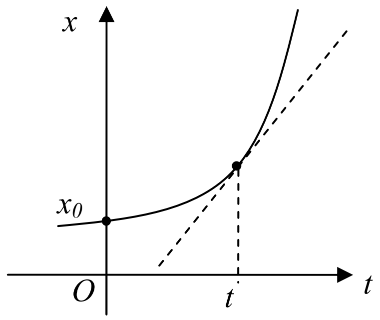
$$\rightarrow v(t) = v_0 + a \cdot t \qquad \rightarrow \text{velocity}$$

Defi. $v(t) = \lim_{\Delta t \to 0} \frac{\Delta x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$

then $$v(t) = \lim_{\Delta t \to 0} \frac{[x_0 + v_0 \cdot (t + \Delta t) + \frac{1}{2} a \cdot (\Delta t)^2] - (x_0 + v_0 \cdot t + \frac{1}{2} a \cdot t^2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(v_0 + a \cdot t) \cdot \Delta t + \frac{1}{2} a \cdot (\Delta t)^2}{\Delta t}$$

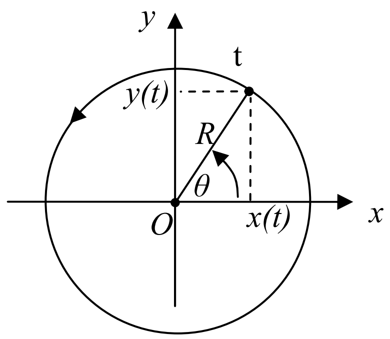
$$= \lim_{\Delta t \to 0} (v_0 + a \cdot t + \frac{1}{2} a \cdot \Delta t) = v_0 + a \cdot t$$



Slope of the tangent line at time *t* is the instaneous velocity *v(t)*.
By similar argument:

defi. $a(t) = \lim_{\Delta t \to 0} \frac{\Delta v(t)}{\Delta t}$, then *a(t)* = *a* = const.

10. Uniform circular motion, two-dimensional motion



Coordinates of the mass point at time t

$x(t) = R \cdot \cos\theta$

$y(t) = R \cdot \sin\theta$

$\theta = \theta(t) = \omega \cdot t + \theta_0$

$\therefore$ at $t = 0$ $\quad \begin{matrix} x = x_0 = R \cdot \cos\theta_0 \\ y = y_0 = R \cdot \sin\theta_0 \end{matrix}$ } initial position

$\Rightarrow \begin{matrix} x(t) = R \cdot \cos(\omega \cdot t + \theta_0) \\ y(t) = R \cdot \sin(\omega \cdot t + \theta_0) \end{matrix}$ } equation of a circle with the time *t* as parameter

$x^2(t) + y^2(t) = R^2$ for any time *t*

What is the velocity *v(t)*, and then acceleration *a(t)*?