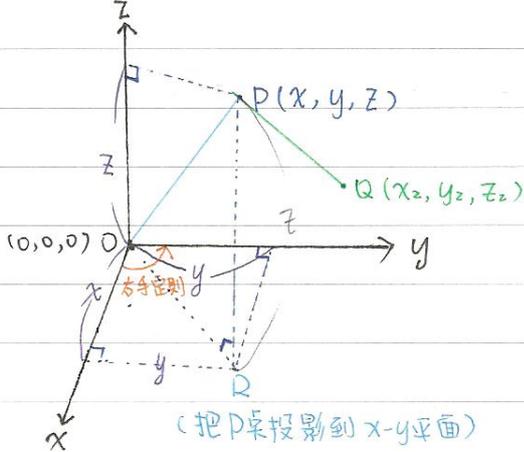


12/15

座標

Cartesian coordinates (直角座標)

1. <三度>



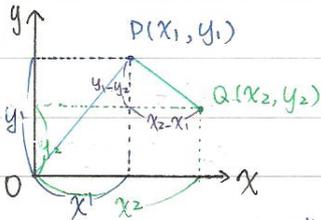
$$OR = \sqrt{x^2 + y^2}$$

$$OP = \sqrt{OR^2 + PR^2} = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \rightarrow x, y, z \text{ 到原点的距離}$$

$$PQ = \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2}$$

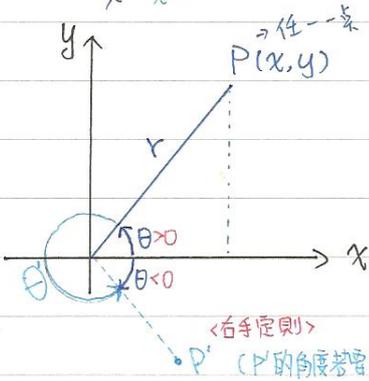
<二度>



$$OP = \sqrt{x_1^2 + y_1^2}$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2.



★ 角度(正)由 x 軸逆時針方向轉

P 點表示法 $\begin{cases} \text{Cartesian coordinates} \\ (x, y) \end{cases} -\infty < x, y < +\infty$

Polar coordinates $\begin{cases} (r, \theta) \\ \text{極座標} \end{cases} \begin{cases} 0 \leq r = \sqrt{x^2 + y^2} < +\infty \\ 0^\circ \leq \theta < 360^\circ \end{cases}$

(0° 跟 360° 是一樣)

$$(r, \theta) \rightarrow \begin{cases} y = r \sin \theta \\ x = r \cos \theta \end{cases}$$

★ 斜率 $\frac{y}{x} = \tan \theta \Rightarrow \theta = \tan^{-1}(\frac{y}{x})$

$$\theta = \pi \text{ rad} = 180^\circ$$

$$\theta = 2\pi \text{ rad} = 360^\circ$$

$$\Rightarrow \theta = \frac{S}{r} \text{ (表 "弧長度" radian)}$$

$$\star 1 \text{ rad} = \frac{180^\circ}{\pi} = 57.2957795^\circ$$

$$30^\circ = 30 \times \frac{\pi}{180} \text{ rad}$$

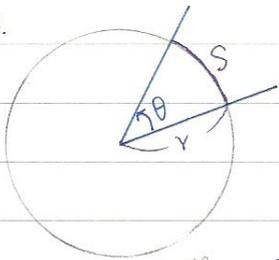
$$\star n! = 1 \times 2 \times 3 \times \dots \times n$$

$$(x^n)' = n \cdot x^{n-1}$$

$$(\sin x)' = \cos x$$

$$\cos x' = -\sin x$$

3.



$$\theta = \frac{S}{2\pi r} \times 360^\circ$$

$$\text{ex. } S = \frac{1}{8} 2\pi r \Rightarrow \theta = 45^\circ$$

$$\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(x) = \frac{x}{1} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots$$

x in rad
(x 要用弧長度)

$$\downarrow 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

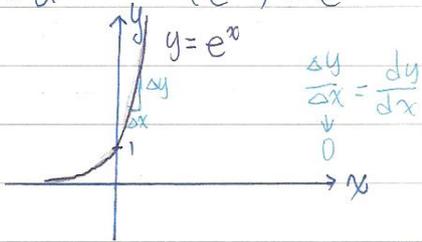
泰勒展開式

★ Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta \quad i = \sqrt{-1} \text{ (虚数)}, i^2 = -1, e = 2.718281828459\dots$$

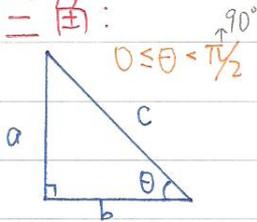
$$a^x \cdot a^y = a^{x+y}$$

$$f(x) = a^x \quad (e^x)' = e^x \Rightarrow \text{当 } a=e \text{ 时, 微分后等于本身}$$



$$★ \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \dots = \frac{2}{\pi}$$

三角:



$$\sin \theta = \frac{a}{c}$$

$$\cos \theta = \frac{b}{c}$$

$$\tan \theta = \frac{a}{b} = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{b}{a}$$

$$\sec \theta = \frac{c}{b}$$

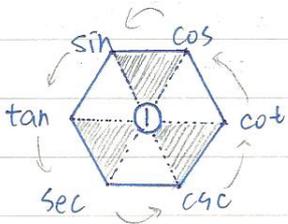
$$\csc \theta = \frac{c}{a}$$

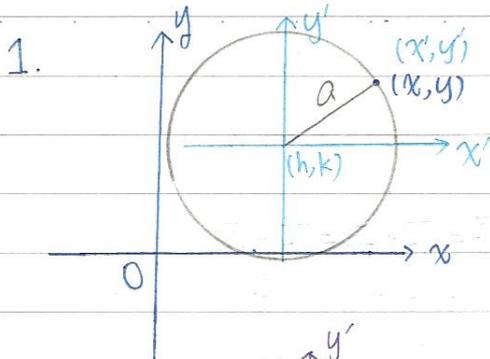
$$\frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$\frac{\tan \theta}{\sin \theta} = \sec \theta$$

$$a^2 + b^2 = c^2$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

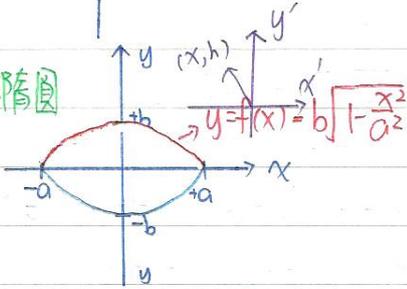




座標平移 $\begin{cases} x' = x - h \\ y' = y - k \end{cases}$

$$a^2 = (x')^2 + (y')^2 = (x-h)^2 + (y-k)^2$$

2. 橢圓



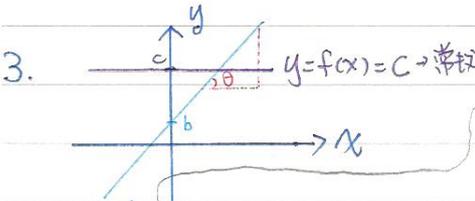
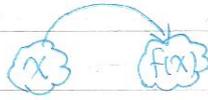
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$

$$\Rightarrow \frac{(x'+h)^2}{a^2} + \frac{(y'+k)^2}{b^2} = 1$$

Domain 定義域 Range 值域

P.S. $f(x): x \rightarrow f(x)$

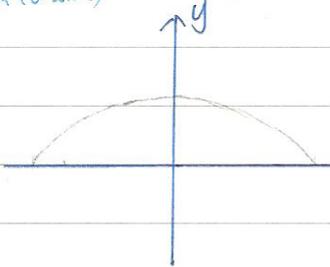


$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x} = \tan \theta = m$$

$$m(x+\Delta x) + b - (mx+b) = m\Delta x \quad \text{slope 斜率}$$

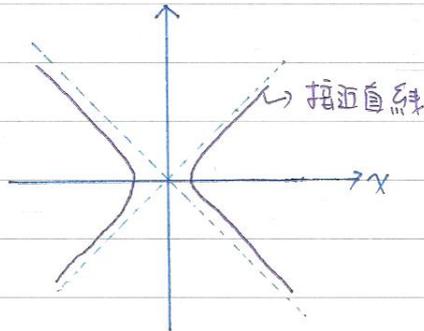
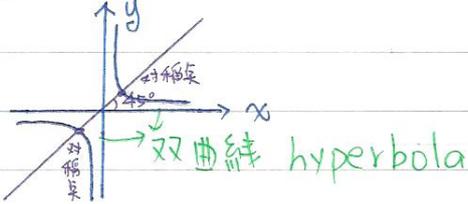
$y = mx + b$ 直線 (a line) \rightarrow y軸 $y = \infty x + b \rightarrow$ 不會這麼寫 $\rightarrow x=0$ (正確)

4.

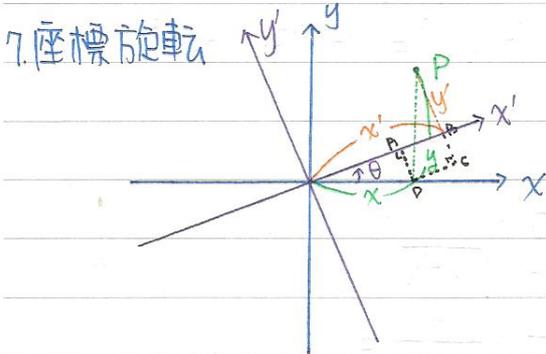
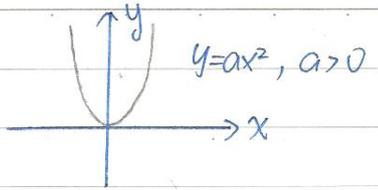


$$f: x \rightarrow b \sqrt{1 - \frac{x^2}{a^2}} \quad f(x) = b \sqrt{1 - \frac{x^2}{a^2}} \quad \begin{matrix} -a \leq x \leq a \\ 0 \leq y \leq b \end{matrix}$$

5. $y = \frac{k}{x}, k > 0$



b. 拋物綫 Parabola $y = ax^2 + bx + c$



$$x' = \overline{OB} = \overline{OA} + \overline{AB} = \overline{OA} + \overline{DC}$$

$$= x \cdot \cos \theta + y \sin \theta$$

$$y' = \overline{PB} = \overline{PC} - \overline{BC} = \overline{PC} - \overline{AD}$$

$$= y \cos \theta - x \sin \theta$$

$$(x', y') = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$$

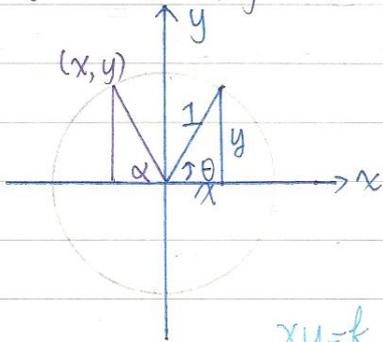
$$\begin{cases} x' = x \cos \theta + y \sin \theta \dots (1) \\ y' = -x \sin \theta + y \cos \theta \dots (2) \end{cases}$$

$$(1) \times \cos \theta - (2) \times \sin \theta \Rightarrow x' \cos \theta - y' \sin \theta = x$$

$$(1) \times \sin \theta + (2) \times \cos \theta \Rightarrow x' \sin \theta + y' \cos \theta = y$$

$$\Rightarrow \begin{cases} x = x' \cos \theta - y' \sin \theta \\ y = x' \sin \theta + y' \cos \theta \end{cases}$$

8.



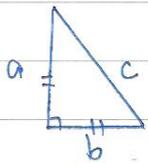
$$\begin{cases} 0 < \theta < \pi/2 \\ x = \cos \theta \\ y = \sin \theta \\ x = -\sin \alpha \\ y = \cos \alpha \end{cases}$$

$$\begin{cases} \theta > \pi/2 \\ \cos \theta = x \\ \sin \theta = y \end{cases}$$

$$\begin{cases} \sin \pi = 0 \\ \cos \pi = -1 \end{cases}$$

$$xy = k \quad (x' \cos \theta - y' \sin \theta)(-x' \sin \theta + y' \cos \theta)$$

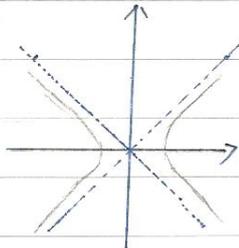
$$= -(x')^2 \cos \theta \sin \theta - (y')^2 \sin \theta \cos \theta + x'y'(\cos^2 \theta - \sin^2 \theta)$$



$$a = b \Rightarrow c = \sqrt{2}a$$

$$\theta = 45^\circ = \frac{\pi}{4} \text{ rad}$$

$$\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$



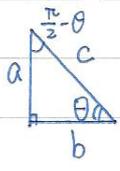
$$\frac{(x')^2}{2k} - \frac{(y')^2}{2k} = k$$

$$\Rightarrow \frac{(x')^2}{2k} - \frac{(y')^2}{2k} = 1$$

雙曲綫

17 1/2

三角函数 trigonometric functions



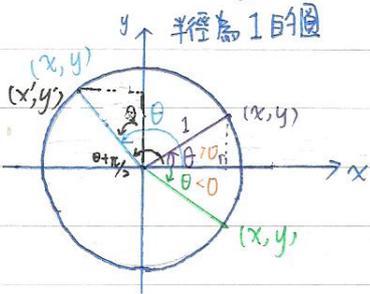
$\sin \theta = \frac{a}{c}$

$\cos \theta = \frac{b}{c}, 0 < \theta < \pi/2$

$\tan \theta = \frac{a}{b}$

$\sin(\frac{\pi}{2} - \theta) = \frac{b}{c} = \cos \theta$

$\cos(\frac{\pi}{2} - \theta) = \frac{a}{c} = \sin \theta = \cos(\theta - \pi/2)$



$0 < \theta < \pi/2$

$x = \cos \theta$

$y = \sin \theta$

For any angle θ

$\cos \theta = x$

$\sin \theta = y$

$n \in \mathbb{Z}$ (整数)

$\sin(\theta + 2n\pi) = \sin \theta$

$\cos(\theta + 2n\pi) = \cos \theta$

週期 = 2π

$\sin(-\theta) = -\sin \theta$ < 奇函数 >

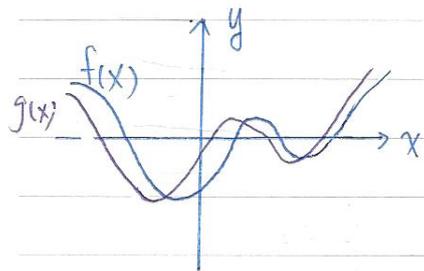
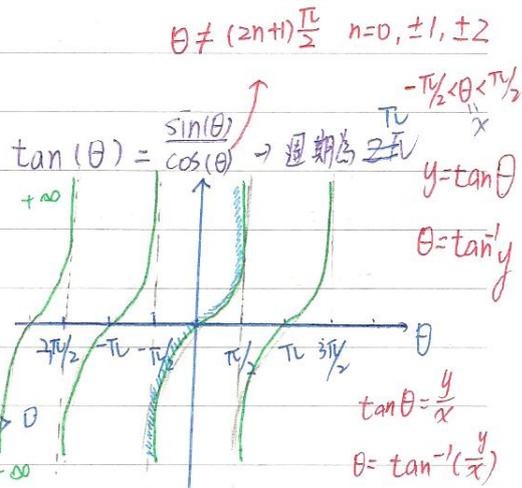
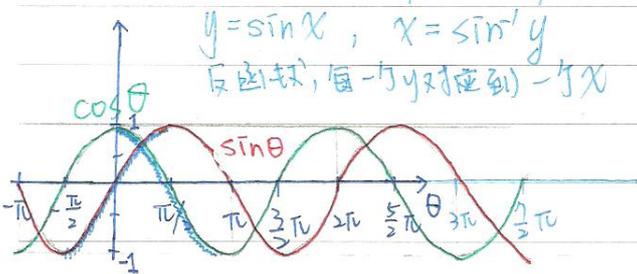
$\cos(-\theta) = \cos \theta$ < 偶函数 >

$\theta \rightarrow \theta + \pi/2$

$-x = y$

$y = \sin(\theta + \pi/2) = x = \cos \theta$

$x = \cos(\theta + \pi/2) = -y = -\sin \theta$



$g(x) = f(x+a)$
 $a > 0$
 $x_0 \rightarrow x_0 - a$

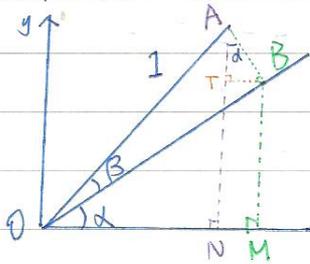
三角反函数

$$y = \arcsin x \equiv \sin^{-1} x \Leftrightarrow x = \sin y \quad -\pi/2 \leq y \leq \pi/2$$

$$y = \arccos x \equiv \cos^{-1} x \Leftrightarrow x = \cos y \quad 0 \leq y \leq \pi$$

$$y = \arctan x \equiv \tan^{-1} x \Leftrightarrow x = \tan y \quad -\pi/2 < y < \pi/2$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



$$\triangle OAN \Rightarrow \sin(\alpha + \beta) = \overline{AN} = \overline{AT} + \overline{TN} = \overline{AT} + \overline{BM}$$

$$\triangle OBM \Rightarrow \overline{BM} = \sin \alpha \cdot \overline{OB}$$

$$\triangle OAB \Rightarrow \overline{OB} = \cos \beta$$

$$\overline{BM} = \sin \alpha \cdot \cos \beta \quad \dots \textcircled{1}$$

$$\frac{\overline{AT}}{\overline{AB}} = \cos \alpha \quad \frac{\overline{AB}}{\overline{OA}} = \sin \beta \Rightarrow \overline{AT} = \cos \alpha \cdot \sin \beta \quad \dots \textcircled{2}$$

$$\sin(\alpha + \beta) = \overline{AT} + \overline{BM} = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\cos(\alpha + \beta) = \frac{\overline{ON}}{\overline{OA}} = \overline{ON} = \overline{OM} - \overline{NM} = \cos \alpha \cdot \overline{OB} - \overline{TN} = \cos \alpha \cdot \overline{OB} - \sin \alpha \cdot \overline{AB}$$

$$= \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\beta = \alpha$$

$$\sin(2\alpha) = 2 \sin \alpha \cdot \cos \alpha$$

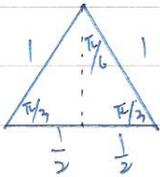
$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

$$\cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}}$$

$$\sin \theta = 0 \quad \text{if } \theta = 0$$

$$\cos \theta = 1 \quad \text{if } \theta = 0$$



$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

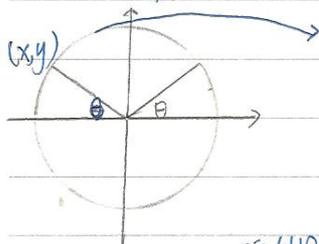
$$\cos \frac{\pi}{3} = \frac{1}{2}$$

$$\sin \frac{\pi}{6} = \frac{1}{2}$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta \quad \beta = \alpha \Rightarrow \sin(2\alpha) = 2\sin\alpha \cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \quad \beta = \alpha \Rightarrow \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$



$$\cos(\pi - \theta) = -\cos\theta$$

$$\sin(\pi - \theta) = \sin\theta$$

$$= 2\cos^2\alpha - 1 \quad \cos\alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

$$= 1 - 2\sin^2\alpha \quad \sin\alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}}$$

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha \cos\beta + \cos\alpha \sin\beta}{\cos\alpha \cos\beta - \sin\alpha \sin\beta} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\begin{cases} \cos 0 = 1 \\ \sin 0 = 0 \end{cases}$$

$$\begin{cases} \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \\ \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \end{cases}$$

$$\begin{cases} \cos \frac{\pi}{2} = 0 \\ \sin \frac{\pi}{2} = 1 \end{cases}$$

$$\begin{cases} \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \sin \frac{\pi}{6} = \frac{1}{2} \end{cases}$$

$$\begin{cases} \cos \frac{\pi}{3} = \frac{1}{2} \\ \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \end{cases}$$

$$\sin \frac{\pi}{12} = \sin(\frac{\pi}{3} - \frac{\pi}{4}) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$$

$$\text{若 } \beta = \alpha \Rightarrow \sin(2\alpha) = 2\sin\alpha \cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\cos\alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

$$\sin\alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}}$$

$$\text{EX. } \sin \frac{\pi}{12} = \frac{1}{2}, \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sqrt{1 - \sqrt{3}} = 1 - \sqrt{3}$$

$$\sqrt{1 - \sqrt{3}} = -1 + \sqrt{3}$$

$$\sin \frac{\pi}{12} = \sqrt{\frac{1 - \cos \frac{\pi}{6}}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2} = \frac{1}{2\sqrt{2}} \sqrt{4 - 2\sqrt{3}} = \frac{1}{2\sqrt{2}} \sqrt{(\sqrt{3} - 1)^2} = \frac{1}{2\sqrt{2}} (\sqrt{3} - 1)$$

$$\cos \frac{\pi}{2^3} = \frac{\sqrt{2}}{2}$$

$$\cos \frac{\pi}{2^3} = \frac{1}{2} \sqrt{2 + 2\cos \frac{\pi}{2^2}} = \frac{1}{2} \sqrt{2 + \sqrt{2}}$$

$$\frac{1}{2} \sqrt{2 + 2\cos \frac{\pi}{2^2}}$$

$$\cos \frac{\pi}{2^4} = \frac{1}{2} \sqrt{2 + 2\cos \frac{\pi}{2^3}} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

$$\Rightarrow \cos \frac{\pi}{2^n} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$$

$$\sin\alpha = 2\cos \frac{\alpha}{2} \sin \frac{\alpha}{2} = 2^2 \cos \frac{\alpha}{2} \cos \frac{\alpha}{2^2} \sin \frac{\alpha}{2^2} = 2^3 \cos \frac{\alpha}{2} \cos \frac{\alpha}{2^2} \cos \frac{\alpha}{2^3} \sin \frac{\alpha}{2^3} \dots$$

$$= 2^n \cos \frac{\alpha}{2} \cos \frac{\alpha}{2^2} \cos \frac{\alpha}{2^3} \dots \cos \frac{\alpha}{2^n} \sin \frac{\alpha}{2^n}$$

$$\alpha = \frac{\pi}{2} \quad \sin \frac{\pi}{2} = 1$$

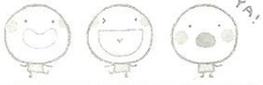
$$\Rightarrow 2^n \cos \frac{\pi}{2^2} \cos \frac{\pi}{2^3} \dots \cos \frac{\pi}{2^{n+1}} \sin \frac{\pi}{2^{n+1}} \approx \frac{\pi}{2^{n+1}} \Rightarrow \frac{2}{\pi} \approx \cos \frac{\pi}{2^2} \cos \frac{\pi}{2^3} \dots \cos \frac{\pi}{2^{n+1}}$$

$$\sin \theta \approx \theta, \quad \theta \ll 1$$

$$\theta = \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$

$$\approx (\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{2+\sqrt{2}}) \dots (\frac{1}{2}\sqrt{2+\sqrt{2+\dots}})$$





$n=1$	$\frac{\pi}{2} \approx \frac{\sqrt{2}}{2}$	$\pi \approx 2\sqrt{2}$
$n=2$		$\pi \approx 3.06$
$n=3$		$\pi \approx 3.12$
$n=5$		$\pi \approx 3.140$
$n=10$		$\pi \approx 3.141591$
$n=50$		$\pi \approx 3.14159265358979 \dots (31位)$

微分

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \dots$$

$$f'(x) = \frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{d \sin x}{dx} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \left\{ \frac{\sin x \cosh - \cos x \sinh - \sin x}{h} \right\} = \cos x$$

$$\frac{d \cos x}{dx} = \lim_{h \rightarrow 0} \left\{ \frac{\cos(x+h) - \cos x}{h} \right\} = \lim_{h \rightarrow 0} \left\{ \frac{\cos x \cosh - \sin x \sinh}{h} \right\} = \lim_{h \rightarrow 0} \left\{ \frac{\cos x [\cosh - 1] - \sin x \sinh}{h} \right\} = -\sin x$$

* Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

* 複數 $a+bi$, a, b 是實數, $i = \sqrt{-1}$

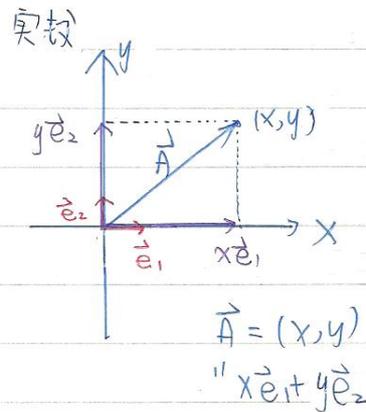
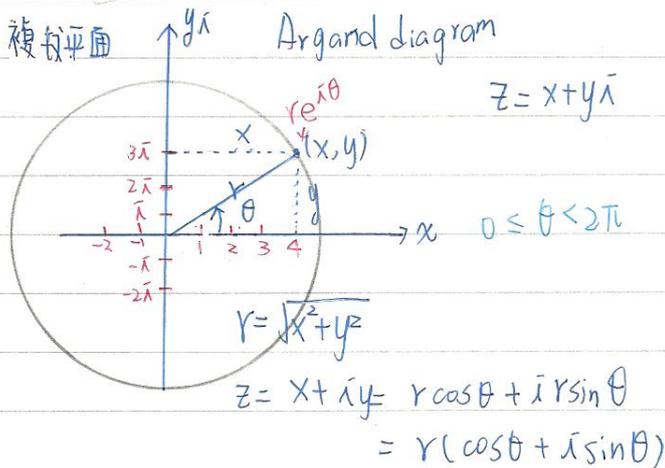
$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad b^2 - 4ac \geq 0 \Rightarrow \text{實根}$$

$$\text{if } \frac{b^2 - 4ac < 0}{-1(4ac - b^2)} \quad \left. \vphantom{\frac{b^2 - 4ac < 0}{-1(4ac - b^2)}} \right\} \text{ then } x = \frac{-b}{2a} \pm \sqrt{4ac - b^2} \cdot i = \frac{-b}{2a} \pm \sqrt{4ac - b^2} i$$

$$z = x + yi = x + yi \quad \text{ex. } x^2 + 1 = 0 \Rightarrow x = \pm i \quad \boxed{i^2 = -1}$$

$$S = \sum_{i=1}^n \frac{1}{x^2}$$



↗ Cartesian form

$$z = x + iy \quad \rightarrow \text{polar form}$$

$$= r e^{i\theta} = r(\cos \theta + i \sin \theta)$$

$$\text{Re}(z) = x \quad \text{Im}(z) = y$$

↑ real ↑ imaginary

$$e^{0i} = 1$$

$$e^{i\pi/4} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$$

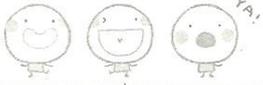
$$= \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}$$

$$e^{i\pi/2} = i$$

$$e^{i\pi} = -1$$

$$e^{i3\pi/2} = -i$$

$$e^{2\pi i} = 1$$



$$\bar{z} = z^* = x - iy \quad z = x + iy$$

$$z_1 = a + bi$$

$$+) z_2 = c + di$$

$$z_1 + z_2 = (a+c) + (b+d)i$$

$$z_1 = a + bi \leftarrow \text{Cartesian form}$$

$$-) z_2 = c + di \leftarrow \text{Cartesian form}$$

$$z_1 - z_2 = (a-c) + (b-d)i$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \quad \text{Polar form VS Cartesian form}$$

$$z_1 \times z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$z = r e^{i\theta}$$

$$z^n = (r e^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

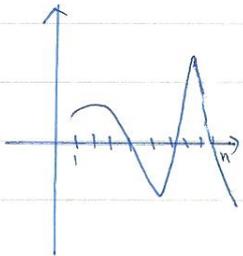
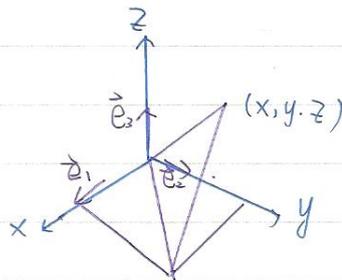
$$z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$$

$$= r_1 (\cos \theta_1 + i \sin \theta_1) + r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y^2 = r^2$$

$$\vec{A} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$$

$$\vec{A} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

↓
n

⇒ 以含权表示向量

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$$

大小 modulus

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$$= \cos \theta + i \sin \theta$$

$$\frac{3+i}{1-2i} = \frac{2}{5} + \frac{7}{5}i$$

$$\hookrightarrow r_2 e^{i\theta_2}$$

$$x^2 + 1 = 0 \Rightarrow x^2 = -1 \quad x = \pm \sqrt{-1} = \pm i$$

$$x^2 - 1 = 0 \Rightarrow x = 1 \quad \text{or} \quad \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$(x-1)(x^2+x+1) = 0$$

實係數 \Rightarrow 必為共軛虛根

$$x^n - 1 = 0$$

$$x^n = 1 = e^{i2k\pi} = e^{i2k\pi/n} \quad k = 0, 1, 2, 3, \dots, n-1 \quad (\because n \text{ 又會有重複})$$

$$x = e^{i2k\pi/n} \quad k = 0, 1, 2, \dots, n-1$$

$$-1, -2, -3, \dots, n-1$$

$$x^3 - 1 = 0$$

$$x = \left\{ e^{i2k\pi/3} \right\}_{k=0,1,2} = \left\{ 1, e^{i2\pi/3}, e^{i4\pi/3} \right\}$$

$$\begin{aligned} (x)^{1/3} &= e^{i\pi/2 + i2k\pi} \\ &= e^{i(\pi/6 + 2k\pi/3)} \end{aligned}$$

$$k=0, e^{i2k\pi} = 1$$

$$\neq k=0$$

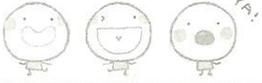
$$\begin{aligned} e^{i\pi/6} &= \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad k=0 \end{aligned}$$

$$z = r e^{i\theta} \quad \bar{z} = r e^{-i\theta}$$

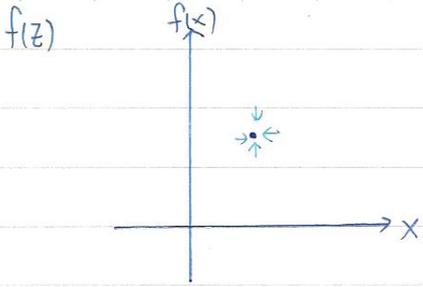
$$z = a + bi \quad \bar{z} = a - bi$$

$$|z| = \sqrt{a^2 + b^2}$$

Give me five!!



Date ♥ ♥



Derivative 導函數 (定義連續)

$$f'(x) \equiv \frac{df(x)}{dx} = \lim_{h \rightarrow 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

$$\Delta f(x) = f(x+\Delta x) - f(x) \quad \frac{\Delta f(x)}{\Delta x} \xrightarrow{\Delta x \rightarrow 0} \frac{df(x)}{dx}$$

$$h \equiv \Delta x$$

$$h \rightarrow 0 \quad \sinh \rightarrow h$$

$$\Delta \sin x = \sin(x+h) - \sin x = \sin x \cdot \cosh + \cos x \cdot \sinh$$

$$\cosh \rightarrow 1 + \frac{a_1 h + a_2 h^2}{h \rightarrow 0}$$

$$(\sin x)' = \cos x$$

$$\cos(x) = \cos x \text{ (偶函數)} \quad \cos x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\sin(-x) = -\sin x \text{ (奇函數)} \quad \sin x = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots$$

$$\begin{cases} (\sin x)' = \cos x \\ (\cos x)' = -\sin x \end{cases}$$

$$\frac{\cos(x+h) - \cos x}{h} = \frac{1}{h} \{ \cos x \cosh - \sin x \sinh - \cos x \}$$

$$f(x) = x^n \quad f(x) = x^2$$

$$\Delta f = (x+h)^2 - x^2 = \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x$$

$$f(x) = x^n$$

$$(x+h)^n = (x+h)(x+h) \dots (x+h) = x^n + n x^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} h^3 + \dots$$

$$f(x) = e^x \quad e = 2.7182818284590 \dots$$

$$\left. \begin{cases} f(x) = e^x \\ f(0) = 1 \end{cases} \right\} \text{定義}$$

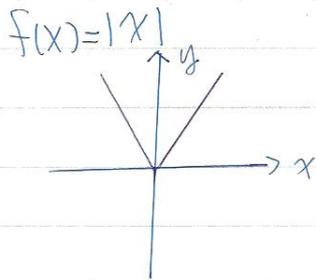
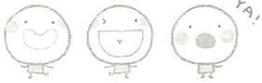
$$(\sin x)'' = (\cos x)' = -\sin x$$

$$(\cos x)'' = (-\sin x)' = -\cos x$$

$$\{f(x) + g(x)\}' = \lim_{h \rightarrow 0} \left\{ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right\}$$

$$\{ \alpha f(x) + \beta g(x) \}' = \lim_{h \rightarrow 0} \left\{ \frac{\alpha f(x+h) - \alpha f(x)}{h} + \frac{\beta g(x+h) - \beta g(x)}{h} \right\}$$

$$\alpha f'(x) + \beta g'(x)$$



$$(|x|)' = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$x=0$ 無導函數

$$y = e^x, \quad x = \ln y = \log_e y$$

$$\ln(e^x) = x \quad e^{\ln x} = x$$

$$\frac{df(x)}{dx} \cdot \frac{dx}{dy} = f'(x) \lambda$$

$$(2^x)' = (e^{\ln 2^x})' = (e^{x \ln 2})' = (e^{x \ln 2}) \cdot \ln 2$$

$$(2^x)' = 2^x \cdot \log_2$$

$$(3^x)' = 3^x \cdot \log_3$$

$$(e^x)' = e^x (\log_e)$$

$$\begin{cases} g(x) = x^2 + 1 \\ f(g(x)) = 2(x^2 + 1)^3 - 1 \end{cases}$$

$$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} = 6(g(x))^2 (2x+1) = 6(x^2+1)^2 (2x+1)$$

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \left\{ \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \right\} = \frac{df(g(x))}{dg(x)} g'(x)$$

$$(e^{kx})' = e^{kx} \cdot \frac{dkx}{dx} = k e^{kx}$$

$$(x^x)' = (e^{\ln(x^x)})' = (e^{x \ln x})' = (e^{x \ln x}) (\ln x + 1)$$

$$(x^x)' = x^x (\ln x + 1)$$

$$\begin{cases} y = \ln x \\ e^y = x \end{cases} \left\{ \frac{dx}{dy} = e^y = x \right\} \frac{dy}{dx} = \frac{1}{x}$$

$$\text{ex, } f(x) = (x^2+1)^{20} (x-1)^{80}$$

$$f'(x) = 20(x^2+1)^{19} \cdot 2x(x-1)^{80} + (x^2+1)^{20} \cdot 80(x-1)^{79} \cdot 1$$

$$(x^n)' = n x^{n-1}$$

$$[\cos(kx)]' = -k \sin(kx)$$

$$(e^x)' = e^x$$

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(e^{kx})' = k e^{kx}$$

$$(x \ln x)' = (x)' \ln x + x (\ln x)'$$

$$[\sin(kx)]' = k \cos(kx)$$

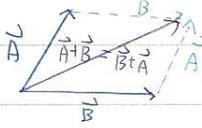
$$\text{三項相乘 } (fgh)' = f'gh + fg'h + fgh' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\begin{aligned} (f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\ \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \end{aligned}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\frac{\sin x}{x^2+1} = (\sin x) \left(\frac{1}{x^2+1}\right)$$

向量与矩阵 Vectors & Matrices.



$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ 交换律

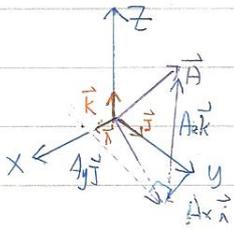


$(-1)\vec{A} = -\vec{A}$ ← A 的反向量

$\vec{A} + (-\vec{A}) = \vec{0}$

$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$ 结合律

$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$, $(\alpha\beta)\vec{A} = \alpha(\beta\vec{A})$ 结合律 $(\alpha + \beta)\vec{A} = \alpha\vec{A} + \beta\vec{A}$ 分配律

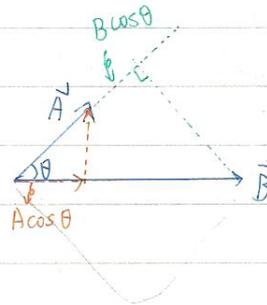


$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$

① $\vec{A} \cdot \vec{B} = AB \cos\theta = \vec{B} \cdot \vec{A}$

$\alpha \cdot \|\vec{A}\| \equiv A$ (向量的大小)

点积 dot product
内积 inner product



$\vec{i} \cdot \vec{j} = 0 = \vec{j} \cdot \vec{i}$

$\vec{j} \cdot \vec{k} = 0 = \vec{k} \cdot \vec{j}$

$\vec{k} \cdot \vec{i} = 0 = \vec{i} \cdot \vec{k}$

② $(\alpha\vec{A}) \cdot \vec{B} = \alpha(\vec{A} \cdot \vec{B})$ $(\alpha\vec{A} + \beta\vec{B}) \cdot \vec{C} = \alpha\vec{A} \cdot \vec{C} + \beta\vec{B} \cdot \vec{C}$

其他项乘起来为0

$\vec{A} \cdot \vec{B} = (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \cdot (B_x \vec{i} + B_y \vec{j} + B_z \vec{k}) = A_x B_x \vec{i} \cdot \vec{i} + A_y B_y \vec{j} \cdot \vec{j} + A_z B_z \vec{k} \cdot \vec{k}$
 $= A_x B_x + A_y B_y + A_z B_z$

③ $\vec{A} \cdot \vec{A} \geq 0$ $\|\vec{A}\|^2 = \vec{A} \cdot \vec{A}$

$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$ (单位向量 $\Rightarrow 1 \times 1 = 1$)

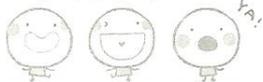
$\vec{A} = \alpha \vec{i} + \beta \vec{j} + \gamma \vec{k}$ (三者要在不同的平面上)

$\vec{A} \rightarrow \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ $\vec{B} \rightarrow \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$

(A_x, A_y, A_z)

矩阵

$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ C_{m1} & C_{m2} & & C_{mn} \end{pmatrix}$



$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times (-1) \\ 3 \times 1 + 4 \times (-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

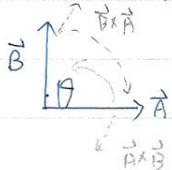
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 5 \\ -1 & 6 & 13 \end{pmatrix}$$

(1) 行的数 = 列的数 才可相乘

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$$

$$\vec{A} \times \vec{B} = (AB \sin \theta) \hat{n} \quad \hat{n} \cdot \vec{A} = \hat{n} \cdot \vec{B} = 0 \Rightarrow \hat{n} \perp \vec{A} \text{ and } \hat{n} \perp \vec{B}$$

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B}) \quad \langle \text{不可交换} \rangle$$



$$(\alpha \vec{A} + \beta \vec{B}) \times \vec{C} = \alpha \vec{A} \times \vec{C} + \beta \vec{B} \times \vec{C} \quad \text{分配律}$$

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \times (B_x \vec{i} + B_y \vec{j} + B_z \vec{k}) \\ &= A_x \vec{i} \times B_x \vec{i} + A_x \vec{i} \times B_y \vec{j} + A_x \vec{i} \times B_z \vec{k} + A_y \vec{j} \times B_x \vec{i} + \\ &\quad A_y \vec{j} \times B_y \vec{j} + A_y \vec{j} \times B_z \vec{k} + A_z \vec{k} \times B_x \vec{i} + A_z \vec{k} \times B_y \vec{j} + A_z \vec{k} \times B_z \vec{k} \\ &= (A_y B_z - A_z B_y) \vec{i} + (A_z B_x - A_x B_z) \vec{j} + (A_x B_y - A_y B_x) \vec{k} \end{aligned}$$

$$\vec{i} \times \vec{i} = 0$$

$$\vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$$

$$\vec{j} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$$

$$\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$$

$$\vec{j} \times \vec{j} = 0$$

$$\vec{k} \times \vec{k} = 0$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{行列式}$$

2x2 矩阵 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2x2 行列式 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

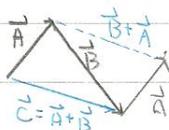
$$\vec{A} = \sum_{i=1}^3 A_i \vec{e}_i \rightarrow A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3 \rightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

张量 $\vec{T} = \sum_{ij} T_{ij} \vec{e}_i \vec{e}_j$

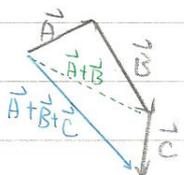
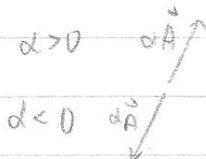
Vectors 向量

matrices 矩陣

R.S.
 純量: 距離, 溫度
 向量: 位移

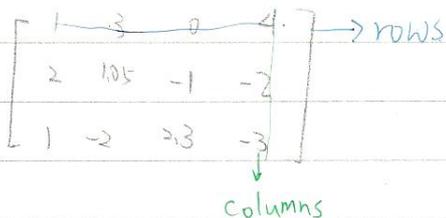


$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ 交換律



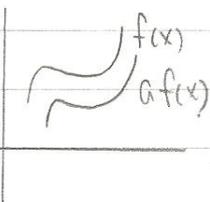
$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$ 結合律

矩陣



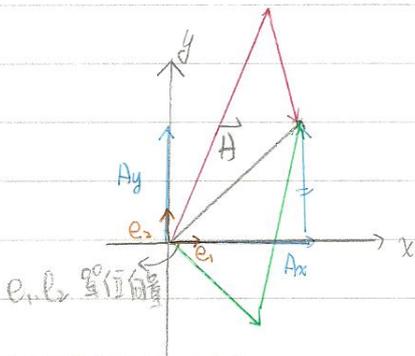
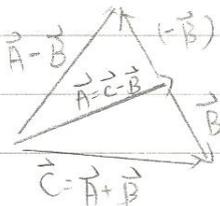
$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $\alpha A = \begin{bmatrix} \alpha 1 & \alpha 2 \\ \alpha 3 & \alpha 5 \end{bmatrix}$

函數



$(-1)\vec{A} = -\vec{A}$

$(\vec{A}) - (\vec{B}) = \vec{A} + (-\vec{B})$



$A_x \times \vec{e}_1 + A_y \times \vec{e}_2 = \vec{A} = (A_x, A_y)$

$\vec{e}_1 \perp \vec{e}_2$

$\|\vec{e}_1\| = \|\vec{e}_2\| = 1$ (長度)

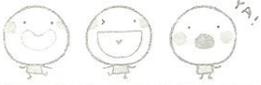
$\|\vec{A}\| = \text{大小} = \sqrt{A_x^2 + A_y^2}$

\vec{A}

分量 Components

A_x, A_y

純量



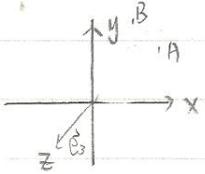
$$\vec{A} = A_x \vec{e}_1 + A_y \vec{e}_2$$

$$\vec{B} = B_x \vec{e}_1 + B_y \vec{e}_2$$

$$\vec{A} + \vec{B} = (A_x + B_x) \vec{e}_1 + (A_y + B_y) \vec{e}_2$$

$$\alpha \vec{A} + \beta \vec{A} = (\alpha + \beta) \vec{A}$$

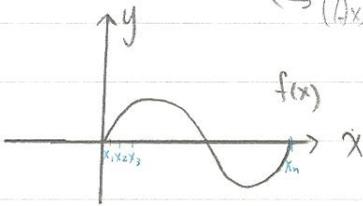
$$\vec{A} \leftrightarrow \begin{bmatrix} A_x \\ A_y \end{bmatrix} \quad \vec{B} \leftrightarrow \begin{bmatrix} B_x \\ B_y \end{bmatrix} \quad \vec{A} + \vec{B} \leftrightarrow \begin{bmatrix} A_x + B_x \\ A_y + B_y \end{bmatrix}$$



$$\vec{A} + \vec{B} = (A_x + B_x) \vec{e}_1 + (A_y + B_y) \vec{e}_2 + (A_z + B_z) \vec{e}_3$$

$$\vec{A} \leftrightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \vec{B} \leftrightarrow \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

$\leftrightarrow (A_x, A_y, A_z)$



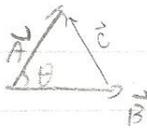
$$|f_n(x)| = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_n) \end{bmatrix}$$

$$\vec{A} \cdot \vec{B} = \|\vec{A}\| \|\vec{B}\| \cos \theta = AB \cos \theta$$

内積
(点積)
Inner product
dot

$$\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2} \equiv A$$

$$\|\vec{B}\| = \sqrt{B_x^2 + B_y^2 + B_z^2} \equiv B$$



$$\vec{A} = \vec{B} + \vec{C} \quad \vec{A} \cdot \vec{B}$$

$$C^2 = A^2 + B^2 - 2AB \cos \theta$$

$$(\vec{A} - \vec{B})^2 = B_x^2 + B_y^2 + B_z^2 - 2(A_x B_x + A_y B_y + A_z B_z)$$

$$\rightarrow (A_x - B_x)^2 + (A_y - B_y)^2 + (A_z - B_z)^2 - 2(A_x B_x + A_y B_y + A_z B_z)$$

$$\vec{A} \cdot \vec{B} = AB \cos \theta = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A} \cdot \vec{B} = 0 \Rightarrow \vec{A} = \vec{0} \text{ or } \vec{B} = \vec{0}$$

$$\quad \quad \quad \perp \vec{A} \perp \vec{B}$$

向量

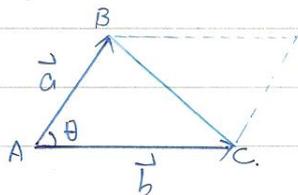
$$\cdot \text{積} \quad \vec{a} \cdot \vec{b} = ab \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{純量}$$

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

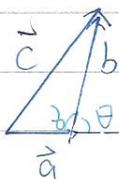
$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\times \text{積} \quad \vec{a} \times \vec{b} = (ab \sin \theta) \hat{n} \quad \text{向量}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



$$|\vec{a} \times \vec{b}| = ab \sin \theta$$



$$(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{c}$$

$$a^2 + 2\vec{a} \cdot \vec{b} + b^2 = c^2$$

$$\Rightarrow c^2 = a^2 + b^2 - 2ab \cos \theta$$

$$0 = (\vec{a} + \vec{b}) \times (\vec{a} + \vec{b})$$

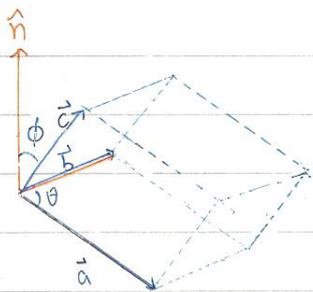
$$= \underbrace{\vec{a} \times \vec{a}}_0 + \underbrace{\vec{a} \times \vec{b} + \vec{b} \times \vec{a}}_0 + \underbrace{\vec{b} \times \vec{b}}_0$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} \cdot \vec{c} = \vec{b} \times \vec{c} \cdot \vec{a} = \vec{c} \times \vec{a} \cdot \vec{b}$$

$$= \vec{a} \cdot \vec{b} \times \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a} = \vec{c} \cdot \vec{a} \times \vec{b}$$

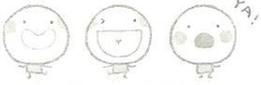
$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



$\vec{a} \times \vec{b} \cdot \vec{c}$ = the volume of the parallelepiped

$$(\vec{a} \times \vec{b}) \cdot \vec{c}$$

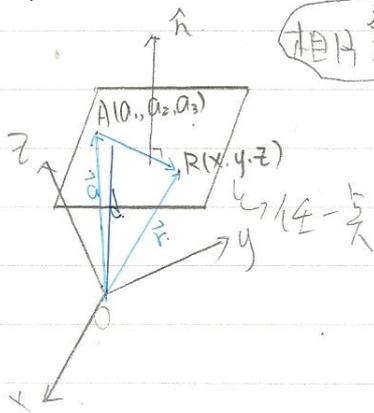
$$= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$



Find the volume of the parallelepiped with sides $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 4\hat{i} + 5\hat{j} + 6\hat{k}$, $\vec{c} = 7\hat{i} + 8\hat{j} + 9\hat{k} \Rightarrow 0$

Find the area of the parallelepiped with sides \vec{a} , \vec{b}

area = $|\vec{a} \times \vec{b}| = |-3\hat{i} + 6\hat{j} - 3\hat{k}| = \sqrt{(-3)^2 + 6^2 + (-3)^2} = \sqrt{54}$ $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$



~~相切~~

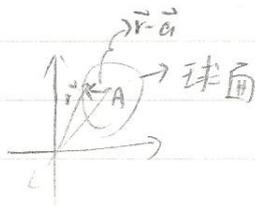
$(\vec{r} - \vec{a}) \cdot \hat{n} = 0$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$

$\vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n} = d$

$lx + my + nz = d \Rightarrow$ 平面方程式



向量

$\vec{a} \cdot \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \equiv [a, b, c]$$

$$= (\vec{a} \times \vec{b}) \cdot \vec{c} = [b, c, a] = [c, a, b]$$

$\vec{v} \equiv \vec{a} \times (\vec{b} \times \vec{c}) = \lambda \vec{b} + \mu \vec{c} = \gamma (\vec{a} \cdot \vec{c}) \vec{b} - \gamma (\vec{a} \cdot \vec{b}) \vec{c}$

$\neq (\vec{a} \times \vec{b}) \times \vec{c}$

$\vec{a} \cdot \vec{v} = 0 = \lambda \vec{a} \cdot \vec{b} + \mu \vec{a} \cdot \vec{c} = 0$

$\frac{\lambda}{\vec{a} \cdot \vec{c}} = -\frac{\mu}{\vec{a} \cdot \vec{b}} = \gamma$

$\gamma = 1$

pf $\gamma = 1$

令 $\vec{b} = \vec{a}$

$\Rightarrow \vec{a} \times (\vec{a} \times \vec{c}) = \gamma \{ (\vec{a} \cdot \vec{c}) \vec{a} - (\vec{a} \cdot \vec{a}) \vec{c} \}$

$\vec{c} \cdot [\vec{a} \times (\vec{a} \times \vec{c})] = \vec{c} \cdot \gamma \{ (\vec{a} \cdot \vec{c}) \vec{a} - (\vec{a} \cdot \vec{a}) \vec{c} \}$

$\parallel = \gamma \{ (\vec{a} \cdot \vec{c})^2 - (\vec{a} \cdot \vec{a}) \vec{c} \cdot \vec{c} \}$

$= -(\vec{a} \times \vec{c}) \cdot (\vec{a} \times \vec{c}) = -a^2 c^2 \sin^2 \theta = \gamma \{ a^2 c^2 \cos^2 \theta - a^2 c^2 \}$

$= -\gamma a^2 c^2 (1 - \cos^2 \theta) \sin^2 \theta$

$\sin^2 \theta = \gamma \cdot \sin^2 \theta \Rightarrow \gamma = 1$

(1) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(2) $\vec{a} \cdot (\alpha \vec{b} + \beta \vec{c}) = \alpha \vec{a} \cdot \vec{b} + \beta \vec{a} \cdot \vec{c}$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$

$-\vec{c} \times (\vec{a} \times \vec{b}) = -(\vec{c} \cdot \vec{b}) \vec{a} + (\vec{c} \cdot \vec{a}) \vec{b}$

$\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ basis $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

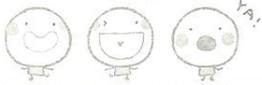
$\vec{v} = \sum_{i=1}^3 v_i \vec{e}_i = v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3$

$= \sum_{j=1}^3 v_j \vec{e}_j$

$\vec{e}_i \cdot \vec{v} = \vec{e}_i \cdot \left(\sum_{j=1}^3 v_j \vec{e}_j \right) = \sum_{j=1}^3 v_j \vec{e}_i \cdot \vec{e}_j = \sum_{j=1}^3 v_j \delta_{ij} = v_i$

$\vec{v} = \sum_{i=1}^3 (\vec{e}_i \cdot \vec{v}) \vec{e}_i = \sum_{i=1}^3 \vec{e}_i \vec{e}_i \cdot \vec{v} = \left(\sum_{i=1}^3 \vec{e}_i \vec{e}_i \right) \cdot \vec{v}$

$\vec{I} \cdot \vec{v} = \vec{v} \quad \vec{v} \cdot \vec{I} = \vec{v}$



$$\vec{v} \rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \vec{e}_1 \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{e}_3 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3$$

$$\rightarrow v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} \rightarrow [a_1 \ a_2 \ a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\vec{b} \equiv |b\rangle \rightarrow \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \langle b| \rightarrow [b_1 \ b_2 \ b_3]$$

$$\vec{a} \cdot \vec{b} = \langle a | b \rangle \quad * \langle \rangle \text{bracket}$$

$$= [a_1 \ a_2 \ a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$|v\rangle = v_1 |e_1\rangle + v_2 |e_2\rangle + v_3 |e_3\rangle$$

$$\langle e_i | v \rangle = v_1 \langle e_i | e_1 \rangle + v_2 \langle e_i | e_2 \rangle + v_3 \langle e_i | e_3 \rangle$$

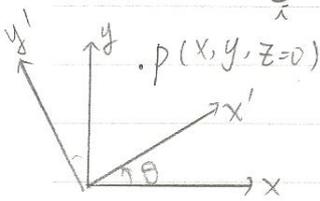
$$v_i = \langle e_i | v \rangle \quad \vec{e}_i \cdot \vec{e}_i$$

$$|v\rangle = \sum_i v_i |e_i\rangle$$

$$= \sum_i \langle e_i | v \rangle |e_i\rangle = \sum_i |e_i\rangle \langle e_i | v \rangle = I |v\rangle$$

$$I \equiv \sum_{i=1}^3 |e_i\rangle \langle e_i| = \sum_{i=1}^3 \vec{e}_i \cdot \vec{e}_i$$

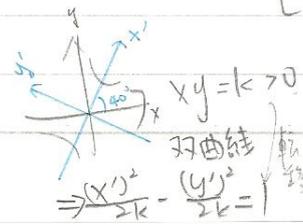
$$\begin{aligned} ax^2 + 2bxy + cy^2 &= P \\ \downarrow \\ \tilde{a}x'^2 + \tilde{c}y'^2 &= \tilde{P} \end{aligned}$$



$$\begin{aligned} |e_1\rangle \langle e_1| & |e_2\rangle \langle e_2| & |e_3\rangle \langle e_3| \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

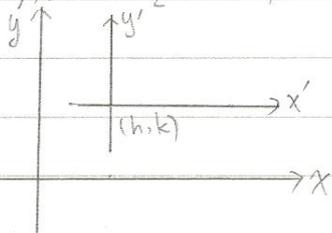
$$\begin{aligned} x' &= x \cos\theta + y \sin\theta \\ y &= -x \sin\theta + y \cos\theta \\ z' &= z \end{aligned}$$



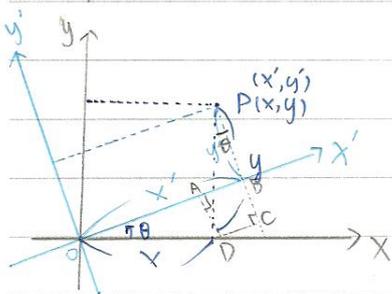
$$\begin{aligned} x' &= \frac{x+y}{\sqrt{2}} & x &= \frac{x'+y'}{\sqrt{2}} \\ y &= \frac{-x+y}{\sqrt{2}} & y &= \frac{x'-y'}{\sqrt{2}} \end{aligned}$$

矩陣

$y = k/x \rightarrow \frac{x^2}{2} - \frac{y^2}{2} = 1$



$\left. \begin{matrix} x' = x - h \\ y' = y - k \end{matrix} \right\} \text{平行 (translation)}$



$x' = \overline{OB} = \overline{OA} + \overline{AB} = \overline{OA} + \overline{DC}$
 $= \overline{OD} \cdot \cos\theta + \overline{PD} \cdot \sin\theta$
 $= x \cdot \cos\theta + y \cdot \sin\theta$

$y' = \overline{PB} = \overline{PC} - \overline{BC} = y \cdot \cos\theta - x \cdot \sin\theta = -x \cdot \sin\theta + y \cdot \cos\theta$
 $= \overline{PC} - \overline{AD}$

$\left\{ \begin{matrix} x' = x \cos\theta + y \sin\theta \\ y' = -x \sin\theta + y \cos\theta \end{matrix} \right\} \text{旋轉 (rotation)}$

$A_{n \times n} \times A_{n \times n}^{-1} = I = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\left\{ \begin{matrix} x = x' \cos\theta - y' \sin\theta \\ y = x' \sin\theta + y' \cos\theta \end{matrix} \right.$

$AB = BA = I \Leftrightarrow B = A^{-1}$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ or } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$
 $\equiv R \begin{pmatrix} x \\ y \end{pmatrix}$

註: $\alpha B = 0 \Rightarrow \alpha = 0 \vee B = 0$

矩陣 $AB = 0 \Rightarrow A = 0 \vee B = 0$ ex $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow A \neq 0, B \neq 0$
 $= 0$

$BA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$

行列式

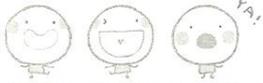
$|A| \neq 0 \rightarrow A^{-1} \text{ exists}$

$AB = 0, |A| \neq 0$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} / |A| = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} / (ad - bc)$

$\Rightarrow A^{-1} A \cdot B = A^{-1} 0 = 0$

$\Rightarrow B = 0$



$$\begin{pmatrix} x \\ y \end{pmatrix} = R^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$R^{-1} = R^T$$

$$R^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R^T$$

反矩阵 转置

R is an orthogonal matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$V = \begin{pmatrix} x \\ y \end{pmatrix} \quad V' = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad V' = R(\theta)V$$

$$V'' = R(\theta_2)V' = R(\theta_2)R(\theta_1)V = R(\theta_1 + \theta_2)V \quad R(\theta_2) = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

$$R(\theta_1) \cdot R(\theta_2) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$

$$F(x, y) = Ax^2 + 2Bxy + Cy^2 = A(x'\cos\theta - y'\sin\theta)^2 + 2B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2$$

$$= (x')^2 [A\cos^2\theta + 2B\sin\theta\cos\theta + C\sin^2\theta] + (y')^2 [A\sin^2\theta - 2B\sin\theta\cos\theta + C\cos^2\theta] + x'y' [2B\cos 2\theta + (C-A)\sin 2\theta]$$

$$\text{令 } 2B\cos 2\theta + (C-A)\sin 2\theta = 0$$

$$F(x, y) = J(x', y') = \alpha(x')^2 + \beta(y')^2$$

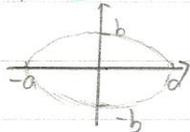
可推得

$$F(x, y) = xy = k$$

$$A=0=C \quad B=\frac{1}{2} \Rightarrow \frac{(x')^2}{2} - \frac{(y')^2}{2} = k$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

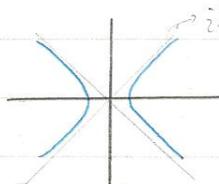
$$a > 0, b > 0$$



橢圓

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

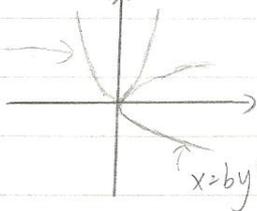
$$a > b > 0$$



雙曲綫

$$y = ax^2$$

$$a > 0$$



$x = by^2$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} / |A|$$

$$C_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \cdot (-1)^{2+1}$$

$\begin{matrix} \uparrow & \uparrow \\ i & j \end{matrix}$

$$AA^{-1} = I$$

$$Ax_i = e_i \quad Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Ax = b \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[A \mid \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right] \rightarrow \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \mid A^{-1} \right]$$

積分

Integration

Integral = anti-derivative 導関数Indefinite Integral 不定積分 $F(x) = \int f(x) dx = \frac{dF(x)}{dx} = f(x)$

$$G(x) = \int f(x) dx$$

$$= \frac{d(F(x)+C)}{dx}$$

 $F(x) - G(x) = C = \text{constant}$ 常数

$$f(x) = 2e^{\lambda x} \sin(ax) + 3x^2/(x^2+1)$$

$$f(x) = \sin(\cos x) \quad f'(x) = \cos(\cos x) \times (-\sin x)$$

$$\frac{d}{dx} \{ \alpha f(x) + \beta g(x) \} = \alpha \frac{df(x)}{dx} + \beta \frac{dg(x)}{dx}$$

$$\int \{ \alpha f(x) + \beta g(x) \} = \alpha \int f(x) dx + \beta \int g(x) dx$$

Linear

$$\frac{d}{dx} [f(x)]^a = \frac{d}{df(x)} [f(x)]^a \frac{df(x)}{dx} = a [f(x)]^{a-1} f'(x)$$

$$\int [f(x)]^a f'(x) dx = \frac{[f(x)]^{a+1}}{a+1} + C, \quad a \neq -1$$

$$\int \frac{1}{f(x)} f'(x) dx = \ln |f(x)| + C \quad \ln = \log_e$$

$$\frac{\ln x}{x^s} \xrightarrow{x \rightarrow \infty} 0 \quad \ln x \xrightarrow{x \rightarrow \infty} \infty$$

$$I = \int \frac{2x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1) + C$$

$$f(x) = x^2+1$$

$$f'(x) = 2x$$

$$\int \frac{x}{(x^2+1)^2} dx = \frac{1}{2} \int \frac{f'(x)}{(f(x))^2} dx = \frac{1}{2} \int \frac{1}{(x^2+1)^2} dx = \frac{1}{2} \int u^{-2} du = \frac{1}{2} \frac{1}{(x^2+1)} + C$$

 $u = \cos x$ change of variable

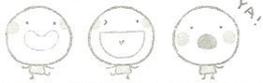
$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = - \int \frac{u'}{u} dx = - \int \frac{du}{u} = - \ln |u| + C =$$

$$u' = \frac{du}{dx}$$

$$= - \ln |\cos x| + C$$

$$du = u dx \quad \Delta f = f(x+\Delta x) - f(x) \Rightarrow df$$

 $\Delta x \rightarrow 0$



(1) change of variables $x \rightarrow u = u(x)$

$$\frac{d \ln|x|}{dx} = \frac{d \ln|x|}{|x|} \cdot \frac{d|x|}{dx} = \frac{1}{|x|} \begin{pmatrix} +1 \\ -1 \end{pmatrix} = \frac{1}{x}$$

$$\int x \sin(x^2) dx = \frac{1}{2} \int \sin u du = -\frac{1}{2} \cos u + C$$

$$\stackrel{\wedge}{=} u = x^2 \quad = -\frac{1}{2} \cos(x^2) + C$$

$$du = 2x dx$$

$$\int \sqrt{\tan x} dx = \frac{1}{2\sqrt{2}} \left\{ -2 \tan^{-1}(-\sqrt{2 \tan x}) + 2 \tan^{-1} \sqrt{2 \tan x} + \ln(\tan x - \sqrt{2 \tan x} + 1) - \ln(\tan x + \sqrt{2 \tan x} + 1) \right\} + C$$

$$\int x e^{-x^2} dx = \frac{1}{2} \int e^{-u} du = -\frac{e^{-u}}{2} + C = -\frac{e^{-x^2}}{2} + C$$

(2) Integration by parts 部分積分

$$f(x) = u(x) v(x)$$

$$\frac{df(x)}{dx} = u'(x) v(x) + u(x) v'(x)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \frac{[u(x+h) - u(x)]v(x+h) + u(x)[v(x+h) - v(x)]}{h}$$

$$\xrightarrow{h \rightarrow 0} v'(x)v(x) + u(x)u'(x)$$

$$\star \int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx$$

$$\Rightarrow \int u dv = uv - \int v du$$

$$\int x e^x dx = x e^x - \int (x)' v dx = x e^x - \int e^x dx = x e^x - e^x + C = (x-1)e^x + C$$

$$u = x, v' = e^x \Rightarrow v = e^x$$

$$de^x = e^x dx$$

$$u = e^x \quad v = x$$

$$\downarrow$$

$$v = \frac{x^2}{2}$$

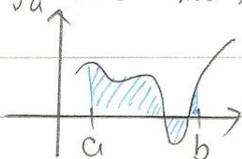
$$I = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x dx$$

積分

$$\begin{aligned}
 I &= \int e^x \sin ax \, dx \quad u = \sin ax \quad v = e^x \\
 &= \int \sin ax \, de^x = (\sin ax) e^x - \int \cos ax \cdot e^x \, dx = (\sin ax) e^x - a \int (\cos ax) e^x \, dx \\
 &= \sin(ax) e^x - \cos(ax) e^x - a^2 \int e^x \sin(ax) e^x \, dx \\
 I &= \frac{1}{a^2+1} [\sin(ax) - \cos(ax)] e^x
 \end{aligned}$$

定積分

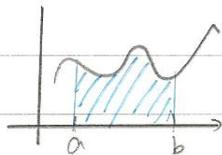
$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i$$



$$\Delta x_i = x_{i+1} - x_i$$

$$\Delta x = \max(x_1 - x_0, x_2 - x_1, \dots)$$

$$\xi_i \in [x_{i+1}, x_i]$$



$$y(x) = \int_a^x f(t) \, dt$$

$$\frac{y(x+\Delta x) - y(x)}{\Delta x} \xrightarrow{\Delta x \rightarrow 0} f(x)$$

$$y(x) = \int_a^x f(t) \, dt \quad \frac{dy(x)}{dx} = f(x)$$

$$F(x) = \int f(x) dx$$

↓
anti-derivative

Definite integral 定積分

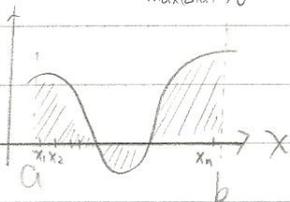
$$\int_a^b f(x) dx = \lim_{\max(\Delta x_i) \rightarrow 0} \sum_{i=1}^n (\xi_i) \cdot \Delta x_i$$

$$\Delta x_i = x_i - x_{i-1}$$

$$\xi_i \in [x_{i-1}, x_i]$$

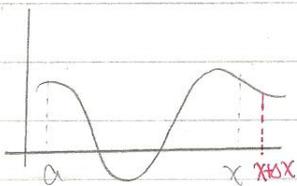
$$F'(x) = f(x)$$

$$\frac{dF(x)}{dx} = f(x)$$



$$dy(x) = \int_a^x f(\tilde{x}) d\tilde{x}$$

$$dy(x) = \int_a^{x+\Delta x} f(\tilde{x}) d\tilde{x}$$



$$\lim_{\Delta x \rightarrow 0} \frac{dy(x+\Delta x) - dy(x)}{\Delta x} \stackrel{\text{f(x)dx}}{=} dy(x) = f(x)$$

任意反導函数

$$\int_a^x f(x) dx = F(x) + C$$

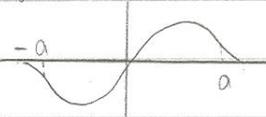
$$\int_a^b f(x) dx = F(b) + C$$

$$\text{set } x=a = F(a) + C = \int_a^a f(x) dx = 0$$

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{the fundamental theorem of integral calculus}$$

$$F(x) \Big|_a^b \equiv [F(x)]_a^b$$

$$-f(x) = f(-x)$$



$$\int_{-a}^a f(x) dx = 0 \quad \text{if } f(-x) = -f(x) \quad \forall x \in [-a, a] \quad \text{奇函数}$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f(x) = f(-x) \quad \forall x \in [-a, a] \quad \text{偶函数}$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$\Delta x > 0$ $\Delta x < 0$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$c \in (a, b)$

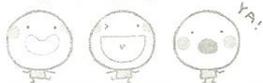
$$\int_a^b [\alpha f(x) + \beta g(x)] dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx \quad \Rightarrow \text{线性关系}$$

$$\int \cos kx dx = \frac{1}{k} \sin kx + C$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \quad n \neq -1$$

$$\int \sin kx dx = -\frac{1}{k} \cos kx + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$



$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\ln|\cos x| + C$$

$$\int \frac{f(x)}{f(x)+1} \, dx = \int \frac{d(f+1)}{f+1} = \ln|f(x)+1| + C$$

$$f(x) = \sqrt{\tan x} \quad f'(x) = \frac{1}{2} \frac{1}{\sqrt{\tan x}} \sec^2 x$$

積分技巧 (1) 部份積分 (2) change of variables (3) 部份分式

$$\int u'v \, dx = uv - \int u'v \, dx = \int u'v \, dx = uv - \int v \, du$$

$$(uv)' = u'v + uv'$$

$$dv = v' \, dx$$

$$\frac{dv}{dx} = v'$$

(2)

$$\int \frac{x \, dx}{x^2+1} = \frac{1}{2} \int \frac{1}{x^2+1} d(x^2+1) = \frac{1}{2} \ln|x^2+1| + C$$

$$I = \int \frac{1}{x^2+1} \, dx \quad I = \int \frac{\sec^2 \theta \, d\theta}{\sec^2 \theta} = \int d\theta = \theta + C$$

$$x = \tan \theta \quad \theta = \arctan x$$

$$dx = \sec^2 \theta \, d\theta$$

$$\frac{d \tan \theta}{d \theta} = \frac{d \sin \theta}{d \theta \cos \theta} = \frac{\cos^2 \theta}{\cos^2 \theta} - \frac{\sin \theta}{\cos^2 \theta} (-\sin \theta) = \frac{1}{\cos^2 \theta}$$

$$(3) \int \frac{1+x+x^2}{(x-1)^3} \, dx = 3 \int \frac{1}{(x-1)^3} \, dx + 3 \int \frac{1}{(x-1)^2} \, dx + 3 \int \frac{1}{(x-1)} \, dx = -\frac{3}{2} \frac{1}{(x-1)^2} - \frac{3}{(x-1)} + \ln|x-1| + C$$

$$f(x) = \frac{1+x+x^2}{(x-1)^3} = \frac{a_0}{(x-1)^3} + \frac{a_1}{(x-1)^2} + \frac{a_2}{(x-1)} = \frac{3}{(x-1)^3} + \frac{3}{(x-1)^2} + \frac{1}{(x-1)}$$

$$I_1 = \int \frac{1+x+x^2}{x^2(x-1)^2} \, dx = -\frac{1}{x} + 3 \ln|x| - \frac{3}{(x-1)} + 3 \ln|x-1| + C$$

$$\frac{a_0}{x^2} + \frac{a_1}{x} + \frac{b_0}{(x-1)^2} + \frac{b_1}{(x-1)}$$

$$= \frac{1}{x^2} + \frac{3}{x} + \frac{3}{(x-1)^2} - \frac{3}{x-1}$$