# Nonlinear Responses of Chiral Fluids from Kinetic Theory

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in collaboration with
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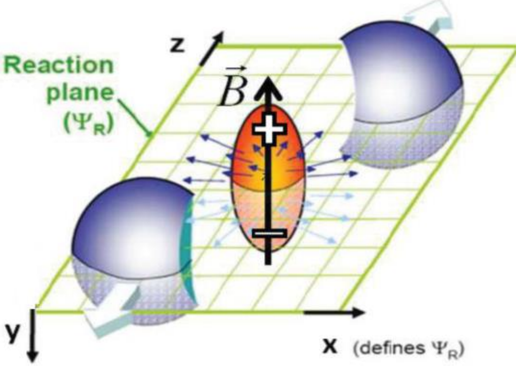
Phys.Rev. D95 (2017) no.9, 091901, arXiv:1612.04630.
arXiv:1710.00278.

# Search for anomalous effects

- Quantum effects associated with anomalies and spins for Weyl fermions :
- Chiral magnetic/vortical effects : CME : $\mathbf{J} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B}$ CVE : $\mathbf{J} = \frac{1}{\pi^2}\mu_5 \mu\, \boldsymbol{\omega}$

## Heavy ion collisions ($m_q \ll T$):

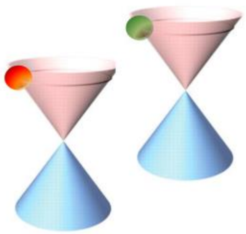


- strong B fields from collisions
- local $n_5$ from topological excitations :

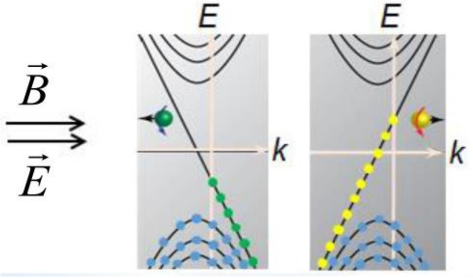
$$Q = \frac{g^2}{32\pi^2}\int d^4x\, F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a$$

- CME signal “might be measurable“ from 3-pt correlations
- strong background : under debate

## Weyl semimetals :

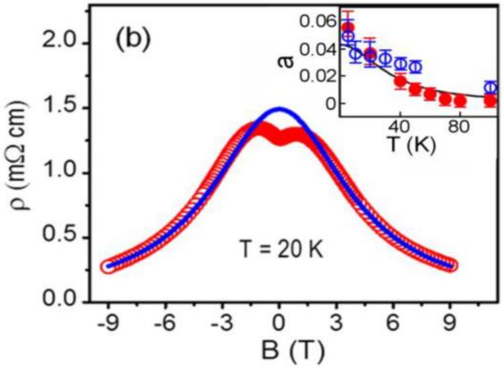


TaAs
NbAs
NbP
TaP



charge pumping via parallel E & B : generate $\mu_5$

$\rho \sim 1/\sigma$



“negative magnetoresistance” (the signal of CME)

Qiang Li, et.al., Nature Phys. 12 (2016) 550-554

# Kinetic theory with quantum anomalies

- Various approaches for studying anomalous transport : hydrodynamics, Kubo formulae, lattice , AdS/CFT, etc.
- The chiral kinetic theory (CKT) : to investigate anomalous transport in and "out of" equilibrium and to manifest the microscopic dynamics.
- Validity : rare collisions (weakly coupled systems)

- The semi-classical approach :

D. T. Son and N. Yamamoto, Phys. Rev. Lett. 109,181602 (2012)
M. Stephanov and Y. Yin, Phys. Rev. Lett. 109, 162001 (2012)

  - Introducing a Berry phase to characterize the quantum $\mathcal{O}(\hbar)$ corrections from anomalies for Weyl fermions.

- Field-theory derivations : equilibrium or with a large chemical potential

J.-W. Chen, et.al. Phys. Rev. Lett. 110, 262301 (2013)    D. T. Son and N. Yamamoto, Phys. Rev. D87, 085016 (2013)

(no collisions: sufficient for equilibrium cases)

- Lorentz symmetry : side jumps

J.-Y. Chen, et.al. Phys. Rev. Lett. 113, 182302 (2014)

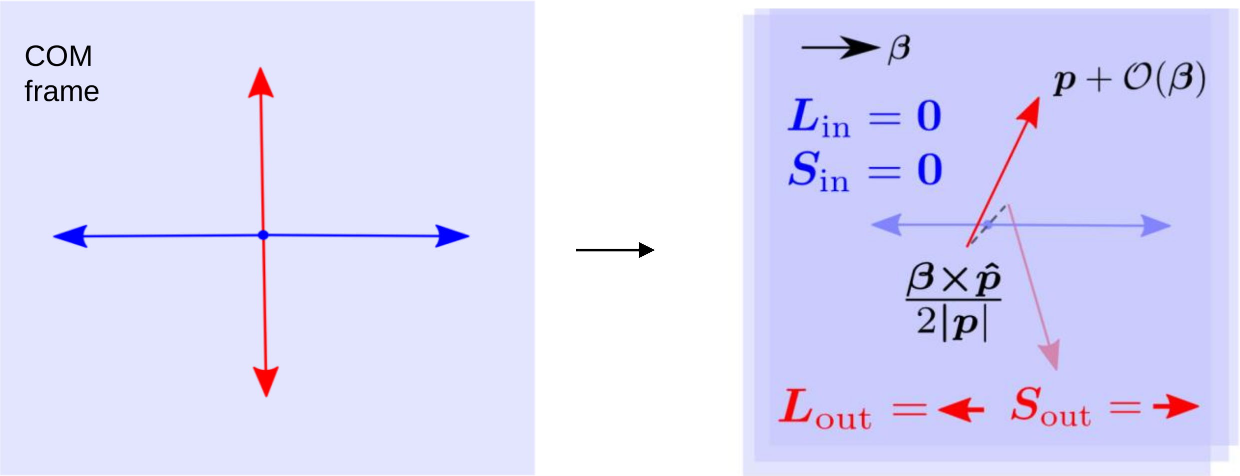
J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. 115, 021601 (2015)

# Side jumps & Lorentz covariance

- Side jumps appear in non-equilibrium cases. J.-Y. Chen, et.al. Phys. Rev. Lett. 113, 182302 (2014) J.-Y. Chen, et.al., Phys. Rev. Lett. 115, 021601 (2015)
  - conservation of angular momentum
  - $f$ is frame dependent
  - The magnetization current is associated with CVE and also the AC conductivity for CME. D. E. Kharzeev, M. A. Stephanov, and H.-U. Yee, Phys. Rev. D95, 051901 (2017)

$$f' - f = -\Delta\cdot\partial f + \int_{BCD} C_{ABCD}\frac{\Delta\cdot\bar{n}}{p\cdot\bar{n}}.$$

$$j^{\mu} = \underbrace{p^{\mu} f}_{\text{normal current}} + \underbrace{\overset{\text{spin tensor}}{\boxed{S^{\mu\nu}}}\partial_{\nu} f}_{\text{magnetization current}} + \underbrace{\int_{BCD} C_{ABCD}\bar{\Delta}^{\mu}}_{\text{jump current}}$$



$$\delta'_{\boldsymbol{\beta}}\boldsymbol{x} = \boldsymbol{\beta}t + \frac{\boldsymbol{\beta}\times\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|};$$

$$\delta t = \boldsymbol{\beta}\cdot\boldsymbol{x}.$$

- ☐ Could we derive it from QFT?

# Two parts in my talk

- We present the CKT from the QFT derivation, which incorporates side jumps and collisions with background fields.

  Phys.Rev. D95 (2017) no.9, 091901, arXiv:1612.04630.

- We apply our formalism to further investigate the nonlinear quantum transport for inviscid chiral fluids, where we find anomalous Hall currents associated with E fields and temperature/chemical-potential gradients.

  arXiv:1710.00278.

# CKT from the Wigner functions

- We consider only right-handed Weyl fermions under U(1) background fields.
- Wigner functions : less (greater) propagators under Wigner transformation.

$$S^>(x,y) = \langle \psi(x)\mathcal{P}\mathcal{U}^\dagger(A_\mu,x,y)\psi^\dagger(y)\rangle$$

$$S^<(x,y) = \langle \psi^\dagger(y)\boxed{\mathcal{P}\mathcal{U}(A_\mu,x,y)}\psi(x)\rangle$$

gauge link

$$\Longrightarrow \quad \grave{S}^{<(>)}(q,X) = \int d^4Y\, e^{\frac{iq\cdot Y}{\hbar}} S^{<(>)}\left(X+\frac{Y}{2}, X-\frac{Y}{2}\right)$$

$X = \frac{x+y}{2}$, $Y = x - y$ $\quad$ $q$ is canonical momentum

H. T. Elze, M. Gyulassy, and D. Vasak, Nucl. Phys. B276, 706 (1986)

fermions & anti-fermions

- Without $\mathcal{O}(\hbar)$ corrections : $\grave{S}^<(q,X) = 2\pi\left(\theta(q^0) - \theta(-q^0)\right)\boxed{f(q,X)}q^\mu\bar{\sigma}_\mu\delta(q^2)$

distribution function

- Wigner functions are always covariant : $J^\mu = \int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\sigma^\mu \grave{S}^<\right)$

$$\Delta_\mu = \partial_\mu + F_{\nu\mu}\partial/\partial q_\nu$$
$$\partial_\mu = \partial/\partial X^\mu$$

- Dirac equations up to $\mathcal{O}(\hbar)$ : (equivalent to gradient expansion )

$$\sigma^\mu\left(q_\mu + \frac{i\hbar}{2}\Delta_\mu\right)\grave{S}^< = \frac{i\hbar}{2}\left(\Sigma^<\grave{S}^> - \Sigma^>\grave{S}^<\right), \quad \left(q_\mu - \frac{i\hbar}{2}\Delta_\mu\right)\grave{S}^<\sigma^\mu = -\frac{i\hbar}{2}\left(\grave{S}^>\Sigma^< - \grave{S}^<\Sigma^>\right)$$

- Caveat : the perturbative solution is subject to weak fields or large momenta.

(strong fields : solve Dirac eq. non-perturbatively.  e.g. Landau levels with large B)

# Perturbative solutions

- Trace and traceless parts of Dirac equations (collisionless) : $\grave{S}^{<} = \bar{\sigma}^{\mu}\grave{S}^{<}_{\mu}$,

$$\Delta_{\mu}\grave{S}^{<\mu} = \Sigma^{<}_{\mu}\grave{S}^{>\mu} - \Sigma^{>}_{\mu}\grave{S}^{<\mu}, \quad q_{\mu}\grave{S}^{<\mu} = 0,$$

$$\Delta_{\mu} = \partial_{\mu} + F_{\rho\mu}\partial^{\rho}_{q}$$

↘ kinetic theory ↘ dispersion relation

$$\hbar\Delta_{[i}\grave{S}^{<}_{0]} - 2\epsilon^{ijk}q_{j}\grave{S}^{<}_{k} = \hbar\left(\Sigma^{<}_{[i}\grave{S}^{>}_{0]} - \Sigma^{>}_{[i}\grave{S}^{<}_{0]}\right),$$

$$\hbar\epsilon^{ijk}\Delta_{j}\grave{S}^{<}_{k} + 2q_{[i}\grave{S}^{<}_{0]} = \hbar\epsilon^{ijk}\left(\Sigma^{<}_{j}\grave{S}^{>}_{k} - \Sigma^{>}_{j}\grave{S}^{<}_{k}\right).$$

→ The traceless part (linear to $\sigma^{i}$) leads to side-jumps

- Perturbative solutions up to $\mathcal{O}(\hbar)$ (without collisions) :

$$\grave{S}^{<\mu}(q) = 2\pi\left[\boxed{\delta(q^2)q^{\mu}f} + \hbar\delta(q^2)\delta^{\mu i}\epsilon^{ijk}\frac{q_j}{2q_0}\Delta_k f + \boxed{\hbar\epsilon^{\mu\nu\alpha\beta}q_{\nu}F_{\alpha\beta}\frac{\partial\delta(q^2)}{2\partial q^2}f}\right]$$

L.O. sol.

CME in equilibrium

  - the side-jump term: $\boxed{\delta\grave{S}^{f<}_{\mu} = 2\pi\delta_{\mu i}\epsilon_{ijk}\frac{\delta(q^2)q_j}{2q_0}\Delta_k f}$

magnetization current
(implies side-jumps in $f$)
Not cov. (frame dep.)

"side jumps originate from a helicity-dependent phase of the wave functions under L.T."

- With collisions : only the side-jump term is modified

$$\delta\grave{S}^{f<}_{\mu} = (2\pi)\delta_{\mu i}\epsilon_{ijk}\delta(q^2)\frac{q_j}{2q_0}\left(\Delta_k f - \boxed{C_k}\right), \qquad C_{\beta}[f] = \Sigma^{<}_{\beta}\bar{f} - \Sigma^{>}_{\beta}f$$

jump current

# Covariant currents and side-jumps

- Introduce the a frame vector $n^\mu$:

$$\grave{S}^{<\mu}(q,X) = 2\pi\bar{\epsilon}(q\cdot n)\left(q^\mu\delta(q^2)f_q^{(n)} + \hbar\delta(q^2)S^{\mu\nu}_{(n)}\mathcal{D}_\nu f_q^{(n)} + \hbar\epsilon^{\mu\nu\alpha\beta}q_\nu F_{\alpha\beta}\frac{\partial\delta(q^2)}{2\partial q^2}f_q^{(n)}\right),$$

$$\mathcal{D}_\beta f_q^{(n)} = \Delta_\beta f_q^{(n)} - \mathcal{C}_\beta, \qquad S^{\mu\nu}_{(n)} = \frac{\epsilon^{\mu\nu\alpha\beta}}{2(q\cdot n)}q_\alpha n_\beta$$

- The modified Lorentz transformation : $f_q^{(n')} = f_q^{(n)} + \frac{\hbar\epsilon^{\nu\mu\alpha\beta}q_\alpha n'_\beta n_\mu}{2(q\cdot u)(q\cdot n')}\mathcal{D}_\nu f_q^{(n)}$
- A more general form of CKT (for $n^\mu = n^\mu(X)$) :

$$\delta\left(q^2 - \hbar\frac{B\cdot q}{q\cdot n}\right)\left[q\cdot\mathcal{D} + \frac{\hbar S^{\mu\nu}_{(n)}E_\mu}{q\cdot n}\mathcal{D}_\nu + \hbar S^{\mu\nu}_{(n)}(\partial_\mu F_{\rho\nu})\partial_q^\rho + \hbar(\partial_\mu S^{\mu\nu}_{(n)})\mathcal{D}_\nu\right]f_q^{(n)} = 0$$

( previous expression of CKT : $n^\mu = (1,\mathbf{0})$ & onshell)

D. T. Son and N. Yamamoto,
Phys. Rev. D87, 085016 (2013)

- Conservation of the angular momentum : COM frame = no-jump frame

J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. 115, 021601 (2015)

E.g. 2-2 Coulomb scattering ( $n_c^\mu = (q^\mu + q'^\mu)/\sqrt{s}$ ) :

$$q^\mu C_\mu\left[f^{(n_c)}\right] = \frac{1}{4}\int_{\mathbf{q}',\mathbf{k},\mathbf{k}'}|\mathcal{M}|^2\left[\bar{f}^{(n_c)}(q)\bar{f}^{(n_c)}(q')f^{(n_c)}(k)f^{(n_c)}(k') - f^{(n_c)}(q)f^{(n_c)}(q')\bar{f}^{(n_c)}(k)\bar{f}^{(n_c)}(k')\right]$$

# Local equilibrium

- Global equilibrium distribution functions (in arbitrary frames) for $C[f] = 0$:

$$f_q^{\mathrm{eq}(n)} = (e^{g}+1)^{-1}, \quad g = \left(\beta q\cdot u - \bar{\mu} + \frac{\hbar S^{\mu\nu}_{(n)}}{2}\partial_\mu(\beta u_\nu)\right)$$

$u^\mu$: fluid velocity

J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. 115, 021601 (2015)

- Global equilibrium Wigner functions :

$$\grave{S}^{<\mu}_{\mathrm{geq}} = 2\pi\bar{\epsilon}(q\cdot n)\boxed{\left[\delta(q^2)\left(q^\mu + \frac{\hbar}{2}\left[u^\mu(q\cdot\omega) - \omega^\mu(q\cdot u)\right]\partial_{q\cdot u}\right) + \hbar\epsilon^{\mu\nu\alpha\beta}q_\nu F_{\alpha\beta}\frac{\partial\delta(q^2)}{2\partial q^2}\right] f_q^{(0)}}$$

vorticity : $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu(\partial_\alpha u_\beta), \quad f_q^{(0)} = (e^{g_0}+1)^{-1}, \quad g_0 = \beta q\cdot u - \bar{\mu}.$

No frame dep.

- A general form for the local equilibrium $f$ : too hard to be found
- Local equilibrium $f$ can be nontrivially defined in the co-moving frame ($n^\mu = u^\mu$) such that $C[f] = 0$ (2 to 2 scattering).
- Wigner functions : $\grave{S}^{<\mu}_{\mathrm{leq}} = 2\pi\bar{\epsilon}(q\cdot u)\delta(q^2)\left(q^\mu + \hbar S^{\mu\nu}_{(u)}\Delta_\nu + \hbar\epsilon^{\mu\nu\alpha\beta}q_\nu F_{\alpha\beta}\frac{\partial\delta(q^2)}{2\partial q^2}\right)f_q^{\mathrm{eq}(u)}$

# Anomalous transport in equilibrium

- Anomalous transport in equilibrium is non-dissipative (irrelevant to interactions).
- Near local equilibrium :

$$T^{\mu\nu}=u^\mu u^\nu\epsilon-p\Theta^{\mu\nu}+\Pi^{\mu\nu}_{\rm non}+\Pi^{\mu\nu}_{\rm dis},\qquad J^\mu=N_0u^\mu+v^\mu_{\rm non}+v^\mu_{\rm dis},\quad \Theta^{\mu\nu}=\eta^{\mu\nu}-u^\mu u^\nu$$

- Reproduce the transport coefficients from Wigner functions :

$$T^{\mu\nu}_{\rm leq}=u^\mu u^\nu\epsilon-p\Theta^{\mu\nu}+\Pi^{\mu\nu}_{\rm non}=\int\frac{d^4q}{(2\pi)^4}\left(q^\mu\grave{S}^{<\nu}_{\rm leq}+q^\nu\grave{S}^{<\mu}_{\rm leq}\right),$$

$$J^\mu_{\rm leq}=N_0u^\mu+v^\mu_{\rm non}=2\int\frac{d^4q}{(2\pi)^4}\grave{S}^{<\mu}_{\rm leq},$$

(agree with different approaches,
e.g. Kubo formulae,
K. Landsteiner, E. Megias, and F. Pena-Benitez,
Lect. Notes Phys. 871, 433 (2013))

⟹

$$\Pi^{\mu\nu}_{\rm non}=\hbar\xi_\omega\left(\omega^\mu u^\nu+\omega^\nu u^\mu\right)+\hbar\xi_B\left(B^\mu u^\nu+B^\nu u^\mu\right)$$

$$v^\mu_{\rm non}=\hbar\sigma_BB^\mu+\hbar\sigma_\omega\omega^\mu$$

$$\sigma_\omega=\frac{T^2}{12}\left(1+\frac{3\bar{\mu}^2}{\pi^2}\right),\quad \sigma_B=\frac{\mu}{4\pi^2},\quad \xi_\omega=\frac{T^3}{6}\left(\bar{\mu}+\frac{\bar{\mu}^3}{\pi^2}\right)=N_0,\quad \xi_B=\frac{T^2}{24}\left(1+\frac{3\bar{\mu}^2}{\pi^2}\right)=\frac{\sigma_\omega}{2}.$$

# Nonlinear responses & anomalous hydro

- Nonlinear responses (in gradient expansions) near local equilibrium : dissipative transport pertinent to interactions.
- CKT with the relaxation-time approximation.

E. V. Gorbar, et.al, Phys.Rev. D93, 105028 (2016), J.-W. Chen, T. Ishii, S. Pu, N. Yamamoto, Phys. Rev. D93, 125023 (2016)

- Previous studies are for open systems : energy-momentum conservation is not preserved when backreaction on the environment is neglected.

(Weyl semimetals : quasiparticle-impurity scattering ≫ scattering between quasiparticles )

- For closed systems : hydrodynamics should be applied, which imposes energy-momentum conservation.

(QGP, Weyl semimetals at higher T as the analog of Dirac fluids in graphene)

- We focus on 2nd order quantum transport by utilizing CKT with RT approx. + anomalous hydro EOM.
- Anomalous hydro : $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B})$

in the local rest frame

$$\frac{\partial_0 T}{T} = \hbar\boldsymbol{\mathcal{E}}\cdot\left(\tilde{T}_B\mathbf{B}+\tilde{T}_\omega\boldsymbol{\omega}T\right), \quad \partial_0\bar{\mu} = \hbar\boldsymbol{\mathcal{E}}\cdot\left(\tilde{\mu}_B\mathbf{B}+\tilde{\mu}_\omega\boldsymbol{\omega}T\right), \quad \partial_0\mathbf{u} = \partial_0\mathbf{u}^{(0)} + \hbar\partial_0\delta\mathbf{u},$$

$$\partial_0\mathbf{u}^{(0)} = -\frac{\nabla T}{T} + \frac{N_0\boldsymbol{\mathcal{E}}}{4p}, \quad \hbar\partial_0\delta\mathbf{u} = \hbar\left(\tilde{U}_E\nabla\times\mathbf{E} + \tilde{U}_T\frac{\mathbf{E}\times\nabla T}{T} + \tilde{U}_{\bar{\mu}}\mathbf{E}\times\nabla\bar{\mu}\right)$$

$$\mathcal{E}_\nu = E_\nu + T\partial_\nu\bar{\mu}$$

# Non-equilibrium distribution functions

- Solving CKT for non-equilibrium fluctuations of $f$ : $f_q^{(u)} - f_q^{\rm eq} = \delta f_q = \delta f_q^{(c)} + \hbar\delta f_q^{(Q)}$

classical quantum

$$\left[q\cdot\Delta + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\Delta_\nu + \hbar S^{\mu\nu}_{(u)}(\partial_\mu F_{\rho\nu})\partial_q^\rho + \hbar\left(\hat\Pi_1^\mu + \hat\Pi_2^\mu\right)\Delta_\mu\right] f_q^{(u)} = \mathcal{C}_{\rm full}, \quad \partial_\mu S^{\mu\nu}_{(u)} = \hat\Pi_1^\nu + \hat\Pi_2^\nu.$$

$B^\mu, E^\mu$, etc. at $O(\partial)$

- The RT approximation : $C_{\rm full} = -\frac{1}{\tau_R}\left(q\cdot u + \hbar\frac{q^\mu\mathcal{A}_\mu}{(q\cdot u)^2}\right)\delta f_q$
- The perturbative solution :

$$\delta f_q = -\frac{\tau_R}{q\cdot u}\left(1 - \frac{\hbar q\cdot\mathcal{A}}{q\cdot u}\right)\left[q\cdot\Delta + \hbar\frac{S^{\mu\nu}_{(u)}E_\mu}{(q\cdot u)}\Delta_\nu + \hbar S^{\mu\nu}_{(u)}(\partial_\mu F_{\rho\nu})\partial_q^\rho + \hbar\left(\hat\Pi_1^\mu + \hat\Pi_2^\mu\right)\Delta_\mu\right] f_q^{\rm eq}$$

- Different contributions for $\hbar\delta f_q^{(Q)} = \delta f_q^{\mathcal{K}} + \delta f_q^{\mathcal{H}} + \delta f_q^{\mathcal{C}}$

from CKT from hydro EOM from collisions

- The non-equilibrium four current :

$$\delta J^\mu_Q = 2\hbar\int\frac{d^4q}{(2\pi)^3}\bar\epsilon(q\cdot u)\delta(q^2)\left[q^\mu\delta f_q^{(Q)} - \frac{1}{2}\left(\epsilon^{\mu\nu\alpha\beta}\frac{u_\nu q_\alpha}{q\cdot u}\Delta_\beta + \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\frac{\partial_{q\nu}}{2}\right)\delta f_q^{(c)}\right]$$

# Anomalous Hall currents

- Non-equilibrium (quantum) charge currents :

$$\delta\boldsymbol{J}_{QR/L} = \mp\frac{\hbar\tau_R}{84\pi^2}\left[\mu_{R/L}\nabla\times\boldsymbol{E} + \frac{7\mu_{R/L}}{2T}\boxed{\boldsymbol{E}\times\nabla T} - \frac{1}{2}\boldsymbol{E}\times\nabla\mu_{R/L} + \frac{12\mu_{R/L}}{T}\boxed{(\nabla\mu_{R/L})\times(\nabla T)}\right], \quad \bar{\mu}_{R/L} \ll 1$$

emerge from hydro.

$$\delta\boldsymbol{J}_{QR/L} = \pm\frac{\hbar\tau_R}{4\pi^2}\left[\frac{\pi^2T^2}{3\mu_{R/L}}\nabla\times\boldsymbol{E} + \frac{2\mu_{R/L}}{3T}\boxed{\boldsymbol{E}\times\nabla T} - \frac{2}{3}\boldsymbol{E}\times\nabla\mu_{R/L} - \frac{2\mu_{R/L}}{3T}\boxed{(\nabla\mu_{R/L})\times(\nabla T)}\right], \quad \bar{\mu}_{R/L} \gg 1$$

- The charge density : $\delta J^0_Q = -\frac{\hbar\tau_R}{6\pi^2}\left[-\boldsymbol{\mathcal{A}}\cdot\boldsymbol{\mathcal{E}} + \bar{\mu}\boldsymbol{\mathcal{A}}\cdot\nabla T + \mu\boldsymbol{\mathcal{A}}\cdot\partial_0\mathbf{u}\right] \xrightarrow{\mathcal{A}^\mu = 0} \delta J^0_Q = 0$

- Anomalous equation from CKT (for general $f$):

$$\partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B}) + \boxed{2\int\frac{d^4q}{(2\pi)^3}\bar{\epsilon}(q\cdot u)\left[\delta(q^2)q\cdot\mathcal{C} + \hbar\epsilon^{\mu\nu\alpha\beta}\mathcal{C}_\mu F_{\alpha\beta}\frac{\partial_{q\nu}\delta(q^2)}{4}\right]}$$

$$\Downarrow$$

matching condition for RT approximations : $\delta J^0 = 0$

# Conclusions & outlook

- We have presented the CKT incorporating side jumps and collisions in the Wigner-function formalism.
- We have applied CKT with the RT approximation to investigate the 2nd order anomalous transport in inviscid chiral fluids.
- We have found anomalous Hall currents, for which the currents propagating along the product of $E$ and $\nabla T$ and of $\nabla T$ and $\nabla\mu$ emerge from hydrodynamics.

- Viscous effects?
- Beyond the RT approximation?
- Applications in phenomenology?