Quantum trajectories and quantum measurement theory in solid-state mesoscopics

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Abstract. Starting from the full Schrödinger equation for a system and an environment (a detector), we present a heuristic derivation of the stochastic Schrödinger (master) equation for a mesoscopic measurement model to illustrate the essential physics of the quantum trajectory theory. This mesoscopic model describes a two-state quantum system, an electron coherently tunneling between two coupled quantum dots, interacting with an environment (a detector), a low transparency point contact or tunnel junction. Then we provide a connection and a unified picture for the quantum trajectory approach and the master equation approach. We show that the master equation for the reduced or "partially" reduced density matrix can be simply obtained when an average or "partial" average is taken on the conditional, stochastic density matrix (or quantum trajectories) over the possible outcomes of the measurements.

1. Introduction

The theory of quantum trajectories or stochastic Schrödinger equations has been developed in last ten years mainly in the quantum optics community to describe open quantum system subject to continuous quantum measurements. But it was introduced to the context of solid-state mesoscopics only recently [1, 2, 3, 4]. Different authors, however, gave somewhat different derivations for the stochastic equations. In Refs. [1, 2] the prescriptions for the quantum trajectories (or selective, conditional, stochastic state evolution) were derived based on the Bayesian formalism, while they were derived in Refs. [3, 4] starting from unconditional master equation. In addition, the interpretations along with the derivations do not seem transparent enough. Hence, many aspects of the theory are still poorly understood in the condensed matter physics community. The main purposes of this paper are (i) to present a simple, heuristic derivation for the same mesoscopic model to illustrate the essential physics of the quantum trajectories. (ii) to provide a unified picture for the quantum trajectory approach and the master equation approach of reduced or "partially" reduced density matrix. Here we refer to the master equation approach of the "partially" reduced density matrix as the approach recently developed in Refs. [5, 6, 7], mainly for the purpose of reading out an initial state of a quantum bit (qubit). We show that the master equation of the reduced or "partially" reduced density matrix can be obtained as a result of ensemble average or "partial" average on the conditional, stochastic master equation of the density matrix (constructed from the conditional system state vector) over all possible detection records.

2. Mesoscopic measurement model and quantum trajectories

2.1. Coupled-quantum-dot and point-contact model

The mesoscopic measurement model considered in Refs. [1, 2, 3, 4, 5] describes a quantum system, an electron coherently tunneling between two coupled quantum dots (CQD's), interacting with an environment (a detector), a low transparency point contact (PC) or tunnel junction. The Hamiltonian describing the whole system can be written as [1, 2, 3, 4, 5]

$$\mathcal{H} = \mathcal{H}_{\rm CQD} + \mathcal{H}_{\rm PC} + \mathcal{H}_{\rm coup} \tag{1}$$

where

$$\mathcal{H}_{\rm CQD} = \hbar\left[\omega_1 c_1^\dagger c_1 + \omega_2 c_2^\dagger c_2 + \Omega(c_1^\dagger c_2 + c_2^\dagger c_1)\right], \tag{2}$$

$$\mathcal{H}_{\rm PC} \;= \hbar\sum_k\left(\omega_k^L a_{Lk}^\dagger a_{Lk} + \omega_k^R a_{Rk}^\dagger a_{Rk}\right) + \sum_{k,q}\left(T_{kq}^* a_{Lk}^\dagger a_{Rq} + T_{qk} a_{Rq}^\dagger a_{Lk}\right), \tag{3}$$

$$\mathcal{H}_{\rm coup} \;= \sum_{k,q} c_1^\dagger c_1\left(\chi_{kq}^* a_{Lk}^\dagger a_{Rq} + \chi_{qk} a_{Rq}^\dagger a_{Lk}\right). \tag{4}$$

$\mathcal{H}_{\rm CQD}$ represents the effective tunneling Hamiltonian for the measured two-state CQD system (mesoscopic charge qubit). The tunneling Hamiltonian for the PC bath (detector) is represented by $\mathcal{H}_{\rm PC}$. Here c_i ($c_i^\dagger$) and $\hbar\omega_i$ represent the electron annihilation (creation) operator and energy for a single electron state in each dot respectively. The coupling between the two dots is given by $\hbar\Omega$. Similarly, a_{Lk}, a_{Rk} and $\hbar\omega_k^L$, $\hbar\omega_k^R$ are respectively the electron annihilation operators and energies for the left and right reservoir states at wave number k. Equation (4), $\mathcal{H}_{\rm coup}$, describes the interaction between the bath (detector) and the CQD qubit system, depending on which dot is occupied. When the electron in the CQD's is located in dot 1, the effective tunneling amplitude of the PC detector changes from $T_{kq} \to T_{kq} + \chi_{kq}$.

2.2. Continuous measurements and quantum trajectory

We will derive, starting from the full Schrödinger equation for the system and environment (bath), the stochastic Schrödinger equation which models the evolution of the CQD system conditioned on continuous in time measurements of the PC bath by a detection device. We follow closely the derivation [8, 9, 10] of the stochastic Schrödinger equation for an atom, driven by classical radiation fields and interacting with the vacuum radiation field, for the CQD/PC model. The basic idea is as follows. Suppose that the combined state $|\Psi\rangle$ of the system and the bath is initially disentangled, $|\Psi(0)\rangle$=$|\psi(0)\rangle|0_B\rangle$. Here the initial "vacuum" state $|0_B\rangle$ of the PC bath is the state where the energy levels in the source (the left PC reservoir) and drain (the right PC reservoir) are filled up to the Fermi energies (chemical potential) μ_L with electron number n_L^0, and μ_R with electron number n_R^0, respectively. We represent the Hilbert space of the PC bath states as the number or Fock states $|n_B\rangle$ of the left and right PC reservoirs. Since the operator $n_L + n_R = \sum_k(a_{Lk}^\dagger a_{Lk} + a_{Rk}^\dagger a_{Rk})$, commutes with the total Hamiltonian Eq. (1), the total electron number of the PC is conserved. We can thus write $|0_B\rangle = |n_L^0\rangle|n_R^0\rangle$ and $|n_B\rangle = |n_L^0 - n\rangle|n_R^0 + n\rangle$, being the states simultaneously having n addition electrons in the drain continuum, and n holes in the source continuum. The effect of $\mathcal{H}_{\rm coup}$ makes the state of the CQD qubit system and the state of the PC reservoirs become entangled so that the total state $|\Psi\rangle$ no longer factorizes as it did at $t = 0$. For the CQD/PC mesoscopic model, the observed quantity or physical observable is the number of electrons tunneling through PC. Hence we can expand the total state on the orthonormal Fock (number) state basis of the PC bath as

$$|\Psi(t)\rangle = \sum_n \beta_n(t)\,|\psi_n(t)\rangle|n_B\rangle. \tag{5}$$

Here we have chosen the CQD qubit system state $|\psi_n(t)\rangle$ to be normalized, but they are, in general, not orthonormal. This is related to the fact that the indirect measurement on the CQD qubit system, discussed later, is not projective. We consider the case that the measurement is performed and repeated within a time interval dt much smaller than the typical system evolution or response time. Hence, the system is effectively continuously monitored. According to the "orthodox" quantum theory of measurements, the possible detector outcomes are the integer eigenvalues of the electron number operator in the right reservoir of the PC at time $t + dt$. Moreover, subsequent to the detection, the bath part of the total wave function $|\Psi(t+dt)\rangle$ collapses to the corresponding eigenstate. That is to say when the measurement identifies the state of the PC bath to be in the particular eigenstate $|n_B\rangle$, the CQD system is in the corresponding pure state $|\psi(t + dt)\rangle = |\psi_n(t + dt)\rangle$. The probability for this to happen is equal to $|\beta_n(t + dt)|^2$. The normalized system state, conditioned on the measurement result just obtained, then serves as the initial state for the next evolution and measurement time interval dt. Thus, according to the measurement record of each experiment run, we obtain one particular evolution of system state $|\psi(t)\rangle$ as a result of a continuous projection of the total state $|\Psi(t)\rangle$ on one of the eigenstates $|n_B\rangle$. Such an evolution $|\psi(t)\rangle$ is called a *quantum trajectory* and its nature is generally stochastic. The stochastic element in the quantum trajectory corresponds exactly to the consequence of the random outcomes of the measurement record.

It is important to realize that the experimenter never makes a direct measurement on the system of interest. Rather, the experimenter observes the number of electrons tunneling through the PC. In the mesoscopic model, the CQD qubit system interacts with the PC reservoirs and their quantum states are entangled. The projective measurement made on the PC bath, however, enables us to disentangle the system and the bath states. We may think the effect on the qubit system state $|\psi(t)\rangle$ is an indirect result of the projective measurement on the PC bath, and we may model it in terms of measurement operators $M_n(dt)$ acting on the qubit system state alone. However, such an effective measurement on the qubit system state is, in principle, not projective. If the initial normalized state of the system is $|\psi(t)\rangle$ immediately before the measurement, the unnormalized state of the measured result being n at the end of the time interval $[t, t+dt)$ of the measurement can be written as $|\tilde{\psi}_n(t+dt)\rangle = M_n(dt)|\psi(t)\rangle$. The corresponding probability and the normalized final state of the system are respectively

$$|\beta_n(t + dt)|^2 = \langle\psi(t)|M_n^\dagger(dt)M_n(dt)|\psi(t)\rangle, \tag{6}$$

$$|\psi_n(t + dt)\rangle = M_n(dt)|\psi(t)\rangle/\sqrt{\langle\psi(t)|M_n^\dagger(dt)M_n(dt)|\psi(t)\rangle}. \tag{7}$$

Here the measurement operator satisfies the completeness condition $\sum_n M_n^\dagger(dt)M_n(dt) = 1$, which is simply a statement of conservation of probability. The question now is to find $|\psi_n(t + dt)\rangle$ or the measurement operators $M_n(dt)$ for the CQD/PC model given that $|\psi(t)\rangle$ is known at time t.

2.3. Jump-free evolution

First, let us consider the case that $|\Psi(0)\rangle$=$|\psi(0)\rangle|0_B\rangle$, and the measurement outcome that no electron tunneling through PC is detected up to time t. We treat the sum of the tunneling Hamiltonian parts in $\mathcal{H}_{PC}$ and $\mathcal{H}_{coup}$ as the interaction Hamiltonian $\mathcal{H}_I$. Then the dynamics of the entire system in the interaction picture, is determined by the time-dependent Hamiltonian [3]

$$H_I(t) = \sum_{k,q} \left(T_{kq}^* + \chi_{kq}^* c_1^\dagger c_1\right) a_{Lk}^\dagger a_{Rq} e^{i(\omega_k^L - \omega_k^R)t} + \mathrm{H.c.}, \tag{8}$$

where H.c. stands for Hermitian conjugate of the entire previous term. The total wave function of the entire system in the interaction picture satisfies

$$\frac{d}{dt}|\Psi(t)\rangle_I = -\frac{i}{\hbar}H_I(t)|\Psi(0)\rangle_I - \frac{1}{\hbar^2}H_I(t)\int_0^t dt' H_I(t')|\Psi(t')\rangle_I. \tag{9}$$

Equation (9) is exact and can be derived from the Schr¨odinger equation. We consider the measurement scheme of counting the number of electrons through the PC. The total wave function in the interaction picture can be expanded in terms of the Fock states of the bath as in Eq. (5) with $|\psi_n(t)\rangle \to |\psi_n(t)\rangle_I$. The effect of zero-count measurement results is that the total state vector $|\Psi(t)\rangle_I$ is repeatedly projected into $|0_B\rangle$ and is renormalized. Hence during the zero-count interval, the state vector $|\psi(t)\rangle_I$ of the system will be in $|\psi_0(t)\rangle_I$. Using Eq. (9) and carrying out the projection onto $|0_B\rangle$, we find for the evolution of the unnormalized state $|\tilde{\psi}_0(t)\rangle_I = \beta_0(t)|\psi_0(t)\rangle_I$ as follows:

$$\frac{d}{dt}|\tilde{\psi}_0(t)\rangle_I = -\frac{1}{\hbar^2}\int_0^t dt' \sum_{k,q}(T^*_{qk} + \chi^*_{qk}n_1)(T_{kq} + \chi_{kq}n_1) \times \theta(\mu_L - \hbar\omega_k^L)\,\theta(\hbar\omega_q^R - \mu_R)e^{i(\omega_k^L-\omega_q^R)(t-t')}|\tilde{\psi}_0(t')\rangle_I, \quad (10)$$

where $n_1 = c_1^\dagger c_1$, and $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. The first term in Eq. (9) does not contribute because $\langle 0_B|H_I(t)|0_B\rangle = 0$. Under the assumption that the correlation time of the bath, $\tau_B \approx \hbar/(\mu_L - \mu_R)$, is much shorter than the typical time constant expected from the system, and provided that $t \gg \tau_B$, we can replace $|\tilde{\psi}_0(t')\rangle_I$ by $|\tilde{\psi}_0(t)\rangle_I$ in the integrand of Eq. (10) and extend the lower limit of the time integration to infinity after the change of variable $\tau = t - t'$. This assumption and resulting simplifications to the integrand of Eq. (10) are known as the Born-Markov approximation.

Carrying out the integration and going back to the Sch¨odinger picture, we find that Eq. (10) becomes

$$\frac{d}{dt}|\tilde{\psi}_0(t)\rangle = -\frac{i}{\hbar}H_{\rm CQD}|\tilde{\psi}_0(t)\rangle - \frac{1}{2}(\mathcal{T}^* + \mathcal{X}^*n_1)(\mathcal{T} + \mathcal{X}n_1)|\tilde{\psi}_0(t)\rangle, \quad (11)$$

where the parameters $\mathcal{T}$ and $\mathcal{X}$ are given by $D = |\mathcal{T}|^2 = 2\pi eVg_Lg_R|T_{00}|^2/\hbar$, and $D' = |\mathcal{T} + \mathcal{X}|^2 = 2\pi eVg_Lg_R|T_{00} + \chi_{00}|^2/\hbar$. Here D and D' are average electron tunneling rates through the PC barrier without and with the presence of the electron in dot 1 respectively, and $eV = \mu_L - \mu_R$ is the external bias applied across the PC. To arrive at Eq. (11), energy-independent tunneling amplitudes represented by T_{00} and χ_{00}, and energy-independent density of states of the reservoirs represented by g_L and g_R are assumed. Requiring that the state $|\psi_0(t)\rangle$ remains normalized so that $d[\langle\psi_0(t)|\psi_0(t)\rangle]/dt = 0$, and with the help of Eq. (11) and the relation $|\tilde{\psi}_0(t)\rangle = \beta_0(t)|\psi_0(t)\rangle$, we obtain

$$d\beta_0(t)/dt = -\mathcal{P}_1(t)\beta_0(t)/2, \quad (12)$$

where $\mathcal{P}_1(t) = \langle(\mathcal{T}^* + \mathcal{X}^*n_1)(\mathcal{T} + \mathcal{X}n_1)\rangle$ and the expectation value $\langle\cdots\rangle$ is taken with respect to the normalized state $|\psi_0(t)\rangle$. This then yields, from Eq. (11), the following evolution equation for the normalized state $|\psi_0(t)\rangle$ of the CQD qubit system under the condition that no electron through PC is detected:

$$\frac{d}{dt}|\psi_0(t)\rangle = -\frac{i}{\hbar}H_{\rm CQD}|\psi_0(t)\rangle - \frac{1}{2}(\mathcal{T}^* + \mathcal{X}^*n_1)(\mathcal{T} + \mathcal{X}n_1)|\psi_0(t)\rangle + \frac{1}{2}\mathcal{P}_1(t)|\psi_0(t)\rangle. \quad (13)$$

We see from Eq. (13) that when the measurement result is *null* (no electron detected), the system changes infinitesimally, but not unitarily. The non-unitary nature of the evolution expresses the fact that the projective measurement on the bath with results of no electrons detected still have effects on the system. In addition, Eq. (13) is non-linear because of the presence of the expectation value term $\mathcal{P}_1(t)$, which ensures that the system state remains normalized.

2.4. Quantum jumps

Next we consider the case that electrons passing through the PC are detected. The probability that detection of electron occurs in time interval $[t,t+dt)$ is equal to the difference between the probability that no electron was detected up to time t and that up to time $t+dt$. This can be directly found from Eq. (12) with the result:

$$|\beta_0(t)|^2 - |\beta_0(t+dt)|^2 = |\beta_0(t)|^2 \mathcal{P}_1(t)\, dt, \tag{14}$$

where the relation $2(d\beta_0/\beta_0) = 2d(\ln\beta_0) = d(\ln\beta_0^2) = d\beta_0^2/\beta_0^2$ has been used. If the detection intervals dt are chosen to be sufficiently short for the detection probability to be small, the probability of two and more electrons could have been detected may be neglected. Hence in each infinitesimal time interval dt, either measurement result is *null* (no electron detected) or there is a *detection* of an electron through the PC barrier. At randomly determined times (conditionally Poisson distributed), when there is a detection of an electron, the system undergoes a finite evolution (or a sudden jump due to collapse), called a *quantum jump*. Given that $|\Psi(t)\rangle = |\psi_0(t)\rangle|0_B\rangle$ at time t and a detection result of an electron at time $t+dt$, the total state vector $|\Psi(t+dt)\rangle$ collapses to the state $|\psi_1(t+dt)\rangle|1_B\rangle$. We know that the probability for the detection to occur is

$$|\beta_1(t+dt)|^2 = 1 - |\beta_0(t+dt)|^2 = \mathcal{P}_1(t)dt. \tag{15}$$

This is obtained from Eq. (14) by setting $|\beta_0(t)|^2 = 1$ since the initial state $|\psi_0(t)\rangle|0_B\rangle$ at time t, as a result of previous measurement, should be normalized. The question now is simply to find the normalized state $|\psi_1(t+dt)\rangle|1_B\rangle$ of the system. It is sufficient to obtain this state to the first order in interaction Hamiltonian. We find

$$|\psi_1(t+dt)\rangle = (\mathcal{T} + \mathcal{X} n_1)|\psi_0(t)\rangle/\sqrt{\mathcal{P}_1(t)}. \tag{16}$$

Equation (16) can also be obtained by means of effective measurement operators. In this case, only two of them, $M_n(dt)$ for $n = 0, 1$, are needed in each quantum-jump measurement interval. Using the relations, $d|\psi(t)\rangle = |\psi(t+dt)\rangle - |\psi(t)\rangle$ and $|\tilde{\psi}_n(t+dt)\rangle = M_n(dt)|\psi(t)\rangle$, we find from Eq. (11) that

$$M_0(dt) = 1 - [(i/\hbar)\mathcal{H}_{\rm CQD} + (1/2)(\mathcal{T}^* + \mathcal{X}^* n_1)(\mathcal{T} + \mathcal{X}\, n_1)]dt. \tag{17}$$

The measurement operator $M_1(dt)$ can be obtained using Eqs. (15) and (6) as

$$M_1(dt) = (\mathcal{T} + \mathcal{X}\, n_1)\sqrt{dt}. \tag{18}$$

The appearance of $\sqrt{dt}$ in $M_1(dt)$ ensures that only a finite number of detections can occur in a finite time interval, since the probability of a detection result is proportional to dt. Then from Eq. (7), the normalized state after a detection Eq. (16) results. One can check that the measurement operators satisfy the completeness condition $\sum_n M_n^\dagger(dt)M_n(dt) = 1$, to first order in time dt.

Right after the detection, the PC reservoirs are immediately reset back to its vacuum state. That is, the detected electron which has tunneled into the right reservoir is destroyed in the electric circuit of the detection device, and an electron from the outer circuit immediately flows to the left reservoir to fill up the hole in the measurement process. So energy levels in the source and drain are again filled up to the Fermi energies (chemical potential) μ_L and μ_R, respectively. In other words, the new state $|\Psi(t+dt)\rangle$ after detection therefore becomes $|\psi_1(t+dt)\rangle|0_B\rangle$. This state then serves as the initial state for next evolution and measurement interval. Then the whole sequence can be repeated to determine the random time evolution of the CQD qubit system state.

We can combine the evolution of the system state for the two possible outcomes of the measurement as

$$|\psi_c(t+dt)\rangle = [1 - dN_c(dt)]|\psi_{0c}(t+dt)\rangle + dN_c(dt)|\psi_{1c}(t+dt)\rangle. \quad (19)$$

Here and from what follows we explicitly use the subscript c to indicate that the quantity to which it is attached is conditioned on previous measurement results, the occurrences (detection records) of the electrons tunneling through the PC barrier in the past. We can think of $dN_c(t)$ as the increment in the number of electrons $N_c(t) = \sum dN_c(t)$ passing through PC barrier in time dt. In the quantum-jump case, $dN_c(t)$ is equal to either zero or one, and hence $[dN_c(t)]^2 = dN_c(t)$. In addition, since the nature of electrons tunneling through the PC is stochastic, $dN_c(t)$ thus represents a classical random process. Formally, the current through the PC can be written as $I_c(t) = e\, dN_c(t)/dt$. It is the variable $N(t)$, the accumulated electron number transmitted through the PC, which is used in Refs. [5, 6, 7]. Intuitively, the ensemble average $E[dN_c(t)]$ of the classical stochastic process $dN_c(t)$ equals $|\beta_{1c}(t+dt)|^2$, the probability (quantum average) of detecting electrons tunneling through the PC barrier in time dt. Hence from Eq. (15), we have

$$E[dN_c(t)] = \mathcal{P}_{1c}(t)dt = [D + (D' - D)\langle n_1\rangle_c(t)]dt, \quad (20)$$

where expectation value $\langle n_1\rangle_c(t)$ is taken with respect to the normalized conditional state $|\psi_c(t)\rangle$ at time t. Equation (20) simply states that the average current is $eD = e|\mathcal{T}|^2$ when dot 1 is empty, and is $eD' = e|\mathcal{T} + \mathcal{X}|^2$ when dot 1 is occupied. Using Eqs. (13) and (16) [or Eqs. (7), (17) and (18)], and expanding and keeping the terms of first order in dt, we obtain from Eq. (19) the quantum-jump stochastic Schr¨odinger equation, conditioned on the observed event in time dt:

$$\begin{aligned} d|\psi_c(t)\rangle = &\left[dN_c(t)\left(\frac{\mathcal{T} + \mathcal{X} n_1}{\sqrt{\mathcal{P}_{1c}(t)}} - 1\right)\right. \\ &\left. -\ dt\left(\frac{i}{\hbar}\mathcal{H}_{CQD} + \frac{(\mathcal{T}^* + \mathcal{X}^* n_1)(\mathcal{T} + \mathcal{X}\, n_1)}{2} - \frac{\mathcal{P}_{1c}(t)}{2}\right)\right]|\psi_c(t)\rangle. \quad (21) \end{aligned}$$

Note that $dN_c(t)$ is of order dt. Hence terms proportional to $dN_c(t)dt$ are ignored in Eq. (21).

To accommodate initial non-pure or mixed states, we express the stochastic Schr¨odinger equations as stochastic master equations of the CQD qubit system density matrix. From Eq. (19), the conditional, stochastic density matrix $\rho_c(t+dt) = |\psi_c(t+dt)\rangle\langle\psi_c(t+dt)|$ satisfies

$$\begin{aligned} \rho_c(t+dt) = &[1 - dN_c(dt)]|\psi_{0c}(t+dt)\rangle\langle\psi_{0c}(t+dt)| \\ &+ dN_c(dt)|\psi_{1c}(t+dt)\rangle\langle\psi_{1c}(t+dt)|, \quad (22) \end{aligned}$$

where the relation $[dN_c(t)]^2 = dN_c(t)$ has been used. Using Eqs. (13) and (16) [or Eqs. (7), (17) and (18)], and keeping terms up to order dt, we obtain the stochastic master equations, conditioned on the observed event in time dt:

$$\begin{aligned} d\rho_c(t) = &\, dN_c(t)\left[\frac{\mathcal{J}[\mathcal{T} + \mathcal{X} n_1]}{\mathcal{P}_{1c}(t)} - 1\right]\rho_c(t) \\ &+\ dt\left\{-\mathcal{A}[\mathcal{T} + \mathcal{X} n_1]\rho_c(t) + \mathcal{P}_{1c}(t)\rho_c(t) - \frac{i}{\hbar}[\mathcal{H}_{CQD}, \rho_c(t)]\right\}, \quad (23) \end{aligned}$$

where $\mathcal{J}[B]\rho = B\rho B^\dagger$, and $\mathcal{A}[B]\rho = (B^\dagger B\rho + \rho B^\dagger B)/2$. One can also derive, with the help of Eq. (21), the stochastic master equation (23) using the stochastic Itô calculus [3] $d\rho_c(t) = d(|\psi_c(t)\rangle\langle\psi_c(t)|) = (d|\psi_c(t)\rangle)\langle\psi_c(t)| + |\psi_c(t)\rangle\, d\langle\psi_c(t)| + (d|\psi_c(t)\rangle)(d\langle\psi_c(t)|)$. Equations (21) and (23) are the same as Eqs. (35) and (33) of Ref. [3]. But the derivation presented here, starting from the Schr¨odinger equation for the combined whole system and illustrating the essential physics of the quantum trajectories, is more transparent and heuristic.

3. Connections to master equation approach

We show next that the master equation for the reduced or "partially" reduced density matrix simply results when an average or "partial" average is taken on the conditional, stochastic density matrix (constructed from the conditional, stochastic CQD qubit system state vector) over the possible outcomes of the measurements on the PC bath. This result provides a unified picture for these seemingly different approaches.

The traditional, unconditional master equation approach is to define a total density matrix $R(t+dt) = |\Psi(t+dt)\rangle\langle\Psi(t+dt)|$, and then trace out the state of the bath. This leads to the reduced density matrix

$$\rho(t+dt) = \mathrm{Tr}_B[R(t+dt)] = \sum_n |\beta_n(t+dt)|^2 |\psi_n(t+dt)\rangle\langle\psi_n(t+dt)| \quad (24)$$

for the CQD qubit system alone. The effect of integrating or tracing out the environmental (detector) degrees of the freedom to obtain the reduced density matrix is equivalent to that of completely ignoring or averaging over the results of all measurement records $dN_c(t)$. This can be seen by taking ensemble average on the conditional, stochastic density matrix Eq. (22), identifying $\rho(t+dt) = E[|\psi_c(t+dt)\rangle\langle\psi_c(t+dt)|]$, and setting $E[dN_c(t)]$ equal to its expected value Eq. (20). Then the resultant equation leads to Eq. (24) with $|\beta_1(t+dt)|^2 = \mathcal{P}_1(t)dt$ and $|\beta_0(t+dt)|^2 = 1 - \mathcal{P}_1(t)dt$ for the quantum-jump case. Furthermore, with the help of Eqs. (13) and (16) [or Eqs. (7), (17) and (18)], we find the unconditional master equation:

$$\dot\rho(t) = -\frac{i}{\hbar}[\mathcal{H}_{CQD}, \rho(t)] + \mathcal{J}[\mathcal{T} + \mathcal{X} n_1]\rho(t) - \mathcal{A}[\mathcal{T} + \mathcal{X} n_1]\rho(t). \quad (25)$$

Note that the $\mathcal{J}$ term originating from the conditional state Eq. (16) or (18) represents the effect, due to a detection of an electron tunneling through the PC, on the CQD density matrix. This is why sometimes $\mathcal{J}$ is called a *jump* superoperator. Equation (25) can also be obtained as in Ref. [3] by taking the ensemble average over the observed stochastic process on Eq. (23) by setting $E[dN_c(t)]$ equal to its expected value Eq. (20). In this approach of master equation of the reduced density matrix, the influence of the PC bath, decoherence effect for example, on the CQD system can be analyzed [3]. But this approach or Eq. (25) does not tell us anything about the experimental observed quantity, namely the electron counts or current through PC. Hence, the PC detector in this approach is treated as a pure environment for the system, rather than a measurement device, which can provide information about the change of the state of the system.

An alternative approach recently developed in Refs. [5, 6, 7] is to take trace over environmental (detector) microscopic degrees of the freedom but keep track of the number of electrons, N, that have tunneled through the PC barrier during time t in the "partially" reduced density matrix. This allows one to extract information about the quantum state of the qubit, by measuring the time average current (N/t) through the PC. The master (rate) equation for this partially reduced density matrix for the CQD qubit system is derived in Ref. [5] from the so-called many-body Schrödinger equation. While it is derived in Refs. [6, 7], by means of the diagrammatic technique in the Keldysh forward and backward in time contour, for a Cooper-pair charge qubit coupled capacitively to a single-electron transistor. Here we show that it can be obtained for the CQD/PC model by taking a "partial" average on the conditional, stochastic CQD qubit system density matrix over the possible outcomes of the measurement on the PC bath. The procedure to take the "partial" average can be described as follows. First, taking the ensemble average on Eq. (23), we obtain Eq. (25). Then to keep track of the number of electrons N that have tunneled, we need to identify the effect of the *jump* superoperator $\mathcal{J}$ term in Eq. (25). If N electrons have tunneled through the PC at time $t + dt$, then the accumulated number of electrons in the drain at the earlier time t, due to the contribution of the *jump* term, should be $(N-1)$. After writing out the number dependence N or $(N-1)$ explicitly, we obtain the master equation for the "partially" reduced density matrix as:

$$\dot{\rho}(N,t) = -\frac{i}{\hbar}[\mathcal{H}_{CQD}, \rho(N,t)] + \mathcal{J}[\mathcal{T}+\mathcal{X}n_1]\rho(N-1,t) - \mathcal{A}[\mathcal{T}+\mathcal{X}n_1]\rho(N,t). \quad (26)$$

Evaluating Eq. (26) in the logical qubit charge state (i.e., perfect localization state of the charge in dot 1 and dot 2, respectively), we obtain the rate equations, the same as Eq. (3.3) of Ref. [5]. If the sum over all possible values of N is taken [i.e. tracing out the detector states completely, $\rho(t) = \sum_N \rho(N,t)$], Eq. (26) then reduces to Eq. (25). This procedure of reducing Eq. (23) or (21) to Eq. (26) and then to Eq. (25) by averaging over (tracing out) more and more available detector information provides a unified picture for these seemingly different approaches reported in the literature. This procedure is particularly simple using our formalism. That is if the (stochastic) master equation is expressed in a form in terms of superoperators $\mathcal{J}$ and $\mathcal{A}$, and the effect of the *jump* superoperator $\mathcal{J}$ term is identified.

To summarize, we have presented a heuristic derivation of the quantum trajectory (stochastic Schrödinger or master) equation starting from the full Schrödinger equation of the system and environment. We focus on the measurement interpretations to the quantum trajectories. Then the concept of quantum trajectories arises quite naturally from an expansion of the total state vector in eigen basis of the operator that represents the physical quantity or observable of the environment that is measured. In the CQD/PC model, the observed quantity of the PC environment is the number of electrons tunneling through the PC barrier. The stochasticity in the quantum trajectory can be view as being due to the randomness in the possible outcomes of the measurement record. We have shown that the quantum trajectory or stochastic Schrödinger equation approach provides us with the most (all) information as far as the system state evolution is concerned. In this approach, we are propagating *in parallel* the information of a conditioned (stochastic) state evolution $|\psi_c(t)\rangle$ and a detection record $dN_c(t)$ in a continuous measurement process. All the information carried away from the system to the reservoirs is recovered and contained by the measurement records $dN_c(t)$ of perfect detection or efficient measurement. This is why the system can be continuously described by a state vector rather than a reduced or "partially" reduced density matrix. We have also shown that the master equations of the reduced or "partially" reduced density matrix can be obtained as a result of taking an ensemble average or partial average over the possible measurement records in the quantum trajectory approach. This provides a unified picture for these seemingly different approaches. The procedure to achieve this unified view is particularly easy to understand using our formalism. Each quantum trajectory and corresponding detection record mimics a possible single run of the continuous in time measurement experiment. We will present elsewhere the simulation results for an initial qubit state readout experiment using both of the quantum trajectory and "partially" reduced density matrix approaches.

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