

Quantum Hall Effect in Quasi-One-Dimensional Conductors: The Roles of Moving FISDW, Finite Temperature, and Edge States

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Abstract. — This paper reviews recent developments in the theory of the quantum Hall effect (QHE) in the magnetic-field-induced spin-density-wave (FISDW) state of the quasi-one-dimensional organic conductors $(\text{TMTSF})_2\text{X}$. The origin and the basic features of the FISDW are reviewed. The QHE in the pinned FISDW state is derived in several simple, transparent ways, including the edge states formulation of the problem. The temperature dependence of the Hall conductivity is found to be the same as the temperature dependence of the Fröhlich current. It is shown that, when the FISDW is free to move, it produces an additional contribution to the Hall conductivity that nullifies the total Hall effect. The paper is written on mathematically simple level, emphasizes physical meaning over sophisticated mathematical technique, and uses inductive, rather than deductive, reasoning.

1 Introduction

Organic metals of the $(\text{TMTSF})_2\text{X}$ family, where TMTSF is tetramethyltetraselenafulvalene and X represents an inorganic anion, are highly anisotropic, quasi-one-dimensional (Q1D) crystals that consist of parallel conducting chains. The overlap of the electron wave functions and the electric conductivity are the highest in the direction of the chains (the **a** direction) and are much smaller in the **b** direction perpendicular to the chains. In this paper, we neglect the coupling between the chains in the third, **c** direction, which is weaker than in the **b** direction. We study the properties of a single layer (the **a-b** plane) that consist of weakly coupled parallel chains, modeling $(\text{TMTSF})_2\text{X}$ as a system of the uncoupled two-dimensional (2D) layers.

The $(\text{TMTSF})_2\text{X}$ materials exhibit very interesting behavior when a strong magnetic field is applied perpendicular to the **a-b** plane. At low temperature below several Kelvin and magnetic field of the order of five Tesla, there is a phase transition from the metallic state to a state, where the spin-density wave appears. This state is called the magnetic-field-induced spin-density-wave (FISDW) state (see Ref. [1] for a review). As the magnetic field is increased further, a

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sequence (cascade) of phase transitions between different FISDWs is observed. Interestingly, within each FISDW phase, the value of the Hall resistance remains constant, independent of the magnetic field, that is, the quantum Hall effect (QHE) is observed. Once the boundary to another FISDW phase is crossed, the value of the Hall resistance jumps to a new value, which remains constant until the next phase boundary is crossed.

It is instructive to compare the QHE in the FISDW state with the conventional QHE observed in 2D semiconductor devices. In both cases, at sufficiently low temperatures, the longitudinal resistivity ρ_{xx} is much smaller than the Hall resistivity ρ_{xy} and is thermally activated with an energy gap, which is equal to 6 K in one of the FISDW phases [2]. The theory says that the Hall conductivity per one layer is quantized: $\sigma_{xy} = 2Ne^2/h$. Whereas in the case of a 2D electron gas in a single semiconducting layer this formula can be directly verified experimentally, in the $(\text{TMTSF})_2\text{X}$ materials the situation is more complicated. What is measured experimentally in $(\text{TMTSF})_2\text{X}$ is the total, bulk Hall resistance of many parallel layers. To find the Hall conductivity per one layer, one needs to know the effective number of layers contributing to the Hall conductivity, which depends on the electric current distribution in the sample and, thus, is somewhat uncertain. So, in $(\text{TMTSF})_2\text{X}$, one can only compare the relative values of the Hall resistances at different plateaus and deduce the integer numbers N from these ratios. For this reason, it is hard to unambiguously discriminate experimentally between the integer and the fractional QHE in the $(\text{TMTSF})_2\text{X}$ materials. Nevertheless, the common belief, strongly influenced by the theory (see the rest of the paper), is that the QHE in $(\text{TMTSF})_2\text{X}$ is the integer one.

Unlike semiconductors, the $(\text{TMTSF})_2\text{X}$ materials have very high, metallic concentration of carriers (one conducting hole per unit cell). Thus, in $(\text{TMTSF})_2\text{X}$, a naively calculated filling factor of the Landau levels in a realistic magnetic field is enormous, of the order of $10^2 - 10^3$, depending on the magnitude of the field. At the same time, the Hall conductivity is quantized with a small, single-digit number N . The discrepancy between the naive filling factor and the value of the Hall conductivity is resolved by the very important fact that the QHE in $(\text{TMTSF})_2\text{X}$ exists solely due to the phase transition into a FISDW state. The FISDW effectively eliminates most of the carriers, reducing the filling factor to the single-digit number N , which manifests itself in the value of the Hall conductivity. In this respect, the QHE in the $(\text{TMTSF})_2\text{X}$ materials significantly differs from the conventional QHE in semiconductors, where the QHE state is not associated with any thermodynamic phase transition and order parameter. In $(\text{TMTSF})_2\text{X}$, the transitions between the QHE plateaus are true thermodynamic phase transitions, accompanied by changes in the FISDW order parameter and observed in the measurements of specific heat [3], magnetization [4], NMR [5], and virtually any other physical quantity. For a given magnetic field, the effective filling factor N is determined by delicate and nontrivial FISDW thermodynamics, which may vary from one material to another. Thus, the ratios of the Hall resistances in the consecutive FISDW phases of $(\text{TMTSF})_2\text{PF}_6$ are typically equal to the ratios of the consecutive integer numbers 1:2:3:4:5 [6], whereas in $(\text{TMTSF})_2\text{ClO}_4$ [7] and $(\text{TMTSF})_2\text{ReO}_4$ [8] the ratios do not follow any simple sequence and may change sign. In this paper, we do not discuss the FISDW thermodynamics in detail. In Sec. 2 we only demonstrate that a FISDW is characterized by an integer number N , and in Sec. 3 we show that this number appears in the expression for the Hall conductivity. However, we do not calculate how the number N depends on the magnetic field and other parameters of the model. These issues are discussed in detail in the theory of the FISDW formation (see Refs. [1, 9] for reviews).

Early theoretical approaches [10, 11] explained the QHE in the FISDW state by counting the number of carriers left after the FISDW gap opens. While this calculation gives correct

answer, it is not completely satisfactory, because the FISDW gap is much smaller than the cyclotron frequency of the magnetic field, which makes the FISDW case totally opposite to the standard semiconductor situation, from where the concept of the calculation is borrowed. Furthermore, the “insulating” FISDW gap and the “Landau” gaps due to the magnetic field hybridize very strongly, which make the situation even more complicated. In Ref. [12], the QHE was derived rigorously, albeit somewhat indirectly, using the Streda formula. In Ref. [13], the QHE was calculated directly, using a manifestly topologically-invariant expression for the Hall conductivity in terms of the electrons wave functions that follows from the Kubo formula. This approach can be straightforwardly generalized [13, 14] to the case where several FISDW order parameters coexist [15].

The present paper is devoted mostly to recent developments in the theory of the QHE in the FISDW state. In Sec. 2, we explain the basics of the FISDW. In Sec. 3, we give yet another derivation of the QHE that emphasizes analogy between the QHE and the Fröhlich conduction of a charge/spin-density wave. In Sec. 4, this analogy is utilized to discuss what happens to the QHE when the FISDW moves. In Sec. 5, the effect of a finite temperature on the QHE is calculated. In Sec. 6, we reformulate the QHE in terms of the edge states. In Sec. 7, conclusions are given. Throughout the paper, we try to keep discussion on mathematically simple level, emphasizing physical meaning over sophisticated mathematical technique and using inductive, rather than deductive, reasoning.

2 Formation of the FISDW

For pedagogical purposes, let us start consideration from a simple one-dimensional (1D) system, where electrons are confined to a chain parallel to the x axis. Suppose the electron dispersion law is parabolic, so the Hamiltonian $\hat{\mathcal{H}}$ can be written as:³⁾

$$\hat{\mathcal{H}} = \hbar^2 k_x^2 / 2m, \quad (1)$$

where $\hbar = h/2\pi$ is the Planck constant, m is the electron mass, and k_x is the electron wave vector along the chain. At zero temperature, the electrons occupy the quantum states with the wave vectors from $-k_F$ to k_F and the energies up to E_F , where k_F and E_F are the Fermi wave vector and energy, which are determined by the concentration of the electrons.

Now, suppose that a periodic potential is present in the system, so that the Hamiltonian is equal to:

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + 2\Delta \cos(Q_x x), \quad (2)$$

where Q_x is the wave vector and $\Delta \ll E_F$ is the amplitude of the periodic potential. As it is well known from quantum mechanics, the periodic potential opens an energy gap of the magnitude 2Δ in the electron spectrum at the wave vectors $k_x = \pm Q_x/2$.⁴⁾ If the wave vector of the periodic potential connects the two Fermi points of the electrons:

$$Q_x = 2k_F, \quad (3)$$

the gap opens right at the Fermi level, so the states below the gap are completely occupied and the states above are completely empty. It is clear that the total energy of the electrons

³⁾The actual form of the longitudinal dispersion law (1) is not very essential.

⁴⁾Smaller gaps, opened at the higher integer multiples of $\pm Q_x/2$, are not essential for our consideration.

is reduced compared to the total energy in the absence of the periodic potential. Thus, if the electrons interact between themselves, they might decide to produce the periodic potential spontaneously, self-consistently in order to reduce the total energy of the system. This phenomenon is called the Peierls instability. Once the periodic potential appears in the system, it modulates the charge or spin density of the electrons, producing a charge- or spin-density wave (CDW/SDW) with the wave vector Q_x (see Ref. [16] for a review). We do not discuss here the details of the interaction between the electrons that leads to formation of the periodic potential. In this paper, we focus only on the mean-field periodic potential experienced by the electrons once the CDW/SDW has been established, presuming that the self-consistency conditions are satisfied. For our purposes, the distinction between the CDW and SDW is not important, so we pay no attention to the spin indices.

Now, to model (TMTSF)₂X, let us consider a 2D system that consists of many chains, parallel to the x axis and equally spaced along the y -axis with the distance b .⁵⁾ The chains are coupled through the electron tunneling of the amplitude t_b , so the electron Hamiltonian is:

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + 2\Delta \cos(Q_x x) + 2t_b \cos(k_y b), \quad (4)$$

where k_y is the electron wave vector across the chains. Hamiltonian (4) is written in the mixed representation, where an electron wave function depends on the coordinate x along the chains and the momentum k_y across the chains. As follows from Eq. (4), the electron energy now depends on the momentum k_y . Strictly speaking, in the presence of many chains, the CDW/SDW potential may also have a certain periodicity across the chains and should be written as $2\Delta \cos(Q_x x + Q_y n b)$, where n is the chain number and Q_y is the wave vector of the CDW/SDW across the chains. To simplify calculations, we consider only the case of $Q_y = 0$, which is not the most realistic case, but the results are qualitatively valid also in a more realistic case of $Q_y \neq 0$. To achieve quantitative agreement between the theory and experiment, it may be necessary to consider a more complicated transverse dispersion law of the electrons and to include the next-nearest-neighbor hopping term $2t'_b \cos(2k_y b)$ in the Hamiltonian. To simplify out qualitative discussion, we neglect this term.

Now, suppose that a magnetic field H is applied along the z -axis perpendicular to the (x, y) -plane. To describe the magnetic field, we select the Landau gauge:⁶⁾

$$A_x = A_z = 0, \quad A_y = Hx, \quad (5)$$

and do the Peierls–Onsager substitution, $k_y \rightarrow k_y - eA_y/c\hbar$, in Hamiltonian (4). The Hamiltonian becomes:

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + 2\Delta \cos(Q_x x) + 2t_b \cos(k_y b - G_x x), \quad (6)$$

where

$$G_x = ebH/\hbar c. \quad (7)$$

Comparing Eqs. (4) and (6), we see that, in the presence of the magnetic field, the hopping *across* the chains becomes a periodic potential *along* the chains with the wave vector G_x (7).

⁵⁾The x and y axes correspond to the \mathbf{a} and \mathbf{b} axes of (TMTSF)₂X.

⁶⁾The fact that we use a specific gauge does not invalidate our results in any way. This gauge is selected to simplify calculations. We can perform the calculations in the most general, arbitrary gauge, but the formulas would be more complicated.

We will refer to this periodic potential as the “hopping potential”. The period of this potential (the magnetic length),

$$l_H = 2\pi/G_x, \quad (8)$$

is determined by the condition that magnetic flux through a 2D cell formed by the magnetic length along the chains, l_H , and the distance between the chains, b , is equal to the flux quantum, ϕ_0 :

$$l_H b H = \phi_0 = hc/e. \quad (9)$$

Eq. (9) is equivalent to Eqs. (7) and (8).

According to Eq. (6), in the presence of both the CDW/SDW and the magnetic field, the electrons experience two periodic potentials with the wave vectors Q_x and G_x . The magnitudes of the two wave vectors are very different. Q_x is big, of the order of $2k_F$, and the corresponding period, $l_{DW} = 2\pi/Q_x$, is short, of the order of the distance between the electrons. On the other hand, in realistic magnetic fields, the magnetic length l_H is much longer than the inter-electron distance, thus the ratio of the wave vectors, G_x/Q_x , is very small, of the order of $10^{-2} - 10^{-3}$, depending on the value of the magnetic field. Thus, the two periodic potentials can be treated as incommensurate. In this case, the energy spectrum is degenerate in k_y , because changing k_y means simply shifting the hopping potential in Eq. (6) along the x axis, which does not change the energy.

To get qualitative picture of the energy spectrum produced by the two periodic potentials, let us assume for a moment that t_b is very small and can be treated as a perturbation. Taken alone, the CDW/SDW potential opens a gap in the electron spectrum at the wave vectors $k_x = \pm Q_x/2$ connected by the CDW/SDW wave vector Q_x . The CDW/SDW potential, combined perturbationally with the hopping potential, opens gaps at the wave vectors $k_x = \pm(Q_x \pm G_x)/2$ connected by the wave vectors $Q_x \pm G_x$ obtained by combining the wave vectors of the two periodic potentials. In the same manner, the CDW/SDW potential, combined n times with the hopping potential, opens gaps at the wave vectors $k_x = \pm(Q_x \pm nG_x)/2$ connected by the combinational wave vectors $Q_x \pm nG_x$. Thus, the electron spectrum contains a sequence of energy gaps, which are equally spaced in momentum k_x with the distance $G_x/2$ [17]. The gaps separate energy bands; each band has the total width $\Delta k_x = G_x$. These bands can be interpreted as the Landau levels broadened into the energy bands (with the dispersion in k_x) by the periodic arrangement of the chains with the period b . The Landau degeneracy in k_y remains in the problem.⁷⁾ The number of states per unit area in each band is equal to

$$\frac{\Delta k_x \Delta k_y}{(2\pi)^2} = \frac{G_x}{(2\pi)^2} \frac{2\pi}{b} = \frac{eH}{hc}, \quad (10)$$

which coincides with the number of states in a Landau level.

In the (TMTSF)₂X materials, the interchain hopping is not that small and, generally speaking, cannot be treated as a perturbation.⁸⁾ Nevertheless, the qualitative picture of the electron energy spectrum outlined above remains valid with the important quantitative difference that some “secondary” gaps, opened at the combinational wave vectors $Q_x + nG_x$, may be bigger than the “primary” gap, opened at Q_x . Since the CDW/SDW potential is produced self-consistently to maximize the energy gain, the electrons would create the CDW/SDW with such a wave vector that the biggest secondary energy gap is located exactly at the Fermi level.

⁷⁾When several FSDW order parameters with different wave vectors (11) coexist, the degeneracy in k_y is lifted.

⁸⁾See Sec. 5 for a nonperturbative treatment of the problem.

In this case, the wave vector of the biggest gap, $Q_x + NG_x$, characterized by some integer number N , must coincide with $2k_F$, the span of the Fermi sea: $Q_x + NG_x = 2k_F$. Thus, the wave vector of the CDW/SDW is determined by the following equation:

$$Q_x = 2k_F - NG_x = 2k_F - NebH/\hbar c. \quad (11)$$

This is the most important equation of this Section. It shows that, in a multichain, 2D system subject to a magnetic field, the longitudinal wave vector of the CDW/SDW is not necessarily equal to $2k_F$, as it was in strictly 1D system (3), but may take many different values (11) labeled by an integer number N [10]. In this paper, we do not calculate which values of N the system selects for a given magnetic field H and given microscopic parameters of the model (t_b , E_F , the amplitude of interaction between the electrons g , etc.). These issues are addressed in reviews [1, 9], as well as in original articles, e.g. [10, 13, 18, 19, 17]. We assume that the value of N is given to us and study the properties of the system in this state, specifically the Hall effect.

The CDW/SDW wave vector (11) changes linearly with the magnetic field H in order to keep the energy gap exactly at the Fermi level. Because the magnetic field is intrinsically involved in the formation of the energy gap at the Fermi level, this kind of density wave in $(\text{TMTSF})_2\text{X}$ is called the magnetic-field-induced spin-density wave (FISDW).⁹⁾ We will use the term FISDW in the rest of the paper.

3 The Quantum Hall Effect

Let us discuss the Hall conductivity of our 2D system in the FISDW state at zero temperature. By naive analogy with conventional semiconductors, one might say [10, 11] that all electron states below the “primary” gap, opened by the FISDW potential at the wave vector Q_x , do not contribute to the Hall conductivity. Thus, the effective number of carriers per one chain is the difference between the total number of carriers, proportional to the size of the Fermi sea $2k_F$, and the number of the “eliminated” carriers, proportional to Q_x :

$$\rho_{\text{eff}} = 2 \frac{2k_F - Q_x}{2\pi b} = \frac{2NeH}{\hbar c}, \quad (12)$$

where the first factor 2 comes from the spin, and the second equality follows from Eq. (11). Substituting Eq. (12) into the conventional formula for the Hall conductivity:

$$\sigma_{xy} = \rho_{\text{eff}} ec/H, \quad (13)$$

we see that the magnetic field cancels out and the Hall conductivity is quantized:

$$\sigma_{xy} = 2Ne^2/h. \quad (14)$$

This derivation can be summarized as follows. The FISDW wave vector (11) adjusts its value to the magnetic field in such a manner that there are always N completely filled Landau bands between the “primary”, “insulating” FISDW gap and the Fermi level. Thus, the Hall conductivity is quantized with the effective number of the Landau bands N . It is by the

⁹⁾The density wave happens to be the spin, not the charge one in $(\text{TMTSF})_2\text{X}$, which is not essential for our discussion.

elimination of almost all of the carriers the FISDW reduces the effective filling factor from $10^2 - 10^3$ to the single-digit number N .

Although the above derivation of σ_{xy} gives correct answer (14), it raises many questions and doubts. Why do we say that the ‘‘primary’’ gap, which is even not the biggest one, ‘‘eliminates’’ the carriers from the Hall effect, whereas the ‘‘secondary’’ gaps do not? Is formula (13) applicable in our situation? To our opinion, the derivation given above is not convincing, and below we give another, rigorous derivation, which is based on the ideas of Refs. [20, 21].

Suppose the electric field E_y is applied perpendicular to the chains. Let us use the following gauge

$$A_x = A_z = 0, \quad A_y = Hx - E_y ct, \quad (15)$$

where t is the time. In the presence of the electric field, the electron Hamiltonian (6) becomes

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + 2\Delta \cos(Q_x x + \Theta) + 2t_b \cos[k_y b - G_x(x - v_{E_y} t)], \quad (16)$$

where, for further purposes, we introduced an arbitrary phase Θ in the FISDW potential, and

$$v_{E_y} = cE_y/H \quad (17)$$

is the drift velocity in the crossed electric and magnetic fields.

We see that, in the presence of the transverse electric field E_y , the hopping potential in Eq. (16) moves along the chains with the velocity v_{E_y} (17) proportional to E_y . Because in the FISDW state all electrons are under the energy gap, by analogy with the Fröhlich conduction produced by motion of a CDW/SDW, the motion of the hopping potential in Eq. (16) should induce some electric current along the chains, j_x , proportional to the velocity v_{E_y} . This is the Hall current, and, once we know j_x , the Hall conductivity can be calculated as $\sigma_{xy} = j_x/E_y$. The difficulty of our problem is that there are two different periodic potentials in Hamiltonian (16), due to the FISDW and due to the hopping. Normally, the FISDW potential is pinned and does not move, whereas the hopping potential moves due to the presence of the electric field E_y in its argument and cannot be pinned.

In order to calculate σ_{xy} , let us consider a more general case where the FISDW potential may also move. To find the Hall conductivity of the pinned FISDW, we will set the FISDW velocity to zero at the end of the calculation. According to Eq. (16), when the FISDW potential moves, the FISDW phase Θ changes in time t , so that the velocity of the motion v_{DW} is proportional to the time derivative $\dot{\Theta}$:

$$v_{\text{DW}} = -\dot{\Theta}/Q_x. \quad (18)$$

We assume that both E_y and $\dot{\Theta}$ are infinitesimal, thus the velocities v_{E_y} (17) and v_{DW} (18) are very small, so the motion of the potentials is adiabatic.

Now, let us calculate the current along the chains produced by the motion of the potentials. Since there is an energy gap at the Fermi level, following the arguments of Laughlin [22], we can say that an integer number of electrons N_1 is transferred from one end of a chain to another, when the FISDW potential is adiabatically shifted along the chain by its period $l_{\text{DW}} = 2\pi/Q_x$. The same is true, with an integer N_2 instead of N_1 , for a displacement of the hopping potential by its period $l_H = 2\pi/G_x$. Because the two potentials are incommensurate, if the first potential is shifted by dx_1 and the second by dx_2 , the total transferred charge dq is the sum of the prorated amounts of N_1 and N_2 :

$$dq = eN_1 \frac{dx_1}{l_{\text{DW}}} + eN_2 \frac{dx_2}{l_H}. \quad (19)$$

Now, suppose that both potentials are shifted by the same displacement $dx = dx_1 = dx_2$. In this case, we can also write that

$$dq = e\rho dx, \quad (20)$$

where $\rho = 4k_F/2\pi$ is the concentration of the electrons. Equating (19) and (20) and substituting the expressions for ρ , l_{DW} (11), and l_H (8), we find the following Diophantine-type equation [20]:

$$4k_F = N_1(2k_F - NG_x) + N_2G_x, \quad (21)$$

where N is the integer that characterizes the FISDW. Since k_F/G_x is, in general, an irrational number, the only possible solution of Eq. (21) for the integers N_1 and N_2 is

$$N_1 = 2, \quad N_2 = N_1N = 2N. \quad (22)$$

Dividing Eq. (19) by the distance between the chains b and by the time increment dt and using expressions (17) and (18) for the velocities and (22) for the integers, we find the density of current along the chains:

$$j_x = -\frac{e}{\pi b}\dot{\Theta} + \frac{2Ne^2}{h}E_y. \quad (23)$$

The first term in Eq. (23) represents the contribution of the FISDW motion, the so-called Fröhlich conductivity [16]. This term vanishes when the FISDW is pinned and does not move ($\dot{\Theta} = 0$). The second term describes the quantum Hall effect. The expression for σ_{xy} that follows from Eq. (23) coincides with Eq. (14). Apparently, the QHE in the FISDW is the integer one, and the derivation given above seems to exclude a possibility of the fractional QHE.

Having derived the QHE in the FISDW state, let us compare it with the conventional integer QHE in semiconductors. In the latter systems, the electron localization due to disorder is thought to play an important role by providing a reservoir of electron states necessary to maintain a constant value of the Hall conductivity with the varying magnetic field. On the other hand, in the FISDW state, we deal with the QHE in a clean, periodic 2D potential. This problem was considered in Ref. [23], either for a tight-binding model, or for two weak sinusoidal potentials in the x and y directions. In the FISDW state, we have an intermediate case, where the chains produce a tight-binding potential in the y direction, whereas the FISDW provides a weak sinusoidal potential in the x direction.¹⁰⁾ It is important that the period of the FISDW potential is not fixed rigidly, but varies with the magnetic field, so that the electrons are redistributed between the Fermi “reservoir” below the “primary” FISDW gap and the “Hall states” above that gap. It is because of this adjustment of the FISDW periodicity the Hall conductivity maintains a constant value with the varying magnetic field. This is in contrast to the models of Ref. [23], where the periodicity of the potentials is fixed, and the Hall conductivity jumps wildly when the magnetic field varies a little. In the FISDW state, impurities are not necessary to produce the QHE, except to pin the FISDW, because, as shown in the next Section, if the FISDW is not pinned and is free to move, the QHE disappears. The (TMTSF)₂X materials seem to be the only substances where the QHE in a 2D periodic potential is realized experimentally.

¹⁰⁾The crystal lattice periodicity in the x direction does not play essential role in our model and may be neglected.

4 Motion of the FISDW

In Sec. 3 we have demonstrated that, when the FISDW is pinned, the Hall conductivity is quantized. This result applies to the case where the applied electric field is weak and time-independent. On the other hand, when the electric field is strong or time-dependent, the FISDW may move. It is interesting to study how this motion would influence the QHE. At first sight, since the density-wave can move only along the chains, this purely 1D motion cannot contribute to the Hall effect, which is essentially a 2D effect. On the other hand, according to Eq. (23), the Fröhlich conductivity due to the motion of the FISDW does contribute to the Hall current along the chains j_x and, in this way, may modify the Hall effect. To solve the problem, we need to find how the velocity of the FISDW, $v_{\text{DW}} \propto \dot{\Theta}$, depends on E_y . It is well known that the electric field along the chains, E_x , may induce the motion of a density wave along the chains. However, it is not obvious whether the electric field *across* the chains E_y may induce the FISDW motion *along* the chains. To study this issue, first we derive the equation of motion of an ideal FISDW and then phenomenologically add pinning and damping of the FISDW to the equation to make it more realistic. We stay within the linear response theory, having in mind depinning of the FISDW by an infinitesimal ac electric field, not by a strong dc field. We study rigid motion of the FISDW, where the phase Θ depends only on the time t , but not on the coordinates x and y . We restrict consideration to the frequencies much lower than the FISDW gap and take into account only collective motion of the FISDW, not single-electron excitations across the gap.

To derive the equation of motion of $\Theta(t)$, we need to know the Lagrangian of the system, L . Two terms in L can be readily recovered by taking into account that the current density j_x (23) is the variational derivative of the Lagrangian with respect to the electromagnetic vector-potential A_x :

$$j_x = c \delta L / \delta A_x. \quad (24)$$

Written in the gauge-invariant form, the recovered part of the Lagrangian is equal to

$$L_1 = - \sum_{i,j,k} \frac{Ne^2}{\hbar c} \varepsilon_{ijk} A_i F_{jk} - \frac{e}{\pi b} \Theta E_x, \quad (25)$$

where ε_{ijk} is the antisymmetric tensor with the indices $i, j, k = t, x, y$; A_i and F_{jk} are the vector-potential and the tensor of the electromagnetic field; and $E_x \equiv F_{tx}$ is the electric field along the chains. The first term in Eq. (25) is the so-called Chern–Simons term responsible for the QHE [13]. The second term describes the interaction of the density-wave condensate with the electric field along the chains [16].

Lagrangian (25) should be supplemented with the kinetic energy of the FISDW condensate, K . Being produced by the instantaneous Coulomb interaction between the electrons, the FISDW potential itself has no inertia. So, K consists of only the kinetic energy of the electrons confined under the FISDW gap. This energy is proportional to the square of the average electron velocity, which, in turn, is proportional to the electric current along the chains:

$$K = \frac{\pi \hbar b}{4v_F e^2} j_x^2, \quad (26)$$

where $v_F = k_F/m$ is the Fermi velocity. Substituting Eq. (23) into Eq. (26), expanding, and omitting the unimportant term proportional to E_y^2 , we obtain the second part of the Lagrangian:

$$L_2 = \frac{\hbar}{4\pi b v_F} \dot{\Theta}^2 - \frac{eN}{2\pi v_F} \dot{\Theta} E_y. \quad (27)$$

The first term in Eq. (27) is the same as the kinetic energy of a purely 1D density wave [16] and is not specific to the FISDW. The most important is the second term which describes the interaction of the FISDW motion and the electric field perpendicular to the chains. This term is allowed by symmetry in the considered system and has the structure of a mixed vector–scalar product:

$$\mathbf{v}_{\text{DW}}[\mathbf{E} \times \mathbf{H}]. \quad (28)$$

Here, \mathbf{v}_{DW} is the velocity of the FISDW, which is proportional to $\dot{\Theta}$ and is directed along the chains, that is, along the x -axis. The magnetic field \mathbf{H} is directed along the z -axis; thus, the electric field \mathbf{E} may enter Eq. (28) only through the component E_y . Comparing formula (28) with the last term in Eq. (27), one should take into account that the magnetic field enters the last term implicitly, through the integer N , which depends on H and changes sign when H changes sign.

Varying the total Lagrangian $L = L_1 + L_2$, given by Eqs. (25) and (27), with respect to Θ , we find the equation of motion of $\Theta(t)$:

$$\ddot{\Theta} = -\frac{2ev_{\text{F}}}{\hbar}E_x + \frac{eNb}{\hbar}\dot{E}_y. \quad (29)$$

In Eq. (29), the first two terms constitute the standard 1D equation of motion of the density wave [16], whereas the last term, proportional to the time derivative of E_y , comes from the last term in Eq. (27) and describes the influence of the electric field across the chains on the motion of the FISDW along the chains.

Setting $E_x = 0$ and integrating Eq. (29) in time, we find that

$$\dot{\Theta} = eNbE_y/\hbar. \quad (30)$$

Substituting Eq. (30) into Eq. (23), we see that the first term in Eq. (23) (the Fröhlich conductivity of the FISDW) precisely cancels the second term (the quantum Hall current), so the total Hall current is equal to zero. This result could have been derived without calculations from the fact that the time dependence $\Theta(t)$ is determined by the principle of minimal action. The relevant part of the action is given, in this case, by Eq. (26), which attains the minimal value at $j_x = 0$. We can say that, if the FISDW is free to move, it adjusts its velocity to “compensate” the external electric field E_y and to keep zero Hall current.

It is instructive to see how the nullification of the Hall conductivity takes place in the case where the electric field is directed along the chains. Varying L (Eqs. (25) and (27)) with respect to A_y , we find the density of current perpendicular to the chains:

$$j_y = -\frac{2Ne^2}{h}E_x - \frac{eN}{2\pi v_{\text{F}}}\ddot{\Theta}. \quad (31)$$

In the r.h.s. of Eq. (31), the first term describes the quantum Hall current, whereas the second term, proportional to the *acceleration* of the FISDW condensate, comes from the last term in Eq. (27) and reflects the contribution of the FISDW motion along the chains to the electric current across the chains. According to the equation of motion (29), the electric field along the chains accelerates the density wave:

$$\ddot{\Theta} = -2ev_{\text{F}}E_x/\hbar, \quad (32)$$

thus, the Hall current (31) vanishes.

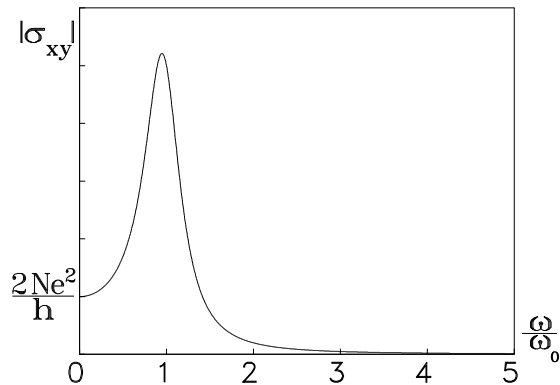


Figure 1: Absolute value of the Hall conductivity in the FISDW state as a function of the frequency ω normalized to the pinning frequency ω_0 , as given by Eq. (34) with $\omega_0\tau = 2$.

However, it is clear that, in stationary, dc measurements, the acceleration of the FISDW, discussed in the previous paragraph, cannot last forever. Any friction or dissipation will inevitably stabilize the motion of the density wave to a steady flow with zero acceleration. In the steady state, the last term in Eq. (31) vanishes, and the current j_y recovers its quantum Hall value. The same is true in the case where the electric field is perpendicular to the chains. In that case, the dissipation eventually stops the motion of the FISDW along the chains and restores j_x (23) to the quantum Hall value. The conclusion is that the contribution of the moving FISDW condensate to the Hall conductivity is essentially nonstationary and cannot be observed in dc measurements.

On the other hand, the effect can be seen in ac measurements. To be realistic, let us add damping and pinning [16] to the equation of motion of the FISDW (29):

$$\ddot{\Theta} + \frac{1}{\tau}\dot{\Theta} + \omega_0^2\Theta = -\frac{2ev_F}{\hbar}E_x + \frac{eNb}{\hbar}\dot{E}_y, \quad (33)$$

where τ is the relaxation time and ω_0 is the pinning frequency. Solving Eq. (33) via the Fourier transformation from the time t to the frequency ω and substituting the result into Eqs. (23) and (31), we find the Hall conductivity as a function of the frequency:

$$\sigma_{xy}(\omega) = \frac{2Ne^2}{h} \frac{\omega_0^2 - i\omega/\tau}{\omega_0^2 - \omega^2 - i\omega/\tau}. \quad (34)$$

The absolute value of the Hall conductivity, $|\sigma_{xy}|$, computed from Eq. (34) is plotted in Fig. 1 as a function of ω/ω_0 for $\omega_0\tau = 2$. The Hall conductivity is quantized at zero frequency and has a resonance at the pinning frequency. At higher frequencies, where the pinning and the damping can be neglected, and the system effectively behaves as an ideal, purely inertial system considered above, the Hall conductivity does decrease toward zero.

Frequency dependence of the Hall conductivity in conventional, semiconductor QHE systems was measured using the technique of crossed wave guides [24], but no measurements have been done in a FISDW system thus far. Such measurements would be very interesting, because

the ac behavior of the FISDW should differentiate the QHE in $(\text{TMTSF})_2\text{X}$ from the conventional QHE in semiconductors. To give a crude estimate of the required frequency range, we quote the value of the pinning frequency $\omega_0 \sim 3 \text{ GHz} \sim 0.1 \text{ K} \sim 10 \text{ cm}$ for a regular SDW (not FISDW) in $(\text{TMTSF})_2\text{PF}_6$ [25].

Theoretically, frequency dependence of the Hall conductivity in a FISDW system was considered in Ref. [30]. This theory fails to produce the QHE at zero frequency; thus, it does not agree with our results. The interplay between the QHE and the motion of the FISDW was discussed in Ref. [31]. Unfortunately, this paper has troubles with calculations and physical interpretation and cannot be considered as a consistent theory. The influence of the FISDW motion on the QHE was described by the present authors in Ref. [32].

Due to the presence of the magnetic field in the problem, we could phenomenologically add a term proportional to E_y to Eq. (33) and a term proportional to $\dot{\Theta}$ to Eq. (31). These terms violate the time reversal symmetry of the equations, which indicate the dissipative nature of these terms. Thus, these terms cannot be derived within the Lagrangian formalism, employed in this Section, and should be obtained from the Boltzmann equation, where the time-reversal symmetry is already broken. Because the dissipation is associated with the normal carriers thermally excited across the FISDW energy gap, these terms should be exponentially small and negligible at low temperatures. If taken into account, these terms would modify the frequency dependence of σ_{xy} (Eq. (34) and Fig. 1) at intermediate frequencies, but not at zero and high frequencies.

In this Section, we did not touch the issue of the FISDW depinning by a strong dc electric field. In that case, the motion of the density wave is controlled by dissipation, which is very difficult to study theoretically on microscopic level. The influence of a steady motion of a regular CDW on the Hall conductivity was studied theoretically in Ref. [26]. The results can be interpreted in the following way: The steady motion of the CDW condensate itself does not contribute to the Hall effect; however, this motion influences the thermally excited normal carriers and, in this way, affects the Hall voltage. This theory is complimentary to our theory, which studies only the condensate contribution at zero temperature. Mathematically, the steady motion of the density wave modifies the Hall effect via the dissipative terms discussed in the previous paragraph. Since the bare value of the Hall conductivity in a regular CDW/SDW system is determined by the normal carriers only, the steady motion of the density wave produces a considerable, of the order of unity, effect on the Hall conductivity, which was observed experimentally [27]. On the other hand, in the case of the FISDW, where the big quantum contribution from the electrons below the gap dominates the Hall conductivity, the contribution of the thermally excited normal carriers to the Hall conductivity should be negligible at low temperatures. Thus, the steady motion of the FISDW should not change considerably the Hall voltage, as, indeed, it was observed experimentally in $(\text{TMTSF})_2\text{ClO}_4$ [28]. More recent measurements in $(\text{TMTSF})_2\text{PF}_6$ [29] show results in some sense opposite to the results of Ref. [28]. The origin of the difference is not clear at the moment.

5 Finite Temperature

The Hall conductivity at a finite temperature is not quantized because of the presence of thermally excited quasiparticles above the energy gap. It is interesting to find how the Hall conductivity evolves with the temperature. Because the QHE at zero temperature is generated by the collective motion of the electrons in the FISDW condensate, the issue here is the temperature dependence of the condensate current. Obviously, the condensate current must

gradually decrease with increasing temperature and vanish at the transition temperature T_c , where the FISDW order parameter disappears. This behavior is qualitatively similar to the temperature dependence of the superconducting condensate density and the inverse magnetic field penetration depth in superconductors.

In order to obtain explicit results, we need to make some approximations. Let us linearize the parabolic dispersion law in Hamiltonian (1) near the Fermi energy:

$$\hbar^2 k_x^2 / 2m - E_F \approx \pm v_F (k_x \mp k_F), \quad (35)$$

and focus on the electrons whose momenta are close $+k_F$ and $-k_F$. Let us count their momenta from $+k_F$ and $-k_F$ and label their wave functions by the index \pm : ψ_+ and ψ_- . In this representation, a complete electron wave function is a spinor (ψ_+, ψ_-) , and the Hamiltonian is a 2×2 matrix, which can be expanded over the Pauli matrices $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3$, and the unity matrix $\hat{1}$ (which we will not write explicitly in the following formulas). Taking into account Eq. (11), we can rewrite Hamiltonian (16) in the spinor representation as

$$\hat{\mathcal{H}} = -i\hbar v_F \hat{\tau}_3 \frac{\partial}{\partial x} + \Delta \hat{\tau}_1 e^{i\hat{\tau}_3 (NG_x x - \Theta)} + 2t_b \cos[k_y b - G_x(x - v_{E_y} t)]. \quad (36)$$

The last term in Eq. (36) can be eliminated by chiral transformation of the electron wave function.¹¹⁾

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \rightarrow \exp \left\{ i\hat{\tau}_3 \frac{2t_b}{\hbar\omega_c} \sin[k_y b - G_x(x - v_{E_y} t)] \right\} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad (37)$$

where

$$\hbar\omega_c = \hbar v_F G_x = ebHv_F/c \quad (38)$$

is the characteristic energy of the magnetic field (the cyclotron frequency), which is equal to the distance in energy between the Landau gaps discussed in Sec. 2. In representation (37), Hamiltonian (36) becomes

$$\hat{\mathcal{H}} = -i\hbar v_F \hat{\tau}_3 \frac{\partial}{\partial x} + \Delta \hat{\tau}_1 \exp\{i\hat{\tau}_3 (NG_x x - \Theta)\} \exp \left\{ i\hat{\tau}_3 \frac{4t_b}{\hbar\omega_c} \sin[k_y b - G_x(x - v_{E_y} t)] \right\}. \quad (39)$$

The chiral transformation (37) has eliminated the hopping potential from Hamiltonian (36) and transformed it into the periodic function multiplying the FISDW potential in Eq. (39). Expanding that periodic function into the Fourier series, we get the following expression:

$$\hat{\mathcal{H}} = -i\hbar v_F \hat{\tau}_3 \frac{\partial}{\partial x} + \Delta \hat{\tau}_1 e^{i\hat{\tau}_3 [N(k_y b + G_x v_{E_y} t) - \Theta]} \sum_n a_{n+N} e^{i\hat{\tau}_3 n [k_y b - G_x(x - v_{E_y} t)]}, \quad (40)$$

where the coefficients of the expansion, a_n , are the Bessel functions:¹²⁾

$$a_n = J_n(4t_b/\hbar\omega_c). \quad (41)$$

The last term in Eq. (40) is the sum of many sinusoidal potentials whose wave vectors are the integer multiples of the magnetic wave vector G_x . Each of these periodic potentials mixes

¹¹⁾This kind of transformation was first introduced in Ref. [33] that started development of the FISDW theory.

¹²⁾General expression (40) is valid even when the FISDW has a nonzero transverse wave vector and the transverse dispersion law of the electrons is more complicated, but expression (41) for the expansion coefficients a_n would be different in that case.

the + and - electrons and opens an energy gap at the electron wave vector k_x shifted from $\pm k_F$ by an integer multiple of $G_x/2$. These multiple gaps are exactly the same gaps that were discussed in Sec. 2.

The term with $n = 0$ in the sum in Eq. (40) does not depend on x and opens a gap right at the Fermi level.¹³⁾ When the temperature T is much lower than the distance between the energy gaps $\hbar\omega_c$:

$$T \ll \hbar\omega_c, \quad (42)$$

only the gap at the Fermi level is important, whereas the other gaps may be neglected. Condition (42) is always satisfied in the relevant temperature range $0 \leq T \leq T_c$ in the weak coupling theory of the FISDW, where $T_c \ll \hbar\omega_c$. Thus, let us omit all the terms in the sum in Eq. (40), except the term with $n = 0$:

$$\hat{\mathcal{H}} = -i\hbar v_F \hat{\tau}_3 \frac{\partial}{\partial x} + \Delta_{\text{eff}} \hat{\tau}_1 e^{i\hat{\tau}_3 [N(k_y b + G_x v_{E_y} t) - \Theta]}, \quad (43)$$

where

$$\Delta_{\text{eff}} = a_N \Delta. \quad (44)$$

This is the so-called single-gap approximation [17]. As explained in Sec. 2, in order to maximize the energy gap at the Fermi level, the system selects such a value of N that maximizes the coefficient a_N in Eq. (44). It follows from Eq. (41) and properties of the Bessel functions that the maximum of a_N is achieved at $N \approx 4t_b/\hbar\omega_c$ [18].¹⁴⁾ It was shown explicitly in Ref. [13] that omission of the gaps located deeply below the Fermi energy does not change the value of the Hall conductivity, at least at zero temperature.

By the above sequence of manipulations, we have combined the two periodic potentials in Eq. (16) into the single effective potential (43) that opens a gap at the Fermi level. It follows from Eq. (43) that the phase φ of this effective potential changes in time at the rate proportional to the transverse electric field E_y :

$$\dot{\varphi} = -NG_x v_{E_y}, \quad (45)$$

which means that the effective potential moves along the chains. Since all electrons are below the energy gap opened by this potential, the motion of the potential induces the Fröhlich current along the chains:

$$j_x = -\frac{e}{\pi b} \dot{\varphi}. \quad (46)$$

Substituting Eqs. (45), (7), and (17) into Eq. (46), we find the QHE in agreement with Eq. (14):

$$j_x = \frac{2Ne^2}{h} E_y. \quad (47)$$

To avoid confusion, we wish to emphasize that, unlike in Sec. 4, here the FISDW is assumed to be immobile, and the FISDW phase Θ in Eq. (43) is time-independent. The effective potential (43) moves, because it is a combination of the stationary FISDW potential and the moving hopping potential (16).

¹³⁾Since, by introducing the \pm electrons, we have already subtracted the wave vectors $\pm k_F$, the actual wave vector that corresponds to this term is $2k_F$.

¹⁴⁾When the transverse wave vector of the FISDW is not zero, the value of N is controlled also by t'_b , the next-nearest-chain hopping integral of electrons [19].

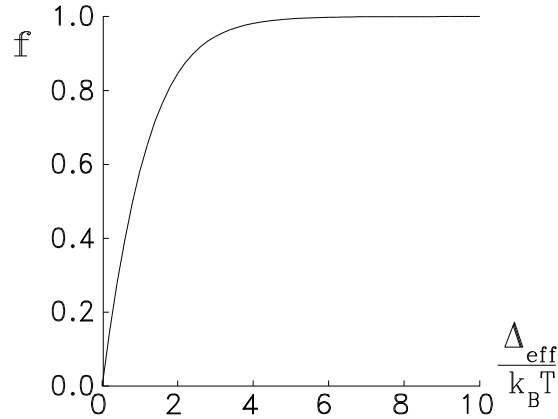


Figure 2: The reduction factor f of the Hall conductivity as a function of the ratio of the energy gap at the Fermi level Δ_{eff} to the temperature T , as given by Eq. (50).

Eq. (46) is a good starting point to discuss the temperature dependence of the QHE. According to the above consideration, the Hall conductivity is the Fröhlich conductivity of the effective periodic potential (43). Thus, the temperature dependence of the QHE must be the same as the temperature dependence of the Fröhlich conductivity. The latter issue was studied in the theory of the CDW [34, 35]. At a finite temperature T , the electric current carried by the CDW condensate is reduced with respect to the zero-temperature value by a factor $f(T)$. The same factor reduces the condensate Hall effect at a finite temperature:

$$\sigma_{xy}(T) = f(T) 2Ne^2/h, \quad (48)$$

$$f(T) = 1 - \int_{-\infty}^{\infty} \frac{dk_x}{\hbar v_F} \left(\frac{\partial E}{\partial k_x} \right)^2 \left[-\frac{\partial n_F(E/k_B T)}{\partial E} \right], \quad (49)$$

where $E = \sqrt{(\hbar v_F k_x)^2 + \Delta_{\text{eff}}^2}$ is the electron dispersion law in the FISDW phase, k_B is the Boltzmann constant, and $n_F(\epsilon) = (e^\epsilon + 1)^{-1}$ is the Fermi distribution function. The last term in Eq. (49) reflects the fact that normal quasiparticles, thermally excited above the energy gap, equilibrate with the immobile crystal lattice; thus, only a fraction of all electrons is carried along the chains by the moving periodic potential, which reduces the Hall/Fröhlich current. A simple, transparent derivation of Eq. (49) is given in Ref. [36].

The function f (49) depends only on the ratio of the energy gap at the Fermi level, Δ_{eff} (44), and the temperature T and can be written as [30, 35]

$$f\left(\frac{\Delta_{\text{eff}}}{k_B T}\right) = \int_0^{\infty} d\zeta \tanh\left(\frac{\Delta_{\text{eff}}}{2k_B T} \cosh \zeta\right) / \cosh^2 \zeta. \quad (50)$$

The function f is plotted in Fig. 2. It is equal to 1 at zero temperature, where Eq. (48) gives the QHE, gradually decreases with increasing T , and vanishes when $T \gg \Delta_{\text{eff}}$. Taking into account that the FISDW order parameter Δ itself depends on T and vanishes at the FISDW transition temperature T_c , it is clear that $f(T)$ and $\sigma_{xy}(T)$ vanish at $T \rightarrow T_c$, where $\sigma_{xy}(T) \propto f(T) \propto \Delta(T) \propto \sqrt{T_c - T}$. Assuming that the temperature dependence $\Delta_{\text{eff}}(T)$ is

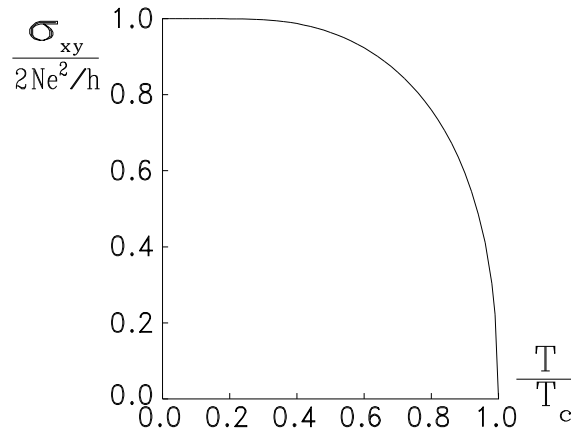


Figure 3: Hall conductivity in the FISDW state as a function of the temperature T normalized to the FISDW transition temperature T_c .

given by the BCS theory [17], we plot the temperature dependence of the Hall conductivity, $\sigma_{xy}(T)$, in Fig. 3. Strictly speaking, Eq. (48) gives only the Hall effect of the FISDW condensate and should be supplemented with the Hall conductivity of the thermally excited normal carriers. Then, at $T \rightarrow T_c$, $\sigma_{xy}(T)$ should not vanish, but approach to the Hall conductivity of the metallic phase. The latter is determined by the distribution of the electron scattering time over the Fermi surface and is small. The curve shown in Fig. 3 should be modified accordingly in a small vicinity of T_c .

The function $f(T)$ (49) is qualitatively similar to the function $f_s(T)$ that describes the temperature reduction of the superconducting condensate density in the London case. Both functions approach 1 at zero temperature, but near T_c the superconducting function behaves differently: $f_s(T) \propto \Delta^2(T) \propto T_c - T$. To understand the origin of the difference between the two functions, one should consider them at a small, but finite frequency ω and wave vector q . Eqs. (49) and (50) represent the limit where $q/\omega = 0$. This is the relevant limit in our case, because the electric field is, supposedly, strictly homogeneous in space ($q = 0$), but may be time-dependent ($\omega \neq 0$). The effective periodic potential (43) is also time-dependent. On the other hand, for the Meissner effect in superconductors, where the magnetic field is stationary ($\omega = 0$), but varies in space ($q \neq 0$), the opposite limit $\omega/q = 0$ is relevant. That is why $f(T)$ and $f_s(T)$ are different.

The function $f(T)$ for the Fröhlich current of a regular CDW/SDW was calculated in Ref. [34] in the form (49) and in Refs. [30, 35] in the form (50). The Hall conductivity in the FISDW state at a finite temperature was discussed in Ref. [30], which failed to produce the QHE at zero temperature. Temperature dependence of the Hall resistance in $(\text{TMTSF})_2\text{X}$ was measured in experiments [37]. However, to compare the experimental results with our theory, it is necessary to convert the Hall resistivity into the Hall conductivity, which requires experimental knowledge of all components of the resistivity tensor.

Because the FISDW phase Θ enters linearly into the phase φ of the effective periodic potential in Eq. (43), the results of Sec. 4 could be immediately generalized to a finite temperature. When the FISDW moves and its phase Θ depends on time, the r.h.s. of Eq. (23) should be multiplied by the function $f(T)$. The frequency-dependent Hall conductivity, given by Eq. (34)

and shown in Fig. 1, should be also multiplied by $f(T)$.¹⁵⁾ Such a simple generalization of the results of this Section to finite frequencies would be possible, because the function f has no frequency dependence for $\omega \ll \Delta_{\text{eff}}$. However, at a finite temperature, the dissipative terms, discussed at the end of Sec. 4, may become comparable with the other terms and significantly change $\sigma_{xy}(\omega)$ beyond multiplication by the factor $f(T)$.

6 Edge States

Thus far we treated the QHE as a bulk phenomenon and did not pay attention to the edges of the crystal. On the other hand, it is known that the theory of the QHE can be reformulated in terms of the gapless edge states located at the boundaries of a Hall sample [38]. The edge states in $(\text{TMTSF})_2\text{X}$ attracted attention in recent studies of the chiral states on the surface of a bulk QHE sample [39]. Let us show how the QHE in the FISDW state can be formulated in terms of the edge states. We will consider a sample that is infinite in the x direction along the chains and has a finite macroscopic size $2L_y$ in the y direction across the chains: $-L_y \leq y \leq L_y$. The edge states are located near the boundaries of the sample at $y = \pm L_y$. The total number of the chains in the crystal, M_{max} , is finite: $M_{\text{max}} = 2L_y/b$.

To introduce the edge states in a most natural way, let us reformulate the FISDW picture using the Wannier representation of the electron wave functions [40]. First, let us find the electron eigenfunctions in the metallic state, in the absence of the FISDW. The Schrödinger equation that corresponds to Hamiltonian (36) with $\Delta = 0$ and $E_y = 0$,

$$[\mp i\hbar v_{\text{F}} \frac{\partial}{\partial x} + 2t_b \cos(k_y b - G_x x)] \psi_{k_x, k_y, \pm}(x) = \varepsilon \psi_{k_x, k_y, \pm}(x), \quad (51)$$

has the following solution:

$$\psi_{k_x, k_y, \pm}(x, n, t) = \exp\left\{i\left[-\frac{\varepsilon t}{\hbar} + k_x x + k_y n b \pm \frac{2t_b}{\hbar\omega_c} \sin(k_y b - G_x x)\right]\right\}, \quad (52)$$

$$\varepsilon = \pm \hbar v_{\text{F}} k_x. \quad (53)$$

In Eq. (52), the wave vectors k_x and k_y are the quantum numbers that label the energy eigenfunctions, whereas x and $n = y/b$ are the running coordinates of the wave functions. Note that the dispersion law (53) is purely 1D: The energy ε depends on k_x , but does not depend on k_y . As mentioned in Sec. 2, this is a consequence of the Landau degeneracy in magnetic field. Because of the degeneracy in k_y , any superposition of eigenstates (52) with different k_y also is an energy eigenstate. Let us superimpose functions (52) with the coefficients of the Fourier transform:

$$\begin{aligned} \psi_{k_x, M, \pm}(x, n, t) &= b \int \frac{dk_y}{2\pi} \psi_{k_x, k_y, \pm}(x, n, t) e^{-ik_y M b} \\ &= e^{i[-\varepsilon t/\hbar + k_x x + (n-M)G_x x]} J_{n-M}(\mp 2t_b/\hbar\omega_c), \end{aligned} \quad (54)$$

where $J_n(\xi)$ is the Bessel function of the n -th order.

The Wannier wave functions (54) form a new complete set of the energy eigenfunctions. These functions are delocalized along the chains, because they are the plane waves in the x direction. The shape of the wave functions across the chains is given by the Bessel function

¹⁵⁾The phenomenological parameters τ and ω_0 may also depend on temperature.

$J_n(2t_b/\hbar\omega_c)$ considered as a function of its index n with the fixed argument $2t_b/\hbar\omega_c$, which is the ratio of the hopping integral between the chains to the cyclotron frequency of the magnetic field. The Bessel function $J_n(2t_b/\hbar\omega_c)$ has a maximum at $n \approx 2t_b/\hbar\omega_c$ and exponentially decreases to zero as n increases further. Thus, the wave functions (54) are localized across the chains with the characteristic width $4t_b/\hbar\omega_c$, which decreases with increasing magnetic field H as $1/H$. Each wave function (54) is centered on a certain chain labeled by the quantum number M .

The wave functions (54) are qualitatively similar to the Landau wave functions of an isotropic particle in magnetic field. The both sets of the wave functions are localized in one direction and delocalized in another, and the energy does not depend on the position where the localized wave function is placed. However, because our problem is strongly anisotropic, the shapes of the wave functions are different: the Bessel function in our case¹⁶⁾ and the Gaussian function in the Landau case.

Since the wave functions $\psi_{k_x, M, \pm}$ form a complete basis, we can use this Wannier basis to describe our system. Let us introduce the operators $\hat{a}_{\pm}^{\pm}(k_x, M)$ and $\hat{a}_{\pm}(k_x, M)$ that create and annihilate an electron on a Wannier chain M in the state $\psi_{k_x, M, \pm}$. Now, let us take into account the FISDW potential $2\Delta \cos(Q_x x)$ in Eq. (6) with the wave vector (11). The matrix elements of the FISDW potential between the states (54) can be easily evaluated. Keeping only the term that opens an energy gap at the Fermi level, we get the following expression for Hamiltonian (6) in the Wannier basis:

$$\begin{aligned} \hat{\mathcal{H}} = & \int \frac{dk_x}{2\pi} \sum_M v_F k_x [\hat{a}_+^{\dagger}(k_x, M) \hat{a}_+(k_x, M) - \hat{a}_-^{\dagger}(k_x, M) \hat{a}_-(k_x, M)] \\ & + \Delta_{\text{eff}} [\hat{a}_+^{\dagger}(k_x, M+N) \hat{a}_-(k_x, M) + \hat{a}_-^{\dagger}(k_x, M) \hat{a}_+(k_x, M+N)], \end{aligned} \quad (55)$$

where $\Delta_{\text{eff}} = \Delta J_N(4t_b/\hbar\omega_c)$ is the same as in Eqs. (44) and (41). There is no single-electron hopping between the Wannier chains in Hamiltonian (55), but the FISDW potential scatters the $-$ electrons into the $+$ electrons and simultaneously displaces them across the chains by N Wannier chains, where N is the parameter of the FISDW. In the Wannier representation, it is very transparent why many different FISDWs are possible in our 2D system in magnetic field. In a purely 1D case, a CDW/SDW may couple the $+$ and $-$ electrons only on the same chain. In our 2D system, the FISDW may couple the $+$ and $-$ electrons on different chains; thus, the FISDW is characterized by the integer distance N between the coupled chains, which may take many different values.

The FISDW potential in Eq. (55) hybridizes the $-$ electrons on the Wannier chain M and the $+$ electrons on the Wannier chain $M+N$ and opens a gap at the Fermi level in their energy spectrum. That procedure works for the chains in the bulk of the crystal. However, the states at the edges of the crystal are exceptional. The $+$ electron on the first N chains on one side of the crystal and the $-$ electrons on the last N chains on the other side of the crystal have no partner chains to couple with, so these electrons remain ungapped. Thus, the one side of the sample possesses N gapless chiral modes propagating along the edge with the velocity v_F , and the other side has N gapless chiral modes propagating in the opposite direction with the velocity $-v_F$.

Now, let us discuss the QHE in this system. Suppose a small electric voltage V_y is applied across the chains. That means that the chemical potential varies across the chains. Because all

¹⁶⁾If the transverse dispersion law of the electrons is more complicated, the shape of the wave function may differ from the Bessel function, but all qualitative features of the Wannier functions, such as the localization across the chains, remain valid.

states in the bulk of the crystal are gapped out, they would not respond to this perturbation. However, since the edge modes are not gapped, the difference of the chemical potentials between the two edges produces an imbalance between the occupation numbers of the modes at the opposite edges, $\delta\rho \propto V_y$, which generates a net current I_x along the chains:

$$I_x = ev_F N \delta\rho = ev_F N \frac{2eV_y}{2\pi\hbar v_F} = \frac{2Ne^2}{h} V_y. \quad (56)$$

Eq. (56) represents the QHE, this time for the Hall conductance, rather than conductivity (14), which coincide in 2D.

The above derivation might have produced impression that the Hall current flows only along the edges of the sample and is zero in the bulk. That is not necessarily the case. Let us show how the bulk and the edge pictures of the QHE connect with each other. Suppose the applied voltage V_y drops homogeneously across the chains, so that there is a tiny voltage drop V_y/M_{\max} between every pair of neighboring chains. Because of the variation of the chemical potential across the chains, the electron concentration and, thus, the Fermi momentum k_F must change from chain to chain. That creates a problem when the FISDW pairs the + and - electrons on different chains, where the Fermi momenta may be different. When the FISDW is pinned and does not move, which we assume to be the case here, the states paired by the FISDW must have exactly opposite momenta. (If the paired momenta are different in absolute values, the total momentum of the electrons under the gap is not zero, which means that the FISDW moves.) So, the momentum distribution of the electrons on each chains must shift in k_x to make $-k_F$ at the chain M equal to $+k_F$ at the chain $M + N$. Since the momentum distribution on each chain is shifted away from the symmetric position, each chain carries an electric current.

Let us illustrate this reasoning quantitatively. The current on a chain M is the difference of the current I_M^+ carried by the electrons with the positive momenta and the current I_M^- of the electrons with the negative momenta. So, the total current is

$$\begin{aligned} I_x = & + I_1^+ - I_1^- \\ & + I_2^+ - I_2^- \\ & \dots \\ & + I_{M_{\max}}^+ - I_{M_{\max}}^-. \end{aligned} \quad (57)$$

Each line in this equation represents the current on a given chain. As it was explained above, because the chains M and $M + N$ are coupled by the pinned FISDW, we have

$$I_M^- = I_{M+N}^+. \quad (58)$$

Substituting Eq. (58) into Eq. (57), we find that the total current is the difference of the edge currents:

$$I_x = \sum_{M=1}^N I_M^+ - \sum_{M=M_{\max}-N+1}^{M_{\max}} I_M^-. \quad (59)$$

At the same time, the current on a given chain M is not zero: $I_M^+ - I_M^- \neq 0$. That means that the total current (56), whose value is given by the difference of the edge terms (59), is spread homogeneously over all chains, so that each chain carries a portion of the total current.

It is easy to see that the Hall current flows in those regions of the crystal where the transverse voltage drops. The total Hall current is always given by Eq. (56); nevertheless,

the actual physical distribution of the Hall voltage and current may be of some interest. In the semiconductor QHE devices, the experiment seems to indicate that all Hall voltage drops near the sample boundaries, but no such measurements were performed in the $(\text{TMTSF})_2\text{X}$ materials.

If the FISDW is allowed to move, it is not required to pair the exactly opposite electron momenta. Then, instead of (58), we can have $I_M^+ = I_M^-$, so the current on each chain and the total current (57) are equal to zero. This is in agreement with the result of Sec. 4 that, if the FISDW is free, there is no Hall effect.

Strictly speaking, expression (54) for the Wannier functions is valid only in the bulk and should be modified near the edges. Although we neglected such complications in the above discussion, we believe that our qualitative results would remain valid in a more accurate theory.

7 Conclusions

Main results reviewed in this paper can be summarized as follows. When a 2D system that consists of parallel conducting chains is placed in a strong magnetic field, the magnetic-field-induced spin-density wave (FISDW) may appear in the system. The FISDW couples the electron states at different chains, thus it is characterized by an integer number N , the distance between the coupled chains. By hybridizing the electron states at the opposite sides of the Fermi surface, the FISDW opens an energy gap at the Fermi level everywhere in the bulk of the crystal. However, on the N chains at the both edges of the crystal, half of the electron states remain ungapped, because they have no partner chains to couple with. The electrons in these N chiral gapless modes propagate with the opposite velocities at the opposite edges.

When an electric field is applied, at zero temperature, the system exhibits the quantum Hall effect (14) with the same integer number N that characterizes the FISDW. As the temperature increases, the Hall conductivity decreases, vanishing at the FISDW transition temperature T_c . The function $f(T)$ that describes the reduction of the Hall effect with the temperature is the same as the temperature reduction function of the Fröhlich current of a regular charge/spin-density wave. If the Hall effect is measured at a high enough frequency, the motion of the FISDW produces an additional contribution to the Hall current, such that the total Hall conductivity vanishes.

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References

- [1] Ishiguro T. and Yamaji K., *Organic Superconductors* (Springer-Verlag, Berlin, 1990) Chapter 9.
- [2] Chamberlin R.V. *et al.*, *Synth. Met.* **27** (1988) B41.
- [3] Pesty F., Garoche P., and Bechgaard K., *Phys. Rev. Lett.* **55** (1985) 2495; Fortune N.A. *et al.*, *ibid.* **64** (1990) 2054.
- [4] Naughton M.J. *et al.*, *Phys. Rev. Lett.* **55** (1985) 969.
- [5] Takahashi T., Jérôme D., and Bechgaard K., *J. Phys. France* **45** (1984) 945.
- [6] Cooper J.R. *et al.*, *Phys. Rev. Lett.* **63** (1988) 1984; Hannahs S.T. *et al.*, *ibid.* **63** (1988) 1988.

- [7] Chamberlin R.V. *et al.*, *Phys. Rev. Lett.* **60** (1988) 1189; Naughton M.J. *et al.*, *ibid.* **61** (1988) 621.
- [8] Kang W., Cooper J.R., and Jérôme D., *Phys. Rev. B* **43** (1991) 11467.
- [9] Montambaux G., *Physica Scripta T* **35** (1991) 188.
- [10] Héritier M., Montambaux G., and Lederer P., *J. Phys. Lett. (Paris)* **45** (1984) L943.
- [11] Azbel M.Y., Bak P., and Chaikin P.M., *Phys. Rev. Lett.* **59** (1987) 926.
- [12] Poilblanc D. *et al.*, *Phys. Rev. Lett.* **58** (1987) 270.
- [13] Yakovenko V.M., *Phys. Rev. B* **43** (1991) 11353.
- [14] Machida K. *et al.*, *Phys. Rev. B* **50** (1994) 921.
- [15] Lebed' A.G., *Pis'ma Zh. Exp. Teor. Fiz.* **51** (1990) 583 (*JETP Lett.* **51** (1990) 663); Machida K. and Nakano M., *J. Phys. Soc. Jpn.* **59** (1990) 4223.
- [16] Grüner G., *Density Waves in Solids* (Addison-Wesley, New York, 1994).
- [17] Poilblanc D. *et al.*, *J. Phys. C* **19** (1986) L321; Virosztek A., Chen L., and Maki K., *Phys. Rev. B* **34** (1986) 3371.
- [18] Lebed' A.G., *Zh. Exp. Teor. Fiz.* **89** (1985) 1034 (*Sov. Phys. JETP* **62** (1985) 595).
- [19] Montambaux G., Héritier M., and Lederer P., *Phys. Rev. Lett.* **55** (1985) 2078; Montambaux G. and Zanchi D., *ibid.* **77** (1996) 366.
- [20] Dana I., Avron Y., and Zak J., *J. Phys. C* **18** (1985) L679.
- [21] Kunz H., *Phys. Rev. Lett.* **57** (1986) 1095; Kohmoto M., *J. Phys. Soc. Jpn.* **62** (1993) 659.
- [22] Laughlin R.B., *Phys. Rev. B* **23** (1981) 5632.
- [23] Thouless D.J., in *The Quantum Hall Effect*, R.E. Prange and S.M. Girvin, Eds. (Springer-Verlag, Berlin, 1987) p. 101; Kohmoto M., *Phys. Rev. B* **39** (1989) 11943.
- [24] Kuchar F. *et al.*, *Phys. Rev. B* **33** (1986) 2965; Galchenkov L.A. *et al.*, *Pis'ma v Zh. Exp. Teor. Fiz.* **46** (1987) 430 [*JETP Lett.* **46** (1987) 542].
- [25] Quinlivan D. *et al.*, *Phys. Rev. Lett.* **65** (1990) 1816.
- [26] Artemenko S. N. and Kruglov A. N., *Fiz. Tverd. Tela* **26** (1984) 2391 [*Sov. Phys. Solid State* **26** (1984) 1448]; Dolgov E. N., *Fiz. Nizk. Temp.* **10** (1984) 911 [*Sov. J. Low Temp. Phys.* **10** (1984) 747].
- [27] Artemenko S. N. *et al.*, *Pis'ma v Zh. Exp. Teor. Fiz.* **39** (1984) 258 [*JETP Lett.* **39** (1984) 308]; Forró L. *et al.*, *Phys. Rev. B* **34** (1986) 9047; Forró L. *et al.*, *Sol. State Comm.* **62** (1987) 715; Trætteberg O., Balicas L., and Kriza G., *J. Phys. IV France, Colloque C2* **3** (1993) 61.
- [28] Osada T. *et al.*, *Phys. Rev. Lett.* **58** (1987) 1563.
- [29] Balicas L., Biškup N., and Kriza G., *J. Phys. IV France, Colloque C2* **3** (1993) 319.
- [30] Virosztek A. and Maki K., *Phys. Rev. B* **39** (1989) 616.
- [31] Petrova T. G. and Rozhavsky A. S., *Fiz. Nizk. Temp.* **18** (1992) 987 and *J. Phys. IV France, Colloque C2* **3** (1993) 303.
- [32] Yakovenko V.M., *J. Phys. IV France, Colloque C2* **3** (1993) 307; *J. Supercond.* **7** (1994) 683; Yakovenko V.M. and Goan H.-S., in *Proceedings of the Physical Phenomena at High Magnetic Fields—II Conference*, Z. Fisk *et al.*, Eds. (World Scientific, Singapore, 1996) p. 116.

- [33] Gor'kov L.P. and Lebed' A.G., *J. Phys. Lett. (Paris)* **45** (1984) L433.
- [34] Lee P.A. and Rice T.M., *Phys. Rev. B* **19** (1979) 3970; Rice T.M., Lee P.A., and Cross M.C., *ibid.* **20** (1979) 1345.
- [35] Maki K. and Virosztek A., *Phys. Rev. B* **41** (1990) 557; **42** (1990) 655.
- [36] Goan H.-S. and Yakovenko V.M., cond-mat/9607199, to be published in *Synth. Met.*
- [37] Kang W. *et al.*, *Phys. Rev. B* **45** (1992) 13566; see also Valfells S. *et al.*, cond-mat/9606212.
- [38] Halperin B.I., *Phys. Rev. B* **25** (1982) 2185.
- [39] Chalker J.T. and Dohmen A., *Phys. Rev. Lett.* **75** (1995) 4496; Balents L. and Fisher M.P.A., *ibid.* **76** (1996) 2782.
- [40] Yakovenko V.M., *Zh. Exp. Teor. Fiz.* **93** (1987) 627 (*Sov. Phys. JETP* **66** (1987) 355) and *Europhys. Lett.* **3** (1987) 1041.