Recent Developments in Exactly Solvable Quantum Mechanics

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Outline

Exactly Solvable Quantum Mechanics

1-d QM, given a Hamiltonian $\mathcal{H} = -\frac{d^2}{dx^2} + U(x), \ \ x_1 < x < x_2,$ $U(x) \in \mathbb{C}^\infty$,

• Eigenvalue problem

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \quad n = 0, 1, 2, \ldots, \ \int_{x_1}^{x_2} \phi_n^2(x)dx < \infty,$$

• all the discrete eigenvalues $\{\mathcal{E}_n\}$ and the corresponding eigenfunctions $\{\phi_n(x)\}$ are exactly calculable

$\Rightarrow$ **Exactly Solvable Quantum Mechanics**

Typical examples of exactly solvable QM I

- harmonic oscillator, $\mathcal{H} = -\frac{d^2}{dx^2} + x^2 - 1$, $-\infty < x < +\infty$,

$\mathcal{E}_n = 2n$, $\phi_n(x) = \phi_0(x)H_n(x)$: Hermite polynomial,
$\phi_0(x) = e^{-x^2/2}$,

$$\int_{-\infty}^{+\infty} \phi_0^2(x)H_n(x)H_m(x)dx \propto \delta_{n\,m}$$

- radial oscillator, $\mathcal{H} = -\frac{d^2}{dx^2} + x^2 + \frac{g(g-1)}{x^2} - (1+2g)$,

$0 < x < +\infty$, $g > 1$, $\mathcal{E}_n = 4n$,
$\phi_n(x) = \phi_0(x)L_n^{(g-1/2)}(x^2)$:Laguerre polynomial,
$\phi_0(x) = e^{-x^2/2}x^g$,

$$\int_0^{+\infty} \phi_0^2(x)L_n^{(g-1/2)}(x^2)L_m^{(g-1/2)}(x^2)dx \propto \delta_{n\,m}$$

Typical examples of exactly solvable QM II

- Pöschl-Teller potential,

$$\mathcal{H} = -\frac{d^2}{dx^2} + x^2 + \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2,$$

$0 < x < \pi/2,\ g > 1,\ \ h > 1,\quad \mathcal{E}_n = 4n(n+g+h),$

$\phi_n(x) = \phi_0(x) P_n^{(g-1/2,h-1/2)}(\cos 2x)$:Jacobi polynomial,

$\phi_0(x) = (\sin x)^g (\cos x)^h,$

$$\int_0^{\pi/2} \phi_0^2(x) P_n^{(g-1/2,h-1/2)}(\cos 2x) P_m^{(g-1/2,h-1/2)}(\cos 2x) dx \propto \delta_{n\,m}$$

Motivations for exactly solvable QM I

1. cornerstones of modern quantum physics
2. Heisenberg operator formalism
   1. creation, annihilation operators
   2. coherent states
   3. dynamical symmetry algebras
3. Schrödinger eq. *i.e.* eigenvalue problem of a *self-adjoint* Hamiltonian
   real eigenvalues and mutually orthogonal eigenfunctions
   $\longrightarrow$ unified framework of classical orthogonal polynomials
4. orthogonality weight function $= \phi_0^2(x)$: square of the ground state eigenfunction

Motivations for exactly solvable QM II

- new exactly solvable QM
  $\Rightarrow$ new orthogonal polynomials with your name on?
  like Hermite, Laguerre or Jacobi?
  (My naïvest motivation for this research)
- Not so$\Longleftarrow$ Bochner's Theorem
  orthogonal polynomials satisfying second order differential equations are *Classical* orthogonal polynomials; Hermite, Laguerre, Jacobi & Bessel

Bochner's Theorem '29

If polynomials $\{p_n(x)\}$ satisfy three term recurrence relations and a second order differential equation

$$\sigma(x)y'' + \tau(x)y' + \lambda_n y = 0,$$

they must be one of the Classical orthogonal polynomials, i.e., the Hermite, Laguerre, Jacobi and Bessel. For $y = p_0(x) =$const, $\Rightarrow \lambda_0 = 0$. For $y = p_1(x) \Rightarrow$ degree$(\tau(x)) \leq 1$. For $y = p_2(x) \Rightarrow$ degree$(\sigma(x)) \leq 2$.

- $\deg(\sigma(x)) = 2$, two equal roots ($x = 0$) $\Rightarrow$ Bessel
- $\deg(\sigma(x)) = 2$, two distinct roots ($x = \pm 1$) $\Rightarrow$ Jacobi
- $\deg(\sigma(x)) = 1$, root at $x = 0$ $\Rightarrow$ Laguerre
- $\deg(\sigma(x)) = 0$ $\Rightarrow$ Hermite

Avoiding Bochner' constraints

- polynomials satisfying difference Schrödinger equation
  differential eq. $\Rightarrow$ difference eq.
  $\Rightarrow$ Wilson, Askey-Wilson, Racah, q-Racah polynomials
- polynomials having holes (three term recurrence is broken) in the degree
- **polynomials starting at degree $\ell \geq 1$**
  (completeness not obvious $\Rightarrow$ experts did not think this option)
- polynomials satisfying difference Schrödinger equation and starting at degree $\ell \geq 1$ and having holes in the degree

Discovery of ∞ Multi-Indexed Orthogonal Polynomials

- Infinitely many *orthogonal polynomials satisfying second order differential equations*, discovered after Hermite, Laguerre and Jacobi polynomials (Gomez-Ullate,Kamran,Milson, Quesne, '08, Odake-RS '09 and others)
- Multi-Indexed orthogonal polynomials $P_{\mathcal{D},n}(x)$, $\mathcal{D} = \{d_1, \ldots, d_M\}$, $d_j \in \mathbb{N}$: degrees of polynomial type seed solutions (virtual state wave functions) employed by multiple Darboux transformations, (n counts nodes in (x_1, x_2))

$$\int_{x_1}^{x_2} P_{\mathcal{D},n}(x) P_{\mathcal{D},m}(x) \mathcal{W}_{\mathcal{D}}(x) dx = h_{\mathcal{D},n} \delta_{n\,m}$$

- degree $\ell + n$ polynomial in x, but forming a complete set,

Discovery of ∞ Multi-Indexed Orthogonal Polynomials II

- No three term recurrence relations
- main part of the eigenfunctions of *exactly solvable Schrödinger eq.*
- when eigenfunctions are employed, $\mathcal{D} = \{d_1, \ldots, d_M\}$, $d_j \in \mathbb{N}$: degrees of the holes
- *global solutions* of (confluent) Fuchsian differential equations with $3 + \ell$ *regular singularities*, all the ℓ extra singularities are apparent and located outside of the orthogonality interval

Basic Ingredients

- Exactly Solvable Quantum Mechanical System

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \quad \mathcal{E}_0 = 0, \quad n = 0, 1, 2, \ldots,$$

- Factorised positive semi-definite Hamiltonian $\mathcal{H} = \mathcal{A}^\dagger\mathcal{A} \geq 0$
- Multiple Darboux-Crum-Krein-Adler transformation

$$\mathcal{H}\psi(x) = \mathcal{E}\psi(x), \quad \mathcal{H}\varphi(x) = \tilde{\mathcal{E}}\varphi(x),$$

$$\Rightarrow \mathcal{H}^{(1)}\psi^{(1)}(x) = \mathcal{E}\psi^{(1)}(x), \quad \mathcal{H}^{(1)} \stackrel{\text{def}}{=} \mathcal{H} - 2\partial_x^2 \log\varphi(x),$$

$$\psi^{(1)}(x) \stackrel{\text{def}}{=} \partial_x\psi(x) - \frac{\partial_x\varphi(x)}{\varphi(x)}\psi(x) = \frac{\mathrm{W}[\varphi,\psi](x)}{\varphi(x)},$$

- Virtual State solutions, $\mathcal{H}\tilde{\varphi}_v(x) = \tilde{\mathcal{E}}_v\tilde{\varphi}_v(x)$, $\tilde{\mathcal{E}}_v < 0$,
  $\tilde{\varphi}_v(x) > 0$, $v \in \mathcal{V}$

Factorised Hamiltonians

Starting point: $\mathcal{H}$ with complete set of eigenvalues and eigenfunctions

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \quad (\phi_n, \phi_m) = h_n\delta_{n\,m}, \quad h_n > 0, \quad n = 0, 1, 2, \ldots,$$

by adjusting the const. of $\mathcal{H} \Rightarrow \mathcal{E}_0 = 0$

$\Rightarrow$ Positive Semi-Definite Hamiltonian $\mathcal{H}$ (Hermitian Matrix)

$$0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots, \quad \Rightarrow \quad \mathcal{H} = \mathcal{A}^\dagger\mathcal{A}$$

$$\mathcal{A} = d/dx - \partial_x\phi_0(x)/\phi_0(x), \quad \mathcal{A}^\dagger = -d/dx - \partial_x\phi_0(x)/\phi_0(x),$$

$\phi_0(x)$: ground state wavefunction, no node ($\phi_0(x) > 0$), square integrable $\mathcal{A}\phi_0(x) = 0$

$$\mathcal{H} = -d^2/dx^2 + V(x), \qquad V(x) = \frac{\partial_x^2\phi_0(x)}{\phi_0(x)}$$

Use virtual state solutions

- rewrite $\mathcal{H}$ by using $\hat{\mathcal{A}}_{d_1}$, $d_1 \in \mathbb{N}$, ($\hat{\mathcal{A}}_{d_1}$ annihilates $\tilde{\varphi}_{d_1}(x)$, $\hat{\mathcal{A}}_{d_1}\tilde{\varphi}_{d_1}(x) = 0$):

$$\hat{\mathcal{A}}_{d_1} \stackrel{\text{def}}{=} d/dx - \partial_x \log\tilde{\varphi}_{d_1}(x), \quad \hat{\mathcal{A}}_{d_1}^{\dagger} = -d/dx - -\partial_x \log\tilde{\varphi}_{d_1}(x),$$

$$\hat{\mathcal{A}}_{d_1}^{\dagger}\hat{\mathcal{A}}_{d_1} = -\frac{d^2}{dx^2} + \left(\frac{\tilde{\varphi}'_{d_1}(x)}{\tilde{\varphi}_{d_1}(x)}\right)^2 + \frac{d}{dx}\left(\frac{\tilde{\varphi}'_{d_1}(x)}{\tilde{\varphi}_{d_1}(x)}\right)$$

$$= -\frac{d^2}{dx^2} + \frac{\tilde{\varphi}''_{d_1}(x)}{\tilde{\varphi}_{d_1}(x)} = -\frac{d^2}{dx^2} + V(x) - \tilde{\mathcal{E}}_{d_1},$$

$$\mathcal{H} = \hat{\mathcal{A}}_{d_1}^{\dagger}\hat{\mathcal{A}}_{d_1} + \tilde{\mathcal{E}}_{d_1}, \quad \hat{\mathcal{A}}_{d_1} : \text{non-singular},$$

- define a new Hamiltonian by changing the order of $\hat{\mathcal{A}}_{d_1}$ and $\hat{\mathcal{A}}_{d_1}^{\dagger}$: $\mathcal{H}_{d_1}^{(1)} \stackrel{\text{def}}{=} \hat{\mathcal{A}}_{d_1}\hat{\mathcal{A}}_{d_1}^{\dagger} + \tilde{\mathcal{E}}_{d_1} = \mathcal{H} - 2\partial_x^2 \log\tilde{\varphi}_{d_1}(x)$

new exactly solvable Hamiltonian

- intertwining relation

$$\mathcal{H}^{(1)}_{d_1}\hat{\mathcal{A}}_{d_1}=(\hat{\mathcal{A}}_{d_1}\hat{\mathcal{A}}^{\dagger}_{d_1}+\tilde{\mathcal{E}}_{d_1})\hat{\mathcal{A}}_{d_1}$$
$$=\hat{\mathcal{A}}_{d_1}(\hat{\mathcal{A}}^{\dagger}_{d_1}\hat{\mathcal{A}}_{d_1}+\tilde{\mathcal{E}}_{d_1})=\hat{\mathcal{A}}_{d_1}\mathcal{H}$$

- $\mathcal{H}^{(1)}_{d_1}$: new exactly solvable isospectral Hamiltonian

$$\phi_{d_1,n}(x)\stackrel{\text{def}}{=}\hat{\mathcal{A}}_{d_1}\phi_n(x)=\frac{\mathrm{W}[\tilde{\varphi}_{d_1},\phi_n](x)}{\tilde{\varphi}_{d_1}},\quad n=0,1,\ldots,$$

$$\tilde{\varphi}_{d_1,\mathrm{v}}(x)\stackrel{\text{def}}{=}\hat{\mathcal{A}}_{d_1}\tilde{\varphi}_{\mathrm{v}}(x)=\frac{\mathrm{W}[\tilde{\varphi}_{d_1},\tilde{\varphi}_{\mathrm{v}}](x)}{\tilde{\varphi}_{d_1}},\quad \mathrm{v}\in\mathcal{D}\backslash d_1$$

$$\mathcal{H}^{(1)}_{d_1}\phi_{d_1,n}(x)=\mathcal{E}_n\phi_{d_1,n}(x),\quad \mathcal{H}^{(1)}_{d_1}\tilde{\varphi}_{d_1,\mathrm{v}}(x)=\tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\varphi}_{d_1,\mathrm{v}}(x),$$

$$(\phi_{d_1,n},\phi_{d_1,m})=(\phi_n,\hat{\mathcal{A}}^{\dagger}_{d_1}\hat{\mathcal{A}}_{d_1}\phi_m)=(\mathcal{E}_n-\tilde{\mathcal{E}}_{\mathrm{v}})h_n\delta_{n\,m}$$

new exactly solvable Hamiltonian 2

- repeat M times by using virtual state solutions specified by $\mathcal{D} = \{d_1, d_2, \ldots, d_M\}$
- $\Rightarrow$ new exactly solvable Hamiltonian with multi-index $\mathcal{D}$

$$\mathcal{H}_{\mathcal{D}}^{(M)} \stackrel{\text{def}}{=} \mathcal{H} - 2\partial_x^2 \log \mathrm{W}[\tilde{\varphi}_{d_1}, \ldots, \tilde{\varphi}_{d_M}](x)$$

$$\phi_{\mathcal{D},n}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\tilde{\varphi}_{d_1}, \ldots, \tilde{\varphi}_{d_M}, \phi_n](x)}{\mathrm{W}[\tilde{\varphi}_{d_1}, \ldots, \tilde{\varphi}_{d_M}](x)}$$

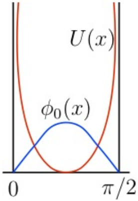
$$\mathcal{H}_{\mathcal{D}}^{(M)} \phi_{\mathcal{D},n}(x) = \mathcal{E}_n \phi_{\mathcal{D},n}(x), \quad n = 0, 1, \ldots,$$

$$(\phi_{\mathcal{D},n}, \phi_{\mathcal{D},m}) = \prod_{j=1}^{M} (\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}) \cdot h_n \delta_{n\,m}$$

positive definite inner products $\mathcal{E}_n > 0$, $\tilde{\mathcal{E}}_{d_j} < 0$.

Example: Pöschl-Teller potential $\Rightarrow$ Jacobi Polynomial

- $\mathcal{H} = -\frac{d^2}{dx^2} + U(x),\ U(x) = \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2,$ regular sing. $x = 0,\ g, 1-g,\ x = \pi/2,\ h, 1-h,\ \boldsymbol{\lambda} = \{g, h\},$
- ground state wavefunct. $\phi_0(x) = (\sin x)^g (\cos x)^h, \quad g, h > 0,$
- $\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(n+g+h), \qquad \eta(x) \stackrel{\text{def}}{=} \cos 2x$
- $\phi_n(x; \boldsymbol{\lambda}) = \phi_0(x) P_n^{(g-1/2, h-1/2)}(\eta(x)), \quad P_n$: Jacobi polynomial



Multi-Indexed Orthogonal Polynomials 1

- Pöschl-Teller potential has virtual state solutions, type I and II, generated by the discrete symmetry of the potential: $g \to 1 - g$, or $h \to 1 - h$
- negative energy and non-square integrable

$$\mathcal{H}\tilde{\phi}_{\text{v}}(x) = \tilde{\mathcal{E}}_{\text{v}}\tilde{\phi}_{\text{v}}(x), \quad \tilde{\mathcal{E}}_{\text{v}} < 0 \quad (\tilde{\phi}_{\text{v}}, \tilde{\phi}_{\text{v}}) = (1/\tilde{\phi}_{\text{v}}, 1/\tilde{\phi}_{\text{v}}) = \infty$$

- they have no zeros in $x \in (0, \pi/2)$
- use these seed solutions $\mathcal{D} \stackrel{\text{def}}{=} \{d_1^{\text{I}}, \ldots, d_M^{\text{I}}, d_1^{\text{II}}, \ldots, d_N^{\text{II}}\}$, $d_j^{\text{I,II}} \geq 1$

Multi-Indexed Orthogonal Polynomials 2

- explicit forms of type I virtual states ($h \to 1-h$)

$$\tilde{\phi}^{\mathrm{I}}_{\mathrm{v}}(x) \stackrel{\text{def}}{=} (\sin x)^{g}(\cos x)^{1-h}\xi^{\mathrm{I}}_{\mathrm{v}}(\eta(x);g,h),$$

$$\xi^{\mathrm{I}}_{\mathrm{v}}(\eta;g,h) \stackrel{\text{def}}{=} P_{\mathrm{v}}(\eta;g,1-h), \quad \mathrm{v}=0,1,\ldots,[h-\tfrac{1}{2}]',$$

$$\tilde{\mathcal{E}}^{\mathrm{I}}_{\mathrm{v}} \stackrel{\text{def}}{=} -4(g+\mathrm{v}+\tfrac{1}{2})(h-\mathrm{v}-\tfrac{1}{2}), \quad \tilde{\boldsymbol{\delta}}^{\mathrm{I}} \stackrel{\text{def}}{=} (-1,1)$$

- explicit forms of type II virtual states ($g \to 1-g$)

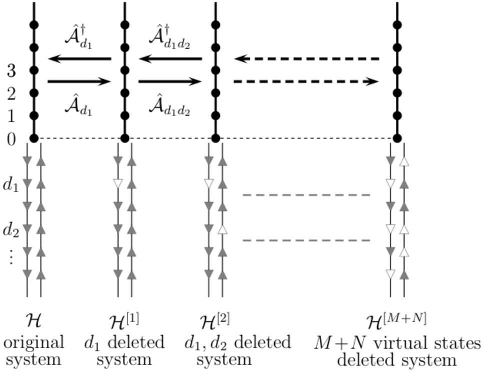
$$\tilde{\phi}^{\mathrm{II}}_{\mathrm{v}}(x) \stackrel{\text{def}}{=} (\sin x)^{1-g}(\cos x)^{h}\xi^{\mathrm{II}}_{\mathrm{v}}(\eta(x);g,h),$$

$$\xi^{\mathrm{II}}_{\mathrm{v}}(\eta;g,h) \stackrel{\text{def}}{=} P_{\mathrm{v}}(\eta;1-g,h), \quad \mathrm{v}=0,1,\ldots,[g-\tfrac{1}{2}]',$$

$$\tilde{\mathcal{E}}^{\mathrm{II}}_{\mathrm{v}} \stackrel{\text{def}}{=} -4(g-\mathrm{v}-\tfrac{1}{2})(h+\mathrm{v}+\tfrac{1}{2}), \quad \tilde{\boldsymbol{\delta}}^{\mathrm{II}} \stackrel{\text{def}}{=} (1,-1)$$

- They are Not symmetries of Jacobi polynomials
- S.Odake & R. Sasaki, Phys. Lett. **B702** (2011) 164-170,

Schematic Picture of Virtual States Deletion



Eigenfunctions etc after Virtual States Deletion

$$\mathcal{H}^{[M]}\phi_n^{[M]}(x) = \mathcal{E}_n\phi_n^{[M]}(x) \quad (n \in \mathbb{Z}_{\geq 0}),$$

$$\mathcal{H}^{[M]}\tilde{\phi}_{\rm v}^{[M]}(x) = \tilde{\mathcal{E}}_{\rm v}\tilde{\phi}_{\rm v}^{[M]}(x) \quad ({\rm v} \in \mathcal{V}\backslash\mathcal{D}),$$

$$\phi_n^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \phi_n](x)}{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)},$$

$$(\phi_m^{[M]}, \phi_n^{[M]}) = \prod_{j=1}^{M}(\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}) \cdot h_n\delta_{m\,n},$$

$$\tilde{\phi}_{\rm v}^{[M]}(x) \stackrel{\text{def}}{=} \frac{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \tilde{\phi}_{\rm v}](x)}{\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)},$$

$$U^{[M]}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \log\bigl|\mathrm{W}[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x)\bigr|.$$

shape inv. exactly solvable $\Rightarrow$ shape inv. exactly solvable

Multi-Indexed Orthogonal Polynomials 3

- Multi-Indexed Orthogonal Polynomials $P_{\mathcal{D},n}(\eta)$:

$$\phi_n^{[M,N]}(x) \equiv \phi_{\mathcal{D},n}(x;\boldsymbol{\lambda}) = (-4)^{M+N}\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta(x);\boldsymbol{\lambda}),$$

$$\psi_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x;\boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta(x);\boldsymbol{\lambda})}, \quad P_{\mathcal{D},0}(\eta;\boldsymbol{\lambda}) \propto \Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})$$

- $\boldsymbol{\lambda}^{[M,N]} = (g+M-N, h-M+N)$,
  $\Xi_{\mathcal{D}}(\eta)$ has no node in $-1<\eta<1$;
- orthogonality

$$\int_{-1}^{1} d\eta\, \frac{W(\eta;\boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})^2} P_{\mathcal{D},m}(\eta;\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) = h_{\mathcal{D},n}\delta_{nm}$$

Multi-Indexed Orthogonal Polynomials 4

- Explicit Forms

$$P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathrm{W}[\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_N,P_n](\eta) \times \Big(\frac{1-\eta}{2}\Big)^{(M+g+\frac{1}{2})N}\Big(\frac{1+\eta}{2}\Big)^{(N+h+\frac{1}{2})M}$$

$$\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathrm{W}[\mu_1,\ldots,\mu_M,\nu_1,\ldots,\nu_N](\eta) \times \Big(\frac{1-\eta}{2}\Big)^{(M+g-\frac{1}{2})N}\Big(\frac{1+\eta}{2}\Big)^{(N+h-\frac{1}{2})M}$$

$$\mu_j = \Big(\frac{1+\eta}{2}\Big)^{\frac{1}{2}-h}\xi^{\mathrm{I}}_{d^{\mathrm{I}}_j}(\eta;g,h), \quad \nu_j = \Big(\frac{1-\eta}{2}\Big)^{\frac{1}{2}-g}\xi^{\mathrm{II}}_{d^{\mathrm{II}}_j}(\eta;g,h)$$

- $P_{\mathcal{D},n}(\eta)$ degree $\ell+n$, $\Xi_{\mathcal{D}}(\eta)$ degree ℓ;

$$\ell = \sum_{j=1}^{M} d^{\mathrm{I}}_j + \sum_{j=1}^{N} d^{\mathrm{II}}_j - \frac{1}{2}M(M-1) - \frac{1}{2}N(N-1) + MN \geq 1$$

How it started: X_1 Jacobi polynomials

- X_1 Jacobi Hamiltonian (Gomez-Ullate et al, Quesne et al, '08)

$$\mathcal{H} = -\frac{d^2}{dx^2} + \frac{g(g+1)}{\sin^2 x} + \frac{h(h+1)}{\cos^2 x} - (2+g+h)^2 + \frac{8(g+h+1)}{1+g+h+(g-h)\cos 2x} - \frac{8(2g+1)(2h+1)}{(1+g+h+(g-h)\cos 2x)^2}$$

$$\phi_0(x) = (\sin x)^{g+1}(\cos x)^{h+1}\frac{3+g+h+(g-h)\cos 2x}{1+g+h+(g-h)\cos 2x} = \frac{P_1^{(g+2-3/2,-h-2-1/2)}(\cos 2x)}{P_1^{(g+1-3/2,-h-1-1/2)}(\cos 2x)}$$

- generalisation

$$w_\ell(x;\boldsymbol{\lambda}) = (g+\ell)\log\sin x + (h+\ell)\log\cos x + \log\frac{\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta;\boldsymbol{\lambda})}$$

$$\xi_\ell(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell^{(g+\ell-3/2,-h-\ell-1/2)}(\eta), \quad \eta = \cos 2x$$

X_ℓ Jacobi Polynomials

- shape invariance can be verified directly

$$(\partial_x w_\ell(x;\boldsymbol{\lambda}))^2 - \partial_x^2 w_\ell(x;\boldsymbol{\lambda}) = (\partial_x w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}))^2 + \partial_x^2 w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + 4(g+h+2\ell+1)$$

- eigenvalues $\mathcal{E}_{\ell,n}(g,h) = \mathcal{E}_n(g+\ell, h+\ell) = 4n(n+g+h+2\ell)$
- eigenfunctions $\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \psi_\ell(x;\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda})$

$$\psi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{e^{w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}}{\xi_\ell(\eta;\boldsymbol{\lambda})}, \quad c_n \stackrel{\text{def}}{=} n+h+1/2$$

$$P_{\ell,n}(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} c_n^{-1}\Big((h+\frac{1}{2})\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})P_n^{(g+\ell-3/2,h+\ell+1/2)}(\eta) + (1+\eta)\xi_\ell(\eta;\boldsymbol{\lambda})\partial_\eta P_n^{(g+\ell-3/2,h+\ell+1/2)}(\eta)\Big)$$

degree $\ell+n$ polynomial

X_ℓ Jacobi Polynomials 2: Fuchsian differential equation

- lowest degree $P_{\ell,0}(\eta;\boldsymbol{\lambda}) \propto \xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})$
- orthogonality

$$\int_0^{\pi/2} \psi_\ell(x;\boldsymbol{\lambda})^2 P_{\ell,n}(\cos 2x;\boldsymbol{\lambda}) P_{\ell,m}(\cos 2x;\boldsymbol{\lambda}) dx = h_{\ell,n}(\boldsymbol{\lambda})\delta_{n\,m}$$

- Fuchsian differential eq.

$$\begin{aligned}
&(1-\eta^2)\partial_\eta^2 P_{\ell,n}(\eta;\boldsymbol{\lambda}) \\
&+\Bigl(h-g-(g+h+2\ell+1)\eta-2\frac{(1-\eta^2)\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda})}{\xi_\ell(\eta;\boldsymbol{\lambda})}\Bigr)\partial_\eta P_{\ell,n}(\eta;\boldsymbol{\lambda}) \\
&+\Bigl(-\frac{2(h+\frac{1}{2})(1-\eta)\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta;\boldsymbol{\lambda})}+\ell(\ell+g-h-1) \\
&\qquad\qquad +n(n+g+h+2\ell)\Bigr)P_{\ell,n}(\eta;\boldsymbol{\lambda})=0
\end{aligned}$$

X_ℓ Jacobi Polynomials 3: Fuchsian differential equation 2

- regular singularities at ℓ roots of $\xi_\ell(\eta;\boldsymbol{\lambda})$, $\eta=\eta_j$:

$$\xi_\ell(\eta_j;\boldsymbol{\lambda})=0,\qquad j=1,2,\ldots,\ell$$

- in the neighbourhood of $\eta=\eta_j$:

$$(1-\eta_j^2)y''-2\frac{(1-\eta_j^2)}{\eta-\eta_j}y'$$
$$-2(h+1/2)(1-\eta_j)\frac{\beta}{\eta-\eta_j}y+\text{regular terms}=0$$

- characteristic eq.: same exponents everywhere

$$\rho(\rho-1)-2\rho=0\quad\Rightarrow\rho=0,3$$

$\rho=0$ corresponds to the polynomial solution

Summary and Outlook

- Question: Why the "New Polynomials" were Not discovered by the experts of the orthogonal polynomials?
- Answers: 'physical thinking' is more suitable for the problem
  1. Schrödinger equation is more general than the equations governing orthogonal polynomials
  2. $g \leftrightarrow 1 - g$, $h \leftrightarrow 1 - h$ are Not the symmetries of the Jacobi (Laguerre) polynomial equations
  3. they are equations for the eigenpolynomials, i e. non-square integrable solutions are discarded
  4. Darboux transformations are defined most generally for the Schrödinger equations

Summary and Outlook 2

- Infinitely many new orthogonal polynomials satisfying second order differential or difference equations are discovered.
  Hopefully they will find many interesting applications.
  At least, they give infinitely many examples of exactly solvable Birth and Death Processes.
- Various concepts and methods of QM have much wider currency and utility in the theory of ordinary differential and difference equations than is usually regarded.
- Various properties of the Askey-scheme of hypergeometric orthogonal polynomials can be understood in a unified fashion, both of a continuous and a discrete variable.
- Multi-variable Multi-Indexed Orthogonal polynomials are the next challenge

Fuchsian Differential Equations 1: overview

$$y'' + f(x)y' + g(x)y = 0, \quad f(x) = \frac{\alpha}{x - x_0} + \sum_{n=0}^{\infty} f_n (x - x_0)^n,$$

$$g(x) = \frac{\beta}{(x - x_0)^2} + \frac{\gamma}{x - x_0} + \sum_{n=0}^{\infty} g_n (x - x_0)^n$$

x_0 : regular singularity

- singular solutions $y_j = (x - x_0)^{\rho_j} (1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n)$
- ρ_1, ρ_2: *characteristic exponents* $\rho(\rho - 1) + \alpha\rho + \beta = 0$
- regular singularities only $\Rightarrow$ Fuchsian equation
- local theory only

Fuchsian Differential Equations 2: examples

- 3 regular singularities at 0, 1, ∞: Gauss hypergeometric eq.

$$x(1-x)y''+(\gamma-(\alpha+\beta+1)x)y'-\alpha\beta y=0$$

- solutions around $x=0$: $\rho_1=0$, $\rho_2=1-\gamma$

$$y_1={}_2F_1(\alpha,\beta;\gamma|x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!},$$
$$y_2=x^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma|x)$$

  hypergeometric function $\Rightarrow$ globally continued

- 4 regular singularities 0, 1, ∞, t: Heun equation
- more than 4 regular singularities: global solution virtually unknown

Fuchsian Differential Equations 3

- if $\rho_2 - \rho_1 = n \in \mathbb{N}$: possible log terms (Frobenius)
- if $\rho_2 - \rho_1 = n \in \mathbb{N}$ and no log terms $\Rightarrow$ apparent singularity
- apparent singularity of Schrödinger eq. (at $x = 0$)

$$\mathcal{H} = -\frac{d^2}{dx^2} + \frac{\alpha}{x^2} + \frac{\beta}{x} + \text{ regular terms}$$

$$\rho = (1 \pm \sqrt{1 + 4\alpha})/2$$

α	$\rho_2 - \rho_1$	
0	1	regular
3/4	2	Painlevé case
2	3	Darboux trans.
15/4	4	??
6	5	Ho-Sasaki-Takemura
⋮	⋮	

- if all extra singularities are apparent $\Rightarrow$ global solutions possible

Parallel History

- shape invariant potentials in discrete QM with pure imaginary shifts; Wilson, Askey-Wilson polynomials etc ('04)
- Heisenberg operator solutions & dynamical symmetry algebras in discrete QM with pure imaginary shifts; Wilson, Askey-Wilson polynomials etc ('06)
- shape invariant potentials in discrete QM with real imaginary shifts; (q-)Racah, (dual) (q-)Hahn, etc ('08)
- Crum's theorem for discrete QM ('09)
- X_ℓ Wilson and Askey-Wilson polynomials ('09)
- Modified Crum's theorem (Krein-Adler transformations) for discrete QM ('10)
- X_ℓ Wilson and Askey-Wilson polynomials derived by Darboux transformations ('10)

Parallel History II

- X_ℓ (q)-Racah polynomials ('11)
- Multi-indexed (q)-Racah polynomials ('12)
- Multi-indexed Wilson and Askey-Wilson polynomials ('12)
- duality between pseudo virtual and eigenstates & Casoratian identities for Wilson and Askey-Wilson polynomials
- non-confining potentials (discrete analogues of Morse, Eckart potentials) ('14)

Work by Satoru Odake and R. Sasaki

- "Equilibria of 'discrete' integrable systems and deformations of classical orthogonal polynomials," J. Phys. **A37** (2004) 11841-11876.
- "Shape Invariant Potentials in 'Discrete Quantum Mechanics,'" J. Nonlinear Mathematical Physics **12** Supplement 1 (2005) 507-521.
- "Calogero-Sutherland-Moser Systems, Ruijsenaars-Schneider-van Diejen Systems and Orthogonal Polynomials," Prog. Theor. Phys. **114** (2005) 1245–1260.
- "Unified Theory of Annihilation-Creation Operators for Solvable ('Discrete') Quantum Mechanics," J. Math. Phys. **47** (2006) 102102.
- "Exact solution in the Heisenberg picture and annihilation-creation operators," Phys. Lett. **B641** (2006) 112–117

Work by Satoru Odake and R. Sasaki, 2

- "Exact Heisenberg operator solutions for multi-particle quantum mechanics," J. Math. Phys. **48** (2007) 082106.
- "Multi-Particle Quasi Exactly Solvable Difference equations," J. Math. Phys. **48** (2007) 122105.
- '*q*-oscillator from the *q*-Hermite polynomial," Phys. Lett. **B663** (2008) 141-145.
- "Orthogonal Polynomials from Hermitian Matrices," J. Math. Phys. **49** (2008) 053503.
- "Exactly solvable 'discrete' quantum mechanics; shape invariance, Heisenberg solutions, annihilation-creation operators and coherent states," Prog. Theor. Phys. **119** (2008) 663-700.

Work by Satoru Odake and R. Sasaki, 3

- "Unified theory of exactly and quasi-exactly solvable 'Discrete' quantum mechanics: I. Formalism," J. Math. Phys. **51** (2010) 083502 (24 pp).
- "Crum's Theorem for Discrete Quantum Mechanics," Prog. Theor. Phys. **122** (2009) 1067-1079.
- "Infinitely many shape invariant potentials and new orthogonal polynomials," Phys. Lett. **B679** (2009) 414-417.
- "Infinitely many shape invariant discrete quantum mechanical systems and new exceptional orthogonal polynomials related to the Wilson and the Askey-Wilson polynomials," Phys. Lett. **B682** (2009) 130-136.
- "Another set of infinitely many exceptional (X_ℓ) Laguerre polynomials," Phys. Lett. **B684** (2010) 173-176.

Work by Satoru Odake and R. Sasaki, 4

- "Infinitely many shape invariant potentials and cubic identities of the Laguerre and Jacobi polynomials." J. Math. Phys. **51** (2010) 053513.
- "Exceptional Askey-Wilson type polynomials through Darboux-Crum transformations," J. Phys. **A43** (2010) 335201 (18pp), arXiv:1004.0544[math-ph].
- "A new family of shape invariantly deformed Darboux-Pöschl-Teller potentials with continuous ℓ," J. Phys. A **44** (2011) 195203, arXiv:1007.3800 [math-ph].
- "Dual Christoffel transformations," Prog. Theor. Phys. **126** (2011) 1-34 arXiv:1101.5468[math-ph].
- "The Exceptional (X_ℓ) (q)-Racah Polynomials," Prog. Theor. Phys. **125** (2011) 851-870. arXiv:1102.0812[math-ph].

Work by Satoru Odake and R. Sasaki, 5

- "Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials," Phys. Lett. **B702** (2011) 164-170. `arXiv:1105.0508[math-ph]`.
- "Discrete quantum mechanics," (Topical Review) J. Phys. **A44** (2011) 353001 (47 pp). `arXiv:1104.0473[math-ph]`.
- "Multi-indexed (q-)Racah polynomials," J. Phys. **A45** (2012) 385201 (21 pp). `arXiv:1203.5868[math-ph]`.
- "Multi-indexed Wilson and Askey-Wilson polynomials," J. Phys. **A46** (2013) 045204 (22 pp). `arXiv:1207.5584[math-ph]`.
- "Extensions of solvable potentials with finitely many discrete eigenstates," J. Phys. A **46** (2013) 235205 (15pp) `arXiv:1301.3980[math-ph]`.

Work by Satoru Odake and R. Sasaki, 6

- "Discrete quantum mechanics," (Topical Review) J. Phys. **A44** (2011) 353001, arXiv:1104.5669
- "Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials," Phys. Lett. **B702** (2011) 164-170, arXiv:1105.0508.
- Leonor García-Gutiérrez, S. Odake and R. Sasaki, "Modification of Crum's Theorem for 'Discrete' Quantum Mechanics," Prog. Theor. Phys. **124** (2010) 1-24, arXiv:1004.0289[math-ph].
- "Multi-indexed (q)-Racah polynomials," J. Phys. **A45** (2012) 385201 (21 pp), arXiv:1203.5868[math-ph].
- "Multi-indexed Wilson & Askey-Wilson polynomials," J. Phys. **A46** (2013) 045204 (22 pp) arXiv:1207.5584[math-ph].

Work by Satoru Odake and R. Sasaki, 7

- “Extensions of solvable potentials with finitely many discrete eigenstates,” J. Phys. A **46** (2013) 235205 (15pp) arXiv:1301.3980[math-ph].
- “Krein-Adler transformations for shape-invariant potentials and pseudo virtual states,” arXiv:1212.6595[math-ph].
- “Non-polynomial extensions of solvable potentials á la Abraham-Moses,” J. Math. Phys. **54** (2013) 102106 (19pp) arXiv:1307.0931[math-ph].
- “Casoratian Identities for the Wilson and Askey-Wilson Polynomials,” J. Approximation Theory (2014) arXiv:1308.4240[math-ph].
- “Solvable Discrete Quantum Mechanics: q-Orthogonal Polynomials with $|q|=1$ and Quantum Dilogarithm,” arXiv:1406.2768[math-ph].

Related Works

- R. Sasaki, "Exactly Solvable Birth and Death Processes," J. Math. Phys. **50** (2009) 103509.
- C-L. Ho, S. Odake and R. Sasaki, "Properties of the exceptional (X_ℓ) Laguerre and Jacobi polynomials," SIGMA **7** (2011) 107 (24pp) `arXiv:0912.5477[math-ph]`.
- R. Sasaki, S. Tsujimoto and A. Zhedanov, "Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations," J. Phys. A **43** (2010) 315204 (20pp), `arXiv:1004.4711[math-ph]`.
- R. Sasaki, "Exactly and Quasi-Exactly Solvable 'Discrete' Quantum Mechanics," Phil. Trans. R. Soc. **A369** (2011) 1301-1318, `arXiv:1004.4712[math-ph]`.

Related Works, 2

- C-L. Ho and R. Sasaki, "Zeros of the exceptional Laguerre and Jacobi polynomials," ISRN Mathematical Physics **2012** 920475 (27pp), doi:10.5402/2012/920475. `arXiv:1102.5669[math-ph]`.
- R. Sasaki and K. Takemura, "Global solutions of certain second order differential equations with a high degree of apparent singularity," SIGMA **8** (2012) 085 (18pp), `arXiv:1207.5302[math.CA]`.
- C.-L. Ho, R. Sasaki and K. Takemura, "Confluence of apparent singularities in multi-indexed orthogonal polynomials: the Jacobi case," J. Phys. A **46** (2013) 115205 (21pp) `arXiv:1210.0207[math.CA]`.

Related Works, 3

- J-C. Lee, C-L. Ho and R. Sasaki, "Scattering Amplitudes for Multi-indexed Extensions of Solvable Potentials," Annals of Physics **343** (2014) 115–131, `arXiv:1309.5471`.
- R. Sasaki, "Exactly solvable potentials with finitely many discrete eigenvalues of arbitrary choice," J. Math. Phys. **55** (2014) 062101 (11pp). `arXiv:1402.5474[math-ph]`.
- R. Sasaki, "Exactly solvable potentials with finitely many discrete eigenvalues of arbitrary choice," J. Math. Phys. **55** (2014) 062101 (11pp). `arXiv:1402.5474[math-ph]`