Mathematics for physicists

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1. GREEN'S THEOREM IN CARTESIAN COORDINATES

For any loop on x-y plane, with $dA = dx \cdot dy$

$$\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \iint_{S} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dA; \qquad S \text{ is the area enclosed by the loop } \Gamma$$

$$\therefore \oint_{\Gamma} \vec{F} \cdot d\vec{r} = 0 \quad \text{if} \quad \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} \quad \text{for any point } \vec{r}$$

OR

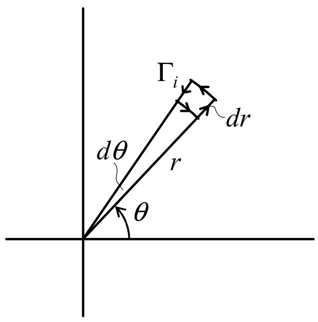
$$\oint_{\Gamma} (F_x \cdot dx + F_y \cdot dy) = \iint_{S} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \cdot dA$$

2. GREEN'S THEOREM IN POLAR COORDINATES

$$\vec{F}(\vec{r}) = \vec{F}(r, \theta) = F_r \cdot \hat{r} + F_\theta \cdot \hat{\theta} \quad , \quad d\vec{r} = dr \cdot \hat{r} + r \cdot d\theta \cdot \hat{\theta}$$

For a differential loop from r to $r + dr$, θ to $\theta + d\theta$

$$\begin{aligned} \oint_{\Gamma_i} \vec{F} \cdot d\vec{r} &= F_r(r, \theta)\, dr + F_\theta(r + dr, \theta) \cdot d\theta \cdot (r + dr) \\ &\quad - F_r(r, \theta + d\theta)\, dr - F_\theta(r, \theta)\, d\theta \cdot r \\ &= \frac{\partial (r \cdot F_\theta)}{\partial r} \cdot dr \cdot d\theta - \frac{\partial F_r}{\partial \theta} \cdot d\theta \cdot dr \\ &= \left(\frac{1}{r} \frac{\partial}{\partial r} (r F_\theta) - \frac{1}{r} \frac{\partial F_r}{\partial \theta} \right) \cdot dA \end{aligned}$$



Then for any loop:

$$\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \sum_i \oint_{\Gamma_i} \vec{F} \cdot d\vec{r} = \iint_{S} \left(\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \cdot dr \cdot d\theta$$

$$\oint_{\Gamma} (F_r dr + r F_\theta d\theta) = \iint_{S} \left(\frac{1}{r} \frac{\partial}{\partial r} (r F_\theta) - \frac{1}{r} \frac{\partial F_r}{\partial \theta} \right) \cdot dA \quad ; \quad dA = r dr d\theta$$

3. CURL IN x-y PLANE

Definition:

$$\left(\text{curl}\,\vec{F}\right)_n = \lim_{\Delta S \to 0} \frac{\oint \vec{F} \cdot d\vec{r}}{\Delta S}\ , \qquad \hat{n} \text{ :normal vector of } \Delta S$$

$$\therefore \left(\text{curl}\,\vec{F}\right)_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \qquad \hat{n} = \hat{k} \text{ for } \Delta S \text{ on } x\text{-}y \text{ plane}$$

$$\Longrightarrow \left(\vec{\nabla} \times \vec{F}\right)_z \qquad z\text{-component of curl}\vec{F}$$

4. CURL IN POLAR COORDINATES IN PLANE

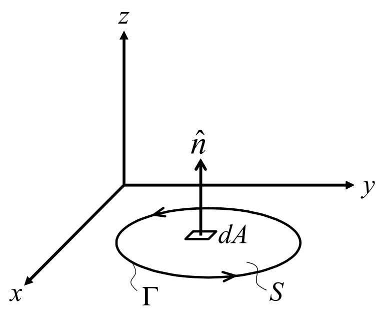
$$\left(\text{curl}\,\vec{F}\right)_z = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right) = \left(\vec{\nabla} \times \vec{F}\right)_z$$

5. FLUX INTEGRAL IN x-y PLANE

$$\iint_S \vec{B} \cdot \hat{n} dA$$

$\hat{n}$: normal vector of area element dA

$\hat{n} = \hat{k}$ for dA on x-y plane, $dA = dxdy$ or $rdrd\theta$



$$\therefore \iint_S \vec{B} \cdot \hat{n} dA = \iint_S B_z \cdot dx \cdot dy \qquad \text{(Cartesian coordinates)}$$

$$\text{or} \iint_S \vec{B} \cdot \hat{n} dA = \iint_S B_z \cdot r \cdot dr \cdot d\theta \qquad \text{(polar coordinates)}$$

6. GREEN'S THEOREM

$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \oint_\Gamma \left(F_x \cdot dx + F_y \cdot dy\right) = \iint_S \left(\vec{\nabla} \times \vec{F}\right)_z \cdot dxdy$$

$$= \iint_S \left(\vec{\nabla} \times \vec{F}\right)_z \cdot \hat{n} dA \qquad , \hat{n} = \hat{k}$$

The circulation of $\vec{F}$ around loop Γ equals the flux of $\left(\vec{\nabla} \times \vec{F}\right)$ through the area S.

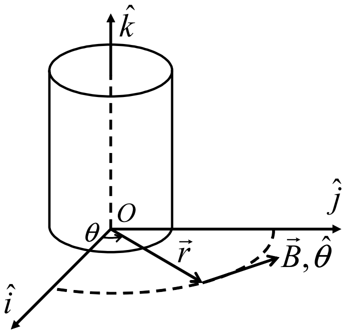
7. MAGNETIC FIELD $\vec{B}$ GENERATED BY A WIRE WITH RADIUS a AND CURRENT I ALONG z-AXIS

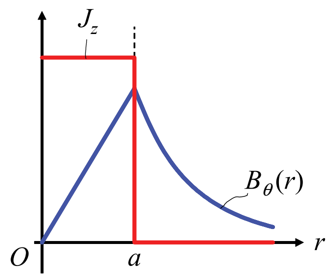
By Ampere law of circulation

$$\oint_{\Gamma} \vec{B} \cdot d\vec{\ell} = \mu \iint_{S} \vec{J} \cdot \hat{n} dA$$

with the current density $\vec{J}$:

$$r \geq a \; ; \quad \vec{J} = 0$$
$$r \leq a \; ; \quad \vec{J} = \hat{k} I / \pi a^2$$





$$\vec{B}\left(\vec{r}\right) = \begin{cases} \dfrac{\mu I}{2\pi r} \cdot \hat{\theta} & r \geq a \\ \\ \dfrac{\mu I r}{2\pi a^2} \cdot \hat{\theta} & r \leq a \end{cases} \qquad \therefore \vec{B}\left(\vec{r}\right) = B_\theta\left(r\right) \cdot \hat{\theta}$$

① For $r \geq a$

$$\left(\operatorname{curl} \vec{B}\right)_z = 0 \qquad \therefore \left(\operatorname{curl} \vec{B}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot B_\theta\right) = 0$$

② For $r \leq a$

$$\left(\operatorname{curl} \vec{B}\right)_z = \mu \cdot J \qquad \therefore \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\mu I r}{2\pi a^2}\right) = \frac{\mu I}{\pi a^2}$$

③ In Cartesian coordinates, $\hat{\theta} = -\sin\theta \hat{\imath} + \cos\theta \hat{\jmath}$

$$\vec{B} = \frac{\mu I}{2\pi a^2}\left(-y\hat{\imath} + x\hat{\jmath}\right) \qquad r \leq a$$

$$\vec{B} = \frac{\mu I}{2\pi}\left(\frac{-y}{\left(x^2+y^2\right)}\hat{\imath} + \frac{x}{\left(x^2+y^2\right)}\hat{\jmath}\right) \qquad r \geq a$$

$$\left(\operatorname{curl} \vec{B}\right)_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{\mu I}{\pi a^2} \qquad r \leq a$$

$$\left(\operatorname{curl} \vec{B}\right)_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \qquad r \geq a$$