(d)**截距式：** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(e)**平行平面：** $E_1: ax+by+cz+d=0$

平行 E_1 **且過點** $P_0(x_0\ ,\ y_0\ ,\ z_0)$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

(f)**過** $E_1: a_1x+b_1y+c_1z+d_1=0$

$E_2: a_2x+b_2y+c_2z+d_2=0$

交線之平面

$(a_1x+b_1y+c_1z+d_1)+k(a_2x+b_2y+c_2z+d_2)=0$

7. **兩平面的夾角：**

(a)**兩平面夾角即為其法向量之夾角**

$E_1: a_1x+b_1y+c_1z+d_1=0 \quad \overrightarrow{n_1}:(a_1\ ,\ b_1\ ,\ c_1)$

$E_2: a_2x+b_2y+c_2z+d_2=0 \quad \overrightarrow{n_2}:(a_2\ ,\ b_2\ ,\ c_2)$

$$\cos q=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

(b) $E_1 \perp E_2 \Leftrightarrow a_1a_2+b_1b_2+c_1c_2=0$

(c) $E_1 // E_2 \Leftrightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\frac{d_1}{d_2}$ (**重合**)

$\neq \frac{d_1}{d_2}$ (**不重合**)

8. (a)**點到平面的距離：** $P:(x_1\ ,\ y_1\ ,\ z_1)$ $E: ax+by+cz+d=0$

若 O **點** $(x_0\ ,\ y_0\ ,\ z_0)$ **為平面上一點,**

$\overrightarrow{OP}:(x_1-x_0\ ,\ y_1-y_0\ ,\ z_1-z_0)$, $\vec{n}:(a\ ,\ b\ ,\ c)$

$$\Rightarrow d=|\overrightarrow{OP}\cdot\hat{n}|=\frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

(b)**兩平行平面：** $E_1: ax+by+cz+d_1=0$, $E_2: ax+by+cz+d_2=0$

若 $P(x_0\ ,\ y_0\ ,\ z_0)$ **為** E_1 **上一點則** $ax_0+by_0+cz_0+d_1=0$

P **到** E_2 **的距離** $d: \frac{|ax_0+by_0+cz_0+d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$

(c)**角平分面：** $E_1: a_1x+b_1y+c_1z+d_1=0$, $E_2: a_2x+b_2y+c_2z+d_2=0$

角平分面 $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$

習題：過點 $A(1,-2,2)$ **與** $B(6,0,-1)$ **且與平面** $E:2x+2y-z-1=0$ **垂直之平面。**

習題：已知平面 $E_1:x+2y-3z+2=0$, $E_2:3x-2y+z-5=0$

若平面 E **過** E_1、E_2 **的交線，且**

(1)**過點** $(1,-1,2)$**，則平面** E **之方程式為？**

(2)**與平面** $E_3:2x+y-z-1=0$ **垂直，則** E**？**

習題：已知空間兩點 $A(-3,-1,1)$, $B(2,-2,3)$ **及平面**

$E:2x+2y-z-6=0$**，則線段** AB **在平面** E **上的投影長度？**

9. **空間直線方程式：**

(a)**參數式：過** $P(x_1,y_1,z_1)$ **與** $Q(x_2,y_2,z_2)$ **之直線方程式**

$$\begin{cases} x=x_1+t(x_2-x_1) \\ y=y_1+t(y_2-y_1) \\ z=z_1+t(z_2-z_1) \end{cases}\text{，其方向向量}(x_2-x_1,y_2-y_1,z_2-z_1)$$

(b)**對稱比例式：過** $P(x_0,y_0,z_0)$ **且方向為** (a,b,c) **之直線**

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

(c)**二面式：相交兩平面** $E_1:a_1x+b_1y+c_1z+d_1=0$

$E_2:a_2x+b_2y+c_2z+d_2=0$

直線方向： (n_1,n_2,n_3)**，則** $\begin{cases} a_1n_1+b_1n_2+c_1n_3=0 \\ a_2n_1+b_2n_2+c_2n_3=0 \end{cases}$

$$\Rightarrow (n_1,n_2,n_3)\propto \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}:\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}:\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

10. (a)**直線與平面的關係：相交於一點，落在平面上，平行**

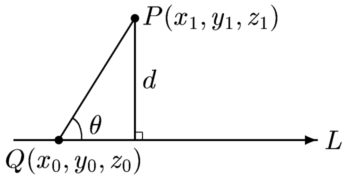
(b)**空間中兩直線之關係：交於一點，平行(重合)，歪斜**

11. **空間中直線的距離公式：**

(a) $P:(x_1,y_1,z_1)$

$$L:\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

$$\overline{PQ}=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}$$



$$\overline{PQ}\cos q = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d = \overline{PQ}\sin q = \sqrt{(\overline{PQ})^2 - (\overline{PQ}\cos q)^2}$$

或求 $\sqrt{[at + (x_0 - x_1)]^2 + [bt + (y_0 - y_1)]^2 + [ct + (z_0 - z_1)^2]^2}$ **之極值**

(b)**兩平行線間之距離：** $L_1 : \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

$$L_2 : \frac{x - x_2}{a} = \frac{y - y_2}{b} = \frac{z - z_2}{c}$$

習題：過點 $A(1, 2, 3)$ **且與直線** $\begin{cases} x + y + z = 0 \\ 2x + y - 2z - 1 = 0 \end{cases}$ **相垂直之平面**

習題：包含兩平行線 $L_1 : \frac{x-1}{3} = \frac{y-3}{-2} = \frac{z+1}{1}$ **與** $L_2 : \frac{x}{3} = \frac{y+1}{-2} = \frac{z-3}{1}$ **之平面方程**

習題：兩歪斜線 $L_1 : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$**，** $L_2 : \frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{4}$

求(1)**包含** L_2 **且與** L_1 **平行之平面**

(2) L_1 **與** L_2 **公垂線之長度**(**距離**)

12. **空間向量的外積：** $\vec{C} = \vec{A} \times \vec{B}$

(a) $\vec{C}$ **是一個向量，** $\vec{C} \perp \vec{A}$, $\vec{C} \perp \vec{B}$ (**右手定則**)

(b) $|\vec{C}|$ **是由** $\vec{A}$, $\vec{B}$ **向量所構成的平行四邊形面積**

由條件(a) $\begin{cases} A_1C_1 + A_2C_2 + A_3C_3 = 0 \\ B_1C_1 + B_2C_2 + B_3C_3 = 0 \end{cases}$

$$\Rightarrow C_1 : C_2 : C_3 = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} : \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix} : \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

由條件(b)**知** $\sqrt{C_1^2 + C_2^2 + C_3^2} = \sqrt{\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix}^2 + \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix}^2 + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}^2}$

$\Rightarrow \vec{C} = (\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix}, \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix}, \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix})$ (**右手定則**)

$$* \ \hat{x}\times\hat{y}=\hat{z},\ \ \hat{y}\times\hat{z}=\hat{x},\ \ \hat{z}\times\hat{x}=\hat{y}$$

$$\begin{aligned}
&(A_1\hat{x}+A_2\hat{y}+A_3\hat{z})\times(B_1\hat{x}+B_2\hat{y}+B_3\hat{z})\\
&=(A_1B_2\hat{z}-A_1B_3\hat{y})+(-A_2B_1\hat{z}+A_2B_3\hat{x})+(A_3B_1\hat{y}-A_3B_2\hat{x})\\
&=(A_2B_3-A_3B_2)\hat{x}+(A_3B_1-A_1B_3)\hat{y}+(A_1B_2-A_2B_1)\hat{z}\\
&=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}
\end{aligned}$$

(c)**線性**$\begin{cases}\vec{A}\times(b_1\vec{B}_1+b_2\vec{B}_2)=b_1(\vec{A}\times\vec{B}_1)+b_2(\vec{A}\times\vec{B}_2)\\ (a_1\vec{A}_1+a_2\vec{A}_2)\times\vec{B}=a(\vec{A}_1\times\vec{B})+a_2(\vec{A}_2\times\vec{B})\end{cases}$

反對稱性$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

(d)Levi-Civita **符號**$\in_{ijk}$, $i,\ j,\ k=1,\ 2,\ 3$

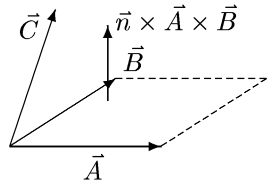
$$\begin{cases}\in_{123}=\in_{231}=\in_{312}=1\\ \in_{321}=\in_{132}=\in_{213}=-1\end{cases}$$

其他都為零， $\vec{C}=\vec{A}\times\vec{B}\Leftrightarrow C_i=\sum_{j,\ k=1}^{3}\in_{ijk}\ A_jB_k$

13. **向量三重積：**$\vec{A},\ \vec{B},\ \vec{C}$ **構成的平行六面體的體積**

$$(\vec{A}\times\vec{B})\cdot\vec{C}=C_1(A_2B_3-A_3B_2)+C_2(A_3B_1-A_1B_3)+C_3(A_1B_2-A_2B_1)$$

$$\Rightarrow(\vec{A}\times\vec{B})\cdot\vec{C}=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$



$\vec{C}\cdot\hat{n}=\vec{C}$ **沿法線方向的長度。**

(a) $(\vec{A}\times\vec{B})\cdot\vec{C}=0$，**三向量共平面**

(b) $(\vec{A}\times\vec{B})\cdot\vec{C}\neq 0$，**三向量不共平面(線性獨立)**

(c) $(\vec{A}\times\vec{B})\cdot\vec{C}=(\vec{B}\times\vec{C})\cdot\vec{A}=(\vec{C}\times\vec{A})\cdot\vec{B}$

$$=\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

14. $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$

$$
\begin{aligned}
&=(A_2B_3-A_3B_2)(C_2D_3-C_3D_2)+(A_3B_1-A_1B_3)(C_3D_1-C_1D_3)+(A_1B_2-A_2B_1)(C_1D_2-C_2D_1)\\
&=A_1C_1(B_2D_2+B_3D_3)+A_2C_2(B_1D_1+B_3D_3)+A_3C_3(B_1D_1+B_2D_2)-B_1C_1(A_2D_2+A_3D_3)\\
&\quad-B_2C_2(A_1D_1+A_3D_3)-B_3C_3(A_1D_1+A_2D_2)\\
&=(A_1C_1+A_2C_2+A_3C_3)(B_1D_1+B_2D_2+B_3D_3)-(B_1C_1+B_2C_2+B_3C_3)(A_1D_1+A_2D_2+A_3D_3)\\
&=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{B}\cdot\vec{C})(\vec{A}\cdot\vec{D})\\
&=\begin{vmatrix}\vec{A}\cdot\vec{C} & \vec{A}\cdot\vec{D}\\ \vec{B}\cdot\vec{C} & \vec{B}\cdot\vec{D}\end{vmatrix}
\end{aligned}
$$

15. $(\vec{A}\times\vec{B})\times\vec{C}=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{B}\cdot\vec{C})\vec{A}$

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$$

習題：證明上式

提示： $\vec{D}=(\vec{A}\times\vec{B})\times\vec{C},\ \vec{D}\perp(\vec{A}\times\vec{B})\ \Rightarrow\vec{D}$ **落在** $\vec{A},\ \vec{B}$ **平面上**

$$\Rightarrow(\vec{A}\times\vec{B})\times\vec{C}=a\,\vec{A}+b\,\vec{B}$$

習題：利用 $(\vec{A}\times\vec{B})\times\vec{C}=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{B}\cdot\vec{C})\vec{A}$ **證明**

$$\sum_{i=1}^{3}\in_{ijk}\in_{imn}=d_{jm}d_{kn}-d_{jn}d_{km}$$

$d_{11}=d_{22}=d_{33}=1$，**其他皆為零** Kronecker Delta **符號**

一次方程組與矩陣

1. **一次方程組的解：**

(a)二元聯立：$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$

$$\Rightarrow \begin{cases} (a_1b_2-b_1a_2)x=(c_1b_2-b_1c_2) \\ (a_2b_1-a_1b_2)y=(c_1a_2-a_1c_2) \end{cases}$$

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

(b)三元聯立：$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$

$$(c_2a_1-c_1a_2)x+(c_2b_1-c_1b_2)y=(c_2d_1-c_1d_2)$$
$$(c_3a_2-c_2a_3)x+(c_3b_2-c_2b_3)y=(c_3d_2-c_2d_3)$$

證明： $x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$

(c)**定義**$\overrightarrow{R}:(x\,,\,y),\ \overrightarrow{A_1}:(a_1\,,\,b_1),\ \overrightarrow{A_2}:(a_2\,,\,b_2)$

$\begin{cases} \overrightarrow{R}\cdot\overrightarrow{A_1}=C_1 \\ \overrightarrow{R}\cdot\overrightarrow{A_2}=C_2 \end{cases}$**，令**$\overrightarrow{R}=a_1\overrightarrow{B_1}+a_2\overrightarrow{B_2}$**求**$a_1$**，**$a_2$

定義$\overrightarrow{B_1}=\dfrac{(b_2\,,-a_2)}{(a_1b_2-b_1a_2)},\ \overrightarrow{B_2}=\dfrac{(b_1\,,-a_1)}{(a_2b_1-b_2a_1)}$

$\begin{cases} \overrightarrow{B_1}\cdot\overrightarrow{A_1}=1,\ \overrightarrow{B_2}\cdot\overrightarrow{A_2}=0 \\ \overrightarrow{B_2}\cdot\overrightarrow{A_1}=0,\ \overrightarrow{B_2}\cdot\overrightarrow{A_2}=1 \end{cases}$**，**$\overrightarrow{B_1}$**，**$\overrightarrow{B_2}$ **為**$\overrightarrow{A_1}$**，**$\overrightarrow{A_2}$ **的對偶基底**

$\Rightarrow \begin{cases} a_1=\overrightarrow{R}\cdot\overrightarrow{A_1}=C_1 \\ a_2=\overrightarrow{R}\cdot\overrightarrow{A_2}=C_2 \end{cases}\quad \overrightarrow{R}=C_1\overrightarrow{B_1}+C_2\overrightarrow{B_2}$