## Home Work Solutions 2

## 1. Electric dipole and atmospheric sprites

Sprites (Fig. 22-9a) are huge flashes that occur far above a large thunderstorm. They were seen for decades by pilots flying at night, but they were so brief and dim that most pilots figured they were just illusions. Then in the 1990s sprites were captured on video. They are still not well understood but are believed to be produced when especially powerful lightning occurs between the ground and storm clouds, particularly when the lightning transfers a huge amount of negative charge $-q$ from the ground to the base of the clouds (Fig. 22-9b).

Just after such a transfer, the ground has a complicated distribution of positive charge. However, we can model the electric field due to the charges in the clouds and the ground by assuming a vertical electric dipole that has charge $-q$ at cloud height $h$ and charge $+q$ at below-ground depth $h$ (Fig. 22-9c). If $q=200 \mathrm{C}$ and $h=6.0 \mathrm{~km}$, what is the magnitude of the dipole's electric field at altitude $z_{1}=30 \mathrm{~km}$ somewhat above the clouds and altitude $z_{2}=60 \mathrm{~km}$ somewhat above the stratosphere?

(a) Photograph of a sprite. (Courtesy NASA)
(b) Lightning in which a large amount of negative charge is transferred from ground to cloud base.
(c) The cloud-ground system modeled as a vertical electric dipole.

## Sol

We can approximate the magnitude $E$ of an electric dipole's electric field on the dipole axis with Eq. 22-8.

## Calculations:

We write that equation as

where $2 h$ is the separation between $-q$ and $+q$ in Fig. $22-9 c$. For the electric field at altitude $z_{1}=30 \mathrm{~km}$, we find

(Answer)

Similarly, for altitude $z_{2}=60 \mathrm{~km}$, we find


As we discuss in Section 22-8, when the magnitude of an electric field exceeds a certain critical value $E_{c}$, the field can pull electrons out of atoms (ionize the atoms), and then the freed electrons can run into other atoms, causing those atoms to emit light. The value of $E_{c}$ depends on the density of the air in which the electric field exists. At altitude $z_{2}=60 \mathrm{~km}$ the density of the air is so low that $E=2.0 \times 10^{2} \mathrm{~N} / \mathrm{C}$ exceeds $E_{c}$, and thus light is emitted by the atoms in the air. That light forms sprites. Lower down, just above the clouds at $z_{1}=30 \mathrm{~km}$, the density of the air is much higher, $E=1.6 \times 10^{3} \mathrm{~N} / \mathrm{C}$ does not exceed $E_{c}$, and no light is emitted. Hence, sprites occur only far above storm clouds.
2. In Fig. 22-8, let both charges be positive. Assuming $z>_{d \text {, show that } E \text { at point } P \text { in that figure is then given by }}$

$$
E=\frac{1}{4 \pi \varepsilon_{0}} \frac{2 q}{z^{2}}
$$

Sol
Since both charges are positive (and aligned along the $z$ axis) we have

$$
\left|E_{\text {net }}^{\mathrm{r}}\right|=\frac{1}{4 \pi \varepsilon_{0}}\left[\frac{q}{(z-d / 2)^{2}}+\frac{q}{(z+d / 2)^{2}}\right] .
$$

For $z \gg d$ we have $(z \pm d / 2)^{-2} \approx z^{-2}$, so
$\left|E_{\text {net }}^{\mathrm{r}}\right| \approx \frac{1}{4 \pi \varepsilon_{0}}\left(\frac{q}{z^{2}}+\frac{q}{z^{2}}\right)=\frac{2 q}{4 \pi \varepsilon_{0} z^{2}}$.
3. In Fig. 22-51, a "semi-infinite" nonconducting rod (that is, infinite in one direction only) has uniform linear charge density $\lambda$. Show that the electric field $\boldsymbol{E}_{\mathbf{p}}$ at point $P$ makes an angle of $45^{\circ}$ with the rod and that this result is independent of the distance $R$. (Hint: Separately find the component of $\boldsymbol{E}_{\mathbf{p}}$ parallel to the rod and the component perpendicular to the rod.)


Figure 22-51 Problem 33.

Sol
Consider an infinitesimal section of the rod of length $d x$, a distance $x$ from the left end, as shown in the following diagram. It contains charge $d q=\lambda d x$ and is a distance $r$ from $P$. The magnitude of the field it produces at $P$ is given by

$$
d E=\frac{1}{4 \pi \varepsilon_{0}} \frac{\lambda d x}{r^{2}}
$$



The $x$ and the $y$ components are

$$
d E_{x}=-\frac{1}{4 \pi \varepsilon_{0}} \frac{\lambda d x}{r^{2}} \sin \theta
$$

and

$$
d E_{y}=-\frac{1}{4 \pi \varepsilon_{0}} \frac{\lambda d x}{r^{2}} \cos \theta
$$

respectively. We use $\theta$ as the variable of integration and substitute $r=R / \cos \theta, x=R \tan \theta$ and $d x=\left(R / \cos ^{2} \theta\right) d \theta$. The limits of integration are 0 and $\pi / 2$ rad. Thus,

$$
E_{x}=-\frac{\lambda}{4 \pi \varepsilon_{0} R} \int_{0}^{\pi / 2} \sin \theta d \theta=\left.\frac{\lambda}{4 \pi \varepsilon_{0} R} \cos \theta\right|_{0} ^{\pi / 2}=-\frac{\lambda}{4 \pi \varepsilon_{0} R}
$$

and

$$
E_{y}=-\frac{\lambda}{4 \pi \varepsilon_{0} R} \int_{0}^{\pi / 2} \cos \theta d \theta=-\left.\frac{\lambda}{4 \pi \varepsilon_{0} R} \sin \theta\right|_{0} ^{\pi / 2}=-\frac{\lambda}{4 \pi \varepsilon_{0} R} .
$$

We notice that $E_{x}=E_{y}$ no matter what the value of $R$. Thus, $\vec{E}$ makes an angle of $45^{\circ}$ with the rod for all values of $R$.
4. A thin nonconducting rod with a uniform distribution of positive charge $Q$ is bent into a circle of radius $R$ (Fig. 22-43). The central perpendicular axis through the ring is a $z$ axis, with the origin at the center of the ring. What is the magnitude of the electric field due to the rod at (a) $z=0$ and (b) $z=\infty$ ? (c) In terms of $R$, at what positive value of $z$ is that magnitude maximum? (d) If $R=2.00 \mathrm{~cm}$ and $Q=4.00 \mu \mathrm{C}$, what is the maximum magnitude?


Figure 22-43 Problem 24.
Sol
(a) It is clear from symmetry (also from Eq. 22-16) that the field vanishes at the center.
(b) The result $(E=0)$ for points infinitely far away can be reasoned directly from Eq. 22-16 (it goes as $1 / z^{2}$ as $z \rightarrow \infty$ ) or by recalling the starting point of its derivation (Eq. 22-11, which makes it clearer that the field strength decreases as $1 / r^{2}$ at distant points).
(c) Differentiating Eq. 22-16 and setting equal to zero (to obtain the location where it is maximum) leads to

$$
\frac{d}{d z}\left(\frac{q z}{4 \pi \varepsilon_{0}\left(z^{2}+R^{2}\right)^{3 / 2}}\right)=\frac{q}{4 \pi \varepsilon_{0}} \frac{R^{2}-2 z^{2}}{\left(z^{2}+R^{2}\right)^{5 / 2}}=0 \Rightarrow z=+\frac{R}{\sqrt{2}}=0.707 R
$$

(d) Plugging this value back into Eq. 22-16 with the values stated in the problem, we find $E_{\max }=$ $4.33 \times 10^{7} \mathrm{~N} / \mathrm{C}$.
5. Find an expression for the oscillation frequency of an electric dipole of dipole moment and rotational inertia $I$ for small amplitudes of oscillation about its equilibrium position in a uniform electric field of magnitude $E$.

Sol
Equation 22-35 $\quad \theta=-p E \sin \theta$ ( captures the sense as well as the magnitude of the effect. That is, this is a restoring torque, trying to bring the tilted dipole back to its aligned equilibrium position. If the amplitude of the motion is small, we may replace $\sin \theta$ with $\theta$ in radians. Thus, $\tau \approx-p E \theta$.
Since this exhibits a simple negative proportionality to the angle of rotation, the dipole oscillates in simple harmonic motion, like a torsional pendulum with torsion constant $\kappa=p E$. The angular frequency $\omega$ is given by

$$
\omega^{2}=\frac{\kappa}{I}=\frac{p E}{I}
$$

where I is the rotational inertia of the dipole. The frequency of oscillation is

$$
f=\frac{\omega}{2 \pi}=\frac{1}{2 \pi} \sqrt{\frac{p E}{I}} .
$$

